Dynamic analysis of Kirchhoff plates

Special structural elements

ME473 Dynamic finite element analysis of structures

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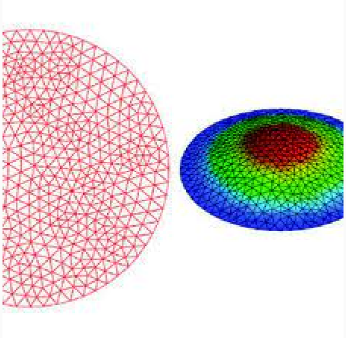
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	
7		Planar beams	Project 1 submission
8		Frames and grids	Project 2 statement
9		Kirchhoff-Love plates	

Summary

- Recap weeks 6 to 8
- Kirchhoff-Love plate theory
- AMC thin plate bending elements
- CR thin plate bending elements

Recommended readings

(P) Petyt, Introduction to finite element vibration analysis (chap. 6)

(N) Neto et al., Engineering Computation of Structures (chap. 6.1)

(O) Ochsner, PDE for classical structural members (chap. 6)

(G) Gmür, Méthode des éléments finis (chap. 3)

Recap weeks 6 to 8
Vibrations of unidimensional structures

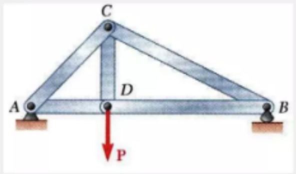
Trusses vs frames

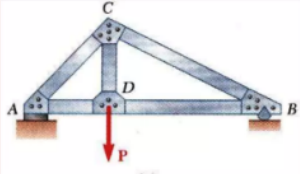
Truss

- Structure composed of oriented bar (rod) elements, connected by frictionless pins, carrying **axial forces** only.

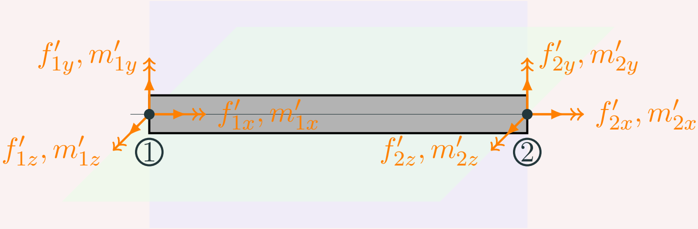
Frame

- Structure composed of oriented beam elements, connected by welding, carrying **transversal, axial forces and torsion**.





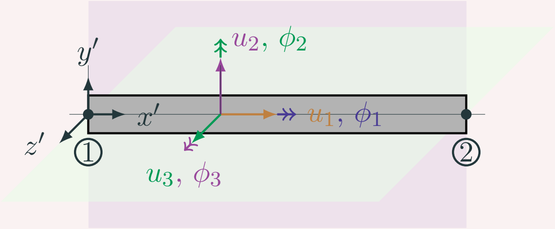
Three-dimensional thin beam structure



Three-dimensional thin beams are uniaxial (slender) element that can support:

- axial loads f'_{ix},
- torsional loads m'_{ix},
- bending in the $x' - y'$ plane: f'_{iy} and m'_{iz},
- bending in the $x' - z'$ plane: f'_{iz} and m'_{iy}.

Differential equations governing the dynamics of thin beams



Bars	$EA\partial^2_{x'x'}u_1(x',t) = \rho A\ddot{u}_1(x',t)$
Shafts	$GJ\partial^2_{x'x'}\phi_1(x',t) = \rho J\ddot{\phi}_1(x',t)$
Planar $(Ox'y')$ *beams*	$\partial^2_{x'x'}\left(EI_{z'}\partial^2_{x'x'}u_2(x',t)\right) + \rho A\ddot{u}_2(x',t) = 0$
Planar $(Ox'z')$ *beams*	$\partial^2_{x'x'}\left(EI_{y'}\partial^2_{x'x'}u_3(x',t)\right) + \rho A\ddot{u}_3(x',t) = 0$

$I_{y'}$ and $I_{z'}$ are the cross-sectional moments of inertia with respect to the axes y and z.

Displacements discretization

Total of **six nodal displacements at each unconstrained node**:

- three translation components q'_{ix}, q'_{iy} and q'_{iz} along the x, y, z axes, and
- three rotational components about these axes ϕ'_{ix}, ϕ'_{iy} and ϕ'_{iz}.

$$\mathbf{q}_{loc}(t) = \begin{bmatrix} q'_{1x}(t) \\ q'_{1y}(t) \\ q'_{1z}(t) \\ \phi'_{1x}(t) \\ \phi'_{1y}(t) \\ \phi'_{1z}(t) \\ q'_{2x}(t) \\ q'_{2y}(t) \\ q'_{2z}(t) \\ \phi'_{2x}(t) \\ \phi'_{2y}(t) \\ \phi'_{2z}(t) \end{bmatrix}$$

$$u^h(x', t) = \mathbf{H}(x')\mathbf{q}_{loc}(t)$$

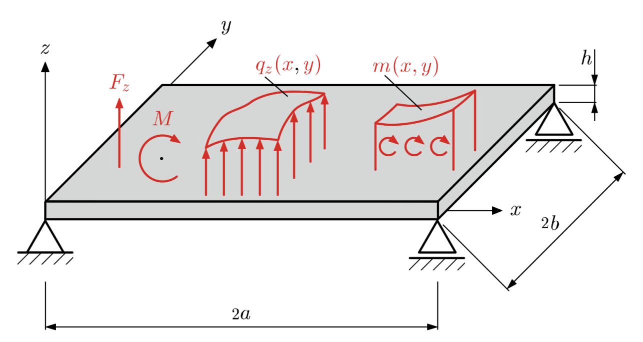
$$\begin{aligned}
h_1(x') &= 1 - x'/\ell & h_7(x') &= x'/\ell \\
h_2(x') &= 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 & h_8(x') &= 3(x'/\ell)^2 - 2(x'/\ell)^3 \\
h_3(x') &= 2(x'/\ell)^3 - 3(x'/\ell)^2 + 1 & h_9(x') &= 3(x'/\ell)^2 - 2(x'/\ell)^3 \\
h_4(x') &= 1 - x'/\ell & h_{10}(x') &= x'/\ell \\
h_5(x') &= x'(1 - x'/\ell)^2 & h_{11}(x') &= x'(x'/\ell)(x'/\ell - 1) \\
h_6(x') &= x'(1 - x'/\ell)^2 & h_{12}(x') &= x'(x'/\ell)(x'/\ell - 1)
\end{aligned}$$

- The stiffness matrix $\mathbf{K}_{loc}$ and mass matrices $\mathbf{M}_{loc}$ for a three-dimensional beam element are written as the superposition of the **axial** matrix, the **torsional** matrix and the **flexural** matrices in the two bending planes.

Classic plate theory

Plate structure

- Plate structures are geometrically similar to structures of the 2D plane stress problem, but it usually carries only transversal loads that lead to **bending** deformation of the plate.
- For example: floors of a building, aerospace and ships structures, etc...



(Credit: (O))

Plate models

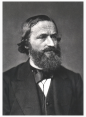
Kirchhoff (1888) and Love (1945)

- *Shear free plates*: thin plates where the contribution of shear force on the deformations is neglected.
- Two-dimensional extension of the Bernoulli-Euler beam theory.

Mindlin (1951) and Reissner (1945)

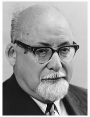
- *Shear deformable plates*: thick plates where the contribution of shear force on deformations is considered.
- Two-dimensional extension of the Timoshenko beam theory.

Gustav Kirchhoff



1888

Augustus Love
Eric Reissner



1945

Raymond Mindlin

1951

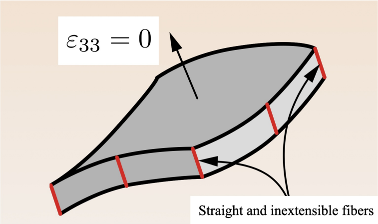
Geometry assumptions

- The thickness of the plate h is constant and much smaller than the planar dimensions a and b:

$$h/a < 0.1 \quad \text{and} \quad h/b < 0.1.$$

- **Inextensibility of transverse fibers**: transverse normal strain is negligible.

$$\varepsilon_{33} = 0.$$



(Credit:(G))

Material and loads assumptions

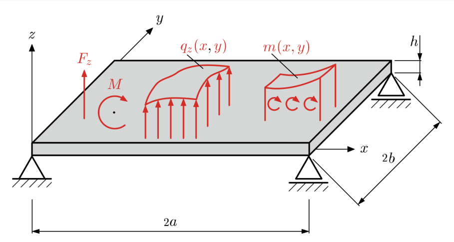
Material

- The material is homogenous and linear-elastic according to Hooke's law for a plane stress state

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0.$$

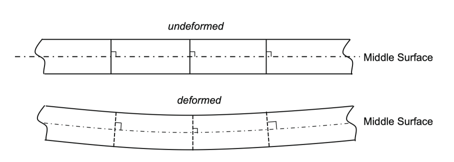
Loads

- External forces act only perpendicular to the $x - y$ plane, the vector of external moments lies within the $x - y$ plane.
- Displacement $u_3(x, y, t)$ is small compared to h: $u_3 \lesssim 0.2h$.



Kirchhoff assumption

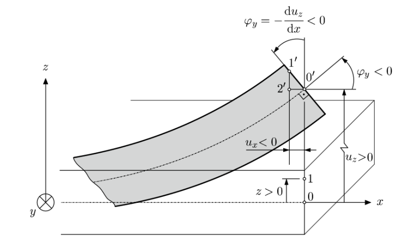
Rectilinearity of the normals: Bernoulli's hypothesis is valid, i.e. a cross-sectional plane stays plane and unwrapped in the deformed state.
This means that the shear strains ε_{13} and ε_{23} due to the distributed shear forces q_x and q_y are neglected.



A straight fiber that is perpendicular to the middle plane of the plate before deformation remain straight and normal to it after deformation.

(Credit: (N))

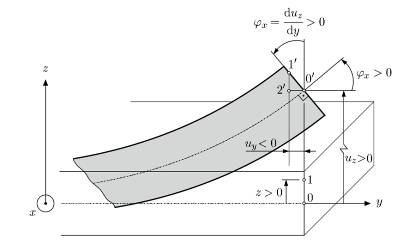
Kinematics assumptions



$$-\varphi_2 \approx \sin(-\varphi_2) = -\frac{u_1}{z}$$

$$-\varphi_2 \approx \tan(-\varphi_2) = \frac{du_3}{dx}$$

$$\implies u_1 = -z\frac{du_3}{dx}$$



$$\varphi_1 \approx \sin(\varphi_1) = -\frac{u_2}{z}$$

$$\varphi_1 \approx \tan(\varphi_1) = \frac{du_3}{dy}$$

$$\implies u_2 = -z\frac{du_3}{dx}$$

Transverse displacement u_3 is the *only* independent variable:

$$\mathbf{u} = \begin{bmatrix} -z\frac{\partial u_3}{\partial x} \\ -z\frac{\partial u_3}{\partial y} \\ u_3 \end{bmatrix}$$

Deformation is exaggerated in the figures for better illustration.

Strain-displacement relation

- Using classical engineering definitions of strain:

$$\varepsilon_{ii} = \partial_i u_i \qquad \text{and} \qquad \gamma_{ij} = \partial_i u_j + \partial_j u_i$$

we obtain

$$\underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}}_{\boldsymbol{\varepsilon}} = -z \underbrace{\begin{bmatrix} \partial^2_{xx} \\ \partial^2_{yy} \\ 2\partial^2_{xy} \end{bmatrix}}_{\boldsymbol{\nabla}_k} u_3 = z \underbrace{\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}}_{\boldsymbol{\kappa}}$$

- $\boldsymbol{\kappa}$ is the matrix that contains the changes in the **curvature of the plate**, given as $\boldsymbol{\kappa} = -\boldsymbol{\nabla}_k u_3$.
- Note that $\varepsilon_{12} = \varepsilon_{23} = 0$ due to Kirchhoff assumptions and $\varepsilon_{33} = 0$ due to the inextensibility of transverse fibers assumption.

Constitutive equation for isotropic material

- Classical plate theory assumes a plane stress state:

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0.$$

- Constitutive equation for *isotropic* material is $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\sigma}$ where

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix},$$

or

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(\nu+1) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}.$$

- $\mathbf{C}$ is the *elasticity matrix* and $\mathbf{D} = \mathbf{C}^{-1}$ is the *elastic compliance matrix*.

Constitutive equation for orthotropic material

- The constitutive equation for *orthotropic* material is $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\sigma}$ where

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

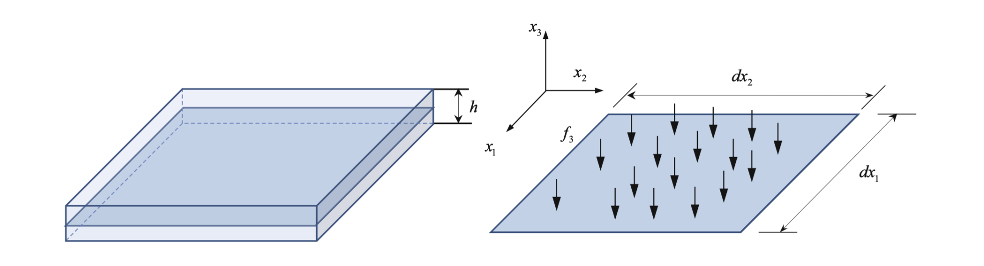
  where

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{33} = G_{12}$$

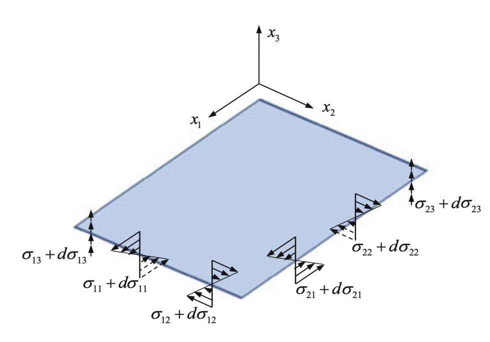
- The orthotropic properties of the lamina are given: E_1, E_2, ν_{12}, G_{12} and $\nu_{21} = \nu_{12}E_2/E_1$ applies.

External forces

Consider a plate cell of dimensions $dx_1 \times dx_2 \times h$ that is submitted to external forces, here denoted by f_3, and inertial force proportional to the material density.



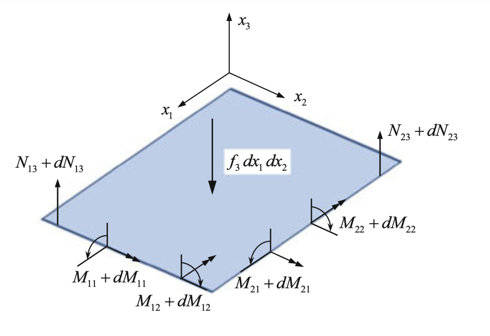
Distributed normal and shear stresses



Normal and shear stresses distributions through the thickness of the plate element:

- linear distributed normal stresses σ_{11} and σ_{22},
- linear distributed shear stresses σ_{12} and σ_{21},
- parabolic distributed shear stresses σ_{23} and σ_{13}.

Moments and shear forces



Moments and shear forces acting along the edge of the plate:

- bending moments M_{11} and M_{22},
- twisting moment M_{12},
- shear forces N_{13} and N_{23}.

$$\mathbf{M} = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\, \boldsymbol{\sigma}\, dx_3 = -\boldsymbol{C}\boldsymbol{\nabla}_k u_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2\, dx_3 = -\frac{h^3}{12}\boldsymbol{C}\boldsymbol{\nabla}_k u_3$$

Dynamic equilibrium equation

- Equilibrium condition for the vertical forces:

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + f_3 - \rho h \ddot{u}_3 = 0$$

- Equilibrium of moments:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} = 0$$

$$\frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - N_{23} = 0$$

Combining the three equations above, we obtain the **dynamic equilibrium equation for Kirchhoff-Love plate bending**:

$$\boldsymbol{\nabla}_k^T \mathbf{M} + f_3 = \rho h \ddot{u}_3$$

Strong form for Kirchhoff-Love plate bending

Let $\Omega = [-a, a] \times [-b, b]$. Find the transverse displacement $u_3 \in C^4(\Omega \times [0, T])$ such that

$$\frac{h^3}{12} \boldsymbol{\nabla}_k^T \boldsymbol{C} \boldsymbol{\nabla}_k u_3 + \rho h \ddot{u}_3 = f_3 \qquad \text{on } \Omega \times]0, T[$$

Boundary conditions (*simply supported*):

$$u_3 = 0 \qquad \text{in } \partial\Omega \times]0, T[$$

$$\mathbf{M}_n = 0 \qquad \text{in } \partial\Omega \times]0, T[$$

Initial conditions:

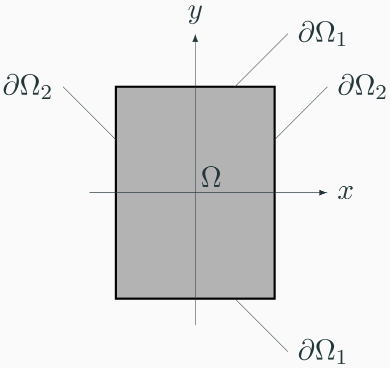
$$u_3(\cdot, 0) = u_0 \quad \text{in } \Omega$$

$$\dot{u}_3(\cdot, 0) = v_0 \quad \text{in } \Omega$$

In case of isotropic material, the strong form equation reduces to

$$D \left(\frac{\partial^4 u_3}{\partial x_1^4} + 2 \frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u_3}{\partial x_2^4} \right) + \rho h \ddot{u}_3 = f_3$$

where $D = Eh^3/(12(1 - \nu^2))$.



Simply supported on all 4 edges:

- No vertical displacement:

$$u_3 = 0 \qquad \text{in } \partial\Omega\times]0,T[$$

- No moment resistance (free to rotate):

$$M_{11} = -D(\partial_x\varphi_2 + \nu\partial_y\varphi_1) = 0 \qquad \text{in } \partial\Omega_1\times]0,T[$$
$$M_{22} = -D(\nu\partial_x\varphi_2 + \partial_y\varphi_1) = 0 \qquad \text{in } \partial\Omega_2\times]0,T[$$

These conditions are replaced by the approximated conditions:

$$\varphi_2 = -\partial_x u_3 = 0 \qquad \text{in } \partial\Omega_1\times]0,T[$$
$$\varphi_1 = \partial_y u_3 = 0 \qquad \text{in } \partial\Omega_2\times]0,T[$$

Weak form for Kirchhoff-Love simply supported plate

The weak form consists of finding the transverse displacement $u_3 \in \mathcal{U}$ such that the following equation is satisfied for every virtual transversal displacement $\delta u_3 \in \mathcal{V}$:

$$\frac{h^3}{12} \int_\Omega (\boldsymbol{\nabla}_k \delta u_3)^T \mathbf{C}\, \boldsymbol{\nabla}_k u_3 \, d\Omega + \int_\Omega \rho h \, \ddot{u}_3 \, \delta u_3 \, d\Omega = \int_\Omega f_3 \, \delta u_3 \, d\Omega$$

The functional spaces are defined as

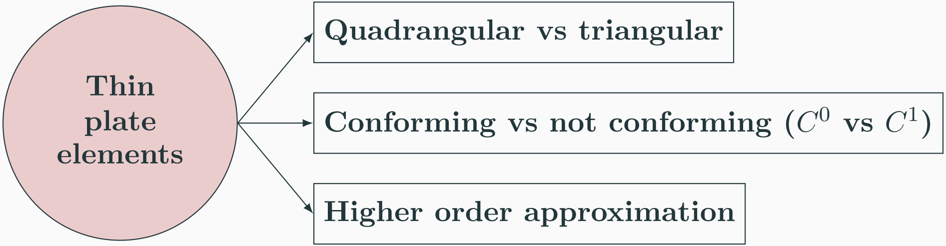
$$\mathcal{U} = \left\{ u_3(\cdot, t) \in H^2(\Omega) \mid u_3 = 0 \text{ in } \partial\Omega \times]0, T[\right\}$$

$$\mathcal{V} = \left\{ \delta u_3 \in H^2(\Omega) \mid \delta u_3 = 0 \text{ in } \partial\Omega \right\}$$

Thin plate bending elements

Overview of thin plate bending elements

Numerous finite elements for plate bending have been developed: **more than 88 distinct types** can be identified.



- **Adini-Melosh-Clough element (AMC)**:
  12 DOFs quadrangular, not conforming, thin plate bending element.
- **Crouzeix–Raviart (CR)**:
  16 DOFs quadrangular, conforming, thin plate bending element.

Finite element approximation of Kirchhoff-Love plate

- Weak form equation

$$\frac{h^3}{12}\int_\Omega (\boldsymbol{\nabla}_k \delta u_3)^T \mathbf{C}\, \boldsymbol{\nabla}_k u_3 \, d\Omega + \int_\Omega \rho h\, \ddot{u}_3\, \delta u_3 \, d\Omega = \int_\Omega f_3\, \delta u_3\, d\Omega$$

- Semi-discrete weak form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

Finite element approximation using polynomial shape functions:

$${}^e u_3^h(x_1, x_2, t) = \sum_{i=1}^{n} {}^e\mathbf{h}_i(x_1, x_2)\, {}^e\mathbf{q}^i(t) = {}^e\mathbf{H}(\mathbf{x})\, {}^e\mathbf{q}(t)$$

Selection of the displacement approximation function

To ensure convergence it is necessary to consider:

Completeness criterion: highest order of derivatives in the weak form is 2.

→

Complete polynomials of at least degree 2:

$$
\begin{aligned}
{}^{e}u_3^h = {} & a_1 + a_2 x + a_3 y + \\
& + a_4 x^2 + a_5 xy + a_6 y^2 + \\
& + \textit{higher order terms}
\end{aligned}
$$

Continuity criterion: ${}^{e}u_3^h$, $\partial_{x_1}{}^{e}u_3^h$, $\partial_{x_2}{}^{e}u_3^h$ are continuous between elements.

→

Each node i has 3 DOFs:

- ${}^{e}d^i$ vertical displacement,
- ${}^{e}\theta_1^i, {}^{e}\theta_2^i$ rotations.

AMC plate bending elements

Selection of the displacement approximation function

				1					Constant
			x_1		x_2				Linear
		x_1^2		$x_1 x_2$		x_2^2			Quadratic
	x_1^3		$x_1^2 x_2$		$x_1 x_2^2$		x_2^3		Cubic
x_1^4		$x_1^3 x_2$		$x_1^2 x_2^2$		$x_1 x_2^3$		x_2^4	Quartic

Displacement approximation for rectangular elements with four nodes and thus 12 DOFs: complete cubic polynomial, augmented with two (*geometrically invariant*) quartic terms

$$
\begin{aligned}
{}^{e}u_3^h(x_1, x_2, t) = {} & a_1 + a_2 x_1 + a_3 x_2 + a_4 x_1^2 + a_5 x_1 x_2 + a_6 x_2^2 + \\
& + a_7 x_1^3 + a_8 x_1^2 x_2 + a_9 x_1 x_2^2 + a_{10} x_2^3 + a_{11} x_1^3 x_2 + a_{12} x_1 x_2^3
\end{aligned}
$$

- ✓ Displacement approximation solves the unloaded strong form equation.
- ✓ Continuity in displacement along the interfaces of the elements.
- ✗ Slopes continuity along the interfaces are not ensured (*not conforming*).

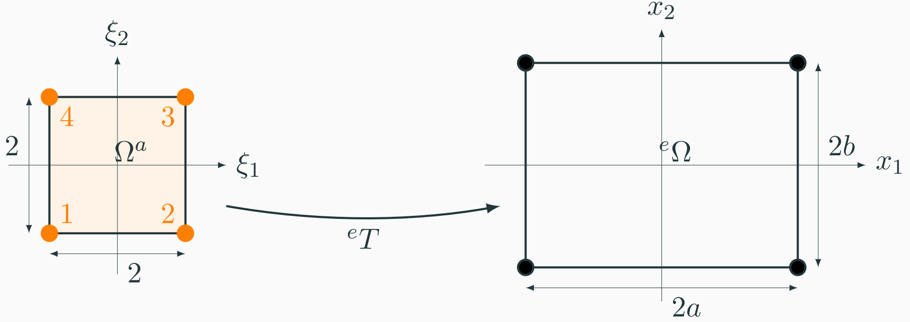
Coordinate transform

The coordinate transformation:

$$^{e}T : \Omega^{a} \to {}^{e}\Omega$$

$$\boldsymbol{\xi} = \{\xi_1, \xi_2\}^T \mapsto \mathbf{x}(\boldsymbol{\xi}) = \{x_1(\boldsymbol{\xi}), x_2(\boldsymbol{\xi})\}^T = \{a\xi_1, b\xi_2\}^T$$

maps any point $\boldsymbol{\xi}$ in $\Omega^a = [-1, 1] \times [-1, 1]$ to its corresponding point of coordinate $\mathbf{x}(\boldsymbol{\xi})$ in ${}^{e}\Omega = [-a, a] \times [-b, b]$:

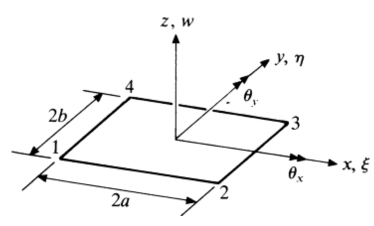


Approximate displacement in local coordinates

$$
\begin{aligned}
{}^{e}u_3^h(\boldsymbol{\xi},t) &= \sum_{i=1}^{4} {}^{a}\mathbf{h}_i(\boldsymbol{\xi})\,{}^{e}\mathbf{q}^i(t) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{q}(t) \\
&= \begin{bmatrix} {}^{a}\mathbf{h}_1(\boldsymbol{\xi}) & {}^{a}\mathbf{h}_2(\boldsymbol{\xi}) & {}^{a}\mathbf{h}_3(\boldsymbol{\xi}) & {}^{a}\mathbf{h}_4(\boldsymbol{\xi}) \end{bmatrix} \begin{bmatrix} {}^{e}\mathbf{q}^1(t) \\ {}^{e}\mathbf{q}^2(t) \\ {}^{e}\mathbf{q}^3(t) \\ {}^{e}\mathbf{q}^4(t) \end{bmatrix}
\end{aligned}
$$

where

- $${}^{e}\mathbf{q}^i(t) = \begin{bmatrix} {}^{e}d^i(t) \\ {}^{e}\theta_1^i(t) \\ {}^{e}\theta_2^i(t) \end{bmatrix} = \begin{bmatrix} {}^{e}u_3^h(\boldsymbol{\xi}^i,t) \\ \partial_{\xi_2}\,{}^{e}u_3^h(\boldsymbol{\xi}^i,t)/b \\ -\partial_{\xi_1}\,{}^{e}u_3^h(\boldsymbol{\xi}^i,t)/a \end{bmatrix}$$
- $\boldsymbol{\xi}^i$ are the local coordinates of node i.



DOFs in local coordinates

- Assumed form of the displacement function:

$$\begin{bmatrix} {}^{e}u_3^h \\ \partial_{\xi_2}\, {}^{e}u_3^h \\ \partial_{\xi_1}\, {}^{e}u_3^h \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_1^2 & \xi_1\xi_2 & \xi_2^2 & \xi_1^3 & \xi_1^2\xi_2 & \xi_1\xi_2^2 & \xi_2^3 & \xi_1^3\xi_2 & \xi_1\xi_2^3 \\ 0 & 0 & 1 & 0 & \xi_1 & 2\xi_2 & 0 & \xi_1^2 & 2\xi_1\xi_2 & 3\xi_2^2 & \xi_1^3 & 3\xi_1\xi_2^2 \\ 0 & 1 & 0 & 2\xi_1 & \xi_2 & 0 & 3\xi_1^2 & 2\xi_1\xi_2 & \xi_2^2 & 0 & 3\xi_1^2\xi_2 & \xi_2^3 \end{bmatrix}}_{\mathbf{P}(\boldsymbol{\xi})} \underbrace{\begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_{12}(t) \end{bmatrix}}_{\boldsymbol{\alpha}(t)}$$

- Then the DOFs in local coordinates are:

$${}^{e}\mathbf{q}^{i}(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & -1/a \end{bmatrix} \mathbf{P}(\boldsymbol{\xi}^i)}_{\bar{\mathbf{P}}(\boldsymbol{\xi}^i)} \ \boldsymbol{\alpha}(t)$$

Shape functions matrix for the AMC element

- Since

$$
{}^e\mathbf{q}(t) = \begin{bmatrix} {}^e\mathbf{q}^1(t) \\ {}^e\mathbf{q}^2(t) \\ {}^e\mathbf{q}^3(t) \\ {}^e\mathbf{q}^4(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{\mathbf{P}}(\boldsymbol{\xi}^1) \\ \bar{\mathbf{P}}(\boldsymbol{\xi}^2) \\ \bar{\mathbf{P}}(\boldsymbol{\xi}^3) \\ \bar{\mathbf{P}}(\boldsymbol{\xi}^4) \end{bmatrix}}_{\mathbf{A}} \boldsymbol{\alpha}(t) \qquad \Rightarrow \qquad \boldsymbol{\alpha}(t) = \mathbf{A}^{-1}\, {}^e\mathbf{q}(t)
$$

- Then the shape functions matrix is ${}^a\mathbf{H} = \begin{bmatrix} {}^a\mathbf{h}_1 & {}^a\mathbf{h}_2 & {}^a\mathbf{h}_3 & {}^a\mathbf{h}_4 \end{bmatrix} = \mathbf{P}_1\mathbf{A}^{-1}$ where

$$
{}^a\mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} (1+\xi_1^i\xi_1)(1+\xi_2^i\xi_2)(2+\xi_1^i\xi_1+\xi_2^i\xi_2-\xi_1^2-\xi_2^2)/8 \\ b(1+\xi_1^i\xi_1)(\xi_2^i+\xi_2)(\xi_2^2-1)/8 \\ -a(\xi_1^i+\xi_1)(\xi_1^2-1)(1+\xi_2^i\xi_2)/8 \end{bmatrix}^T
$$

and $\boldsymbol{\xi}^i = \{\xi_1^i, \xi_2^i\}^T$ are the local coordinates of node i.

Deformation, stiffness, mass matrices and loads vector

- ${}^a\mathbf{B} = \nabla_k {}^a\mathbf{H}$ is a (3×12) matrix:

$$ {}^a\mathbf{B} = \begin{bmatrix} \frac{1}{a^2}\partial^2_{\xi_1} \\ \frac{1}{b^2}\partial^2_{\xi_2} \\ \frac{2}{ab}\partial^2_{\xi_1\xi_2} \end{bmatrix} \begin{bmatrix} {}^a\mathbf{h}_1 & {}^a\mathbf{h}_2 & {}^a\mathbf{h}_3 & {}^a\mathbf{h}_4 \end{bmatrix} = \left[\begin{bmatrix} \frac{1}{a^2}\partial^2_{\xi_1} \\ \frac{1}{b^2}\partial^2_{\xi_2} \\ \frac{2}{ab}\partial^2_{\xi_1\xi_2} \end{bmatrix} {}^a\mathbf{h}_1 \quad \dots \quad \begin{bmatrix} \frac{1}{a^2}\partial^2_{\xi_1} \\ \frac{1}{b^2}\partial^2_{\xi_2} \\ \frac{2}{ab}\partial^2_{\xi_1\xi_2} \end{bmatrix} {}^a\mathbf{h}_4 \right] $$

- ${}^e\mathbf{K}$ and ${}^e\mathbf{M}$ are (12×12) matrices and ${}^e\mathbf{r}$ is a (12×1) vector:

$$ {}^e\mathbf{K} = \frac{h^3}{12} \int_\Omega {}^a\mathbf{B}^T \mathbf{C} {}^a\mathbf{B}\, d\Omega = \frac{h^3 ab}{12} \int_{-1}^{1}\int_{-1}^{1} {}^a\mathbf{B}^T \mathbf{C} {}^a\mathbf{B}\, d\xi_1 d\xi_2, $$

$$ {}^e\mathbf{M} = \int_\Omega \rho h\, {}^a\mathbf{H}^T {}^a\mathbf{H}\, d\Omega = \rho h ab \int_{-1}^{1}\int_{-1}^{1} {}^a\mathbf{H}^T {}^a\mathbf{H}\, d\xi_1 d\xi_2, $$

$$ \mathbf{r}(t) = \int_\Omega {}^a\mathbf{H}^T f_3(t)\, d\Omega = ab f_3(t) \int_{-1}^{1}\int_{-1}^{1} {}^a\mathbf{H}^T\, d\xi_1 d\xi_2. $$

Post processing

- The **approximated stresses** ${}^e\sigma^h_{11}$ and ${}^e\sigma^h_{22}$ and ${}^e\sigma^h_{12}$ at any point (x_1, x_2, x_3) of the element e are given in term of the nodal displacements:

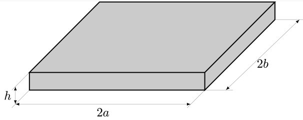
$$ {}^e\boldsymbol{\sigma}^h = \mathbf{C}\,{}^e\boldsymbol{\varepsilon}^h = -x_3 \mathbf{C} \boldsymbol{\nabla}_k {}^e u^h_3 = -x_3 \mathbf{C}\,{}^e\mathbf{B}\,{}^e\mathbf{q}(t) $$

- The **approximated bending moments** ${}^eM^h_{11}$ and ${}^eM^h_{22}$ and twisting moment ${}^eM^h_{12}$ per unit length are given by:

$$ {}^e\mathbf{M} = \begin{bmatrix} {}^eM^h_{11} \\ {}^eM^h_{22} \\ {}^eM^h_{12} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \begin{bmatrix} {}^e\sigma^h_{11} \\ {}^e\sigma^h_{22} \\ {}^e\sigma^h_{12} \end{bmatrix} dx_3 = -\frac{h^3}{12} \boldsymbol{C}\,{}^e\mathbf{B}\,{}^e\mathbf{q}(t) $$

Example: isotropic square plate in free vibrations

Use AMC elements to estimate the five lowest frequencies of an isotropic square plate ($\ell \times \ell$) which is simply supported on all four edges (SSSS).



Compare the results with the analytical solution

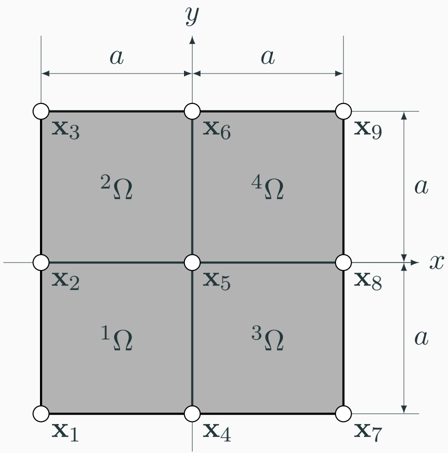
$$\omega_{m,n} = \pi^2 \frac{m^2 + n^2}{\ell^2} \sqrt{\frac{D}{\rho h}} \text{ rad/s},$$

where ℓ is the length of each side and (m, n) are the number of half-waves in the x- and y-directions and D is flexural rigidity of the plate:

$$D = \frac{Eh^3}{12(1 - \nu^2)}.$$

Simply supported isotropic square plate

Discretization with a rectangular mesh of 2×2 AMC elements.



- $2a = 1$ length $[m]$
- $2a = 1$ height $[m]$
- $h = 0.01$ thickness $[m]$
- $E = 210$ Young's modulus $[GPa]$
- $\nu = 0.3$ Poisson's ratio
- $\rho = 7800$ material density $[kg/m^3]$

Symmetry and boundary conditions

We exploit the symmetry of the problem by modeling only one-quarter of the plate with 4 AMC elements, applying appropriate boundary conditions to account for the symmetric and antisymmetric modes along the edges.

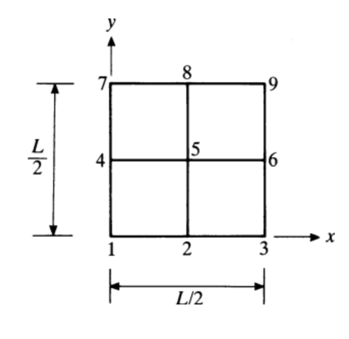


Figure 1: One-quarter of the plate represented by four rectangular elements.

- **Simple support**:
  - sides 1-3: $u_3 = \theta_2 = 0$ at nodes 1, 2, and 3,
  - sides 1-7: $u_3 = \theta_1 = 0$ at nodes 1, 4, and 7.
- **Symmetric modes**:
  - with respect to side 3-9: $\theta_2 = 0$ at nodes 3, 6, and 9,
  - with respect to side 7-9: $\theta_1 = 0$ at nodes 7, 8, and 9.
- **Antisymmetric modes**:
  - with respect to side 3-9: $u_3 = \theta_1 = 0$ at nodes 3, 6, and 9.
  - with respect to side 7-9: $u_3 = \theta_2 = 0$ at nodes 7, 8, and 9.

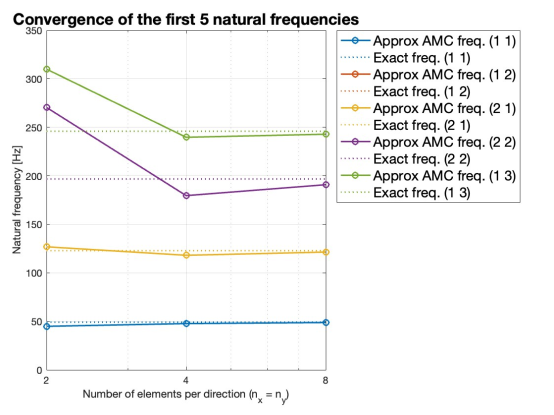
Frequencies approximation for square plate with AMC elements

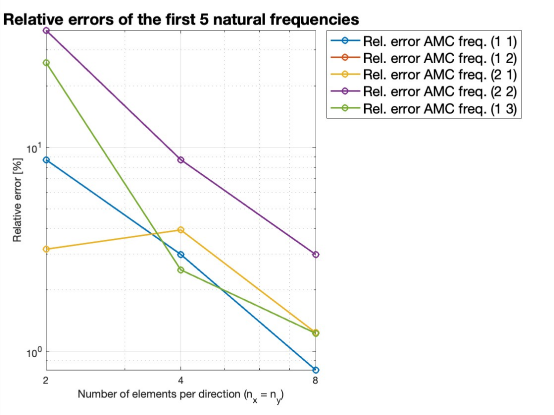
Comparison of exact and approximate natural frequencies with a mesh of 2×2 AMC elements for a thin SSSS square plate.

m	n	Exact Freq. [Hz]	Approx. Freq. [Hz]	Rel. Error (%)
1	1	49.171	44.892	-8.7032
1	2	122.93	126.82	3.1687
2	1	122.93	126.82	3.1687
2	2	196.69	270.44	37.496
1	3	245.86	309.92	26.059
3	1	245.86	517.41	110.45
2	3	319.61	517.41	61.886

Frequencies approximation for square plate with AMC elements

In these figures, we present the convergence and relative errors of the first five natural frequencies of a square plate, obtained using AMC elements for different mesh refinements.





Modal shapes

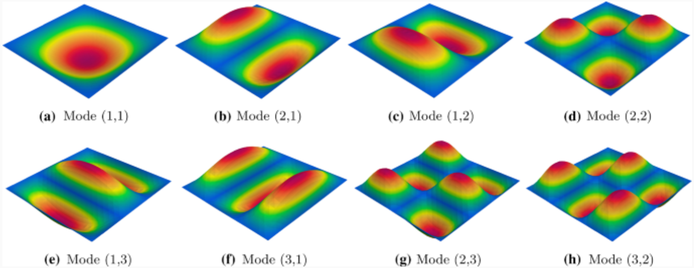
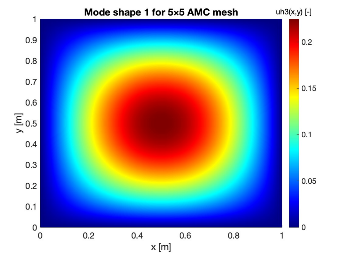
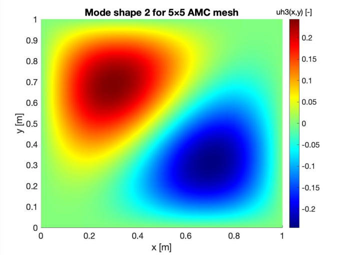


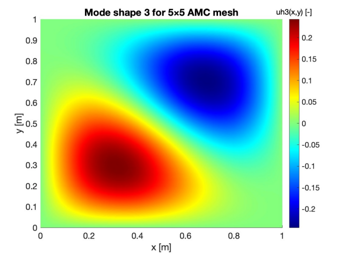
Figure 2: Mode shapes for a simply supported (SSSS) square plate.

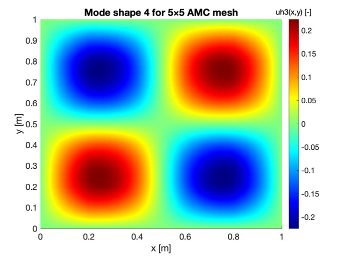
(Credit: Pagani, Azzara, Carrera - Geometrically nonlinear analysis and vibration of in-plane-loaded variable angle tow composite plates and shells)

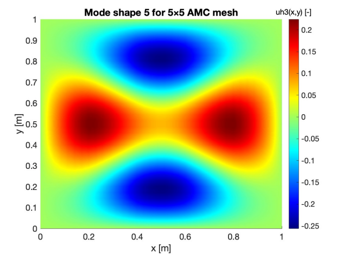
Modal patterns











CR thin plate bending elements

Selection of the displacement approximation function

$$
\begin{array}{ccccccc|l}
 & & & 1 & & & & \text{Constant} \\
 & & x_1 & & x_2 & & & \text{Linear} \\
 & x_1^2 & & x_1x_2 & & x_2^2 & & \text{Quadratic} \\
x_1^3 & & x_1^2x_2 & & x_1x_2^2 & & x_2^3 & \text{Cubic} \\
 & x_1^3x_2 & & x_1^2x_2^2 & & x_1x_2^3 & & \text{Quartic} \\
 & & x_1^3x_2^2 & & x_1^2x_2^3 & & & \text{Quintic} \\
 & & & x_1^3x_2^3 & & & & \text{Sextic}
\end{array}
$$

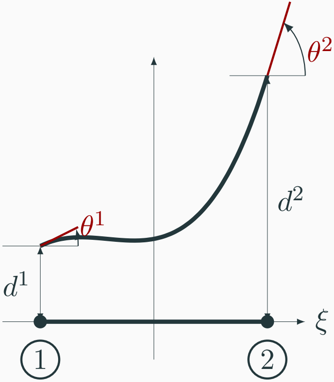
Displacement approximation for rectangular elements with four nodes and 16 DOFs (4 DOFs per node): complete cubic polynomial, augmented with three quartic, two quintic and one sextic terms:

$$
\begin{aligned}
{}^{e}u_3^h(x_1, x_2, t) = & \; a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_1x_2 + a_6x_2^2 + \\
& + a_7x_1^3 + a_8x_1^2x_2 + a_9x_1x_2^2 + a_{10}x_2^3 + \\
& + a_{11}x_1^3x_2 + a_{12}x_1x_2^3 + a_{13}x^2y^2 + a_{14}x^3y^2 + a_{15}x^2y^3 + a_{16}x^3y^3
\end{aligned}
$$

Recall: transversal bending for Euler-Bernoulli beam

Approximated transversal displacement u_2^has a function of the nodal DOFs:

$$ {}^e u_2^h(\xi, t) = \mathbf{H}(\xi)\mathbf{q}(t) = [h_1(\xi),\ h_2(\xi),\ h_3(\xi),\ h_4(\xi)] \begin{bmatrix} d^1(t) \\ \theta^1(t) \\ d^2(t) \\ \theta^2(t) \end{bmatrix} $$



C^1 **Hermite shape functions on** $[-1, 1]$**:**

$$ h_1(\xi) = (\xi^3 - 3\xi + 2)/4 $$
$$ h_3(\xi) = (-\xi^3 + 3\xi + 2)/4 $$
$$ h_2(\xi) = (\xi^3 - \xi^2 - \xi + 1)/4 $$
$$ h_4(\xi) = (\xi^3 + \xi^2 - \xi - 1)/4 $$

Shape functions matrix: a first try

- Let ξ^i be the coordinate of the node i. Then the Hermite shape functions matrix is $\mathbf{H}(\xi) = [f_1 \ g_1 \ f_2 \ g_2]$ where

$$f_i(\xi) = (-\xi^i\xi^3 + 3\xi^i\xi + 2)/4 \qquad g_i(\xi) = (\xi^3 + \xi^i\xi^2 - \xi - \xi^i)/4.$$

- The shape functions matrix for the plate bending element is a product of Hermite functions:

$$^a\mathbf{H} = \begin{bmatrix} ^a\mathbf{h}_1 & ^a\mathbf{h}_2 & ^a\mathbf{h}_3 & ^a\mathbf{h}_4 \end{bmatrix}$$

where

$$^a\mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} f_i(\xi_1)f_i(\xi_2) \\ bf_i(\xi_1)g_i(\xi_2) \\ -ag_i(\xi_1)f_i(\xi_2) \end{bmatrix}^T.$$

Zero-twist constraint

- The approximate displacement in local coordinate is defined as:

$$ {}^{e}u_3^h(\boldsymbol{\xi}, t) = \sum_{i=1}^{4} {}^{a}\mathbf{h}_i(\xi_1, \xi_2)\, {}^{e}\mathbf{q}^i(t) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\, {}^{e}\mathbf{q}(t) $$

✗ The twist

$$ \frac{\partial^2}{\partial \xi_1 \partial \xi_2} {}^{e}u_3^h $$

is zero at the four nodal points. Thus the plate will tend to a zero twist condition as an increasing number of elements are used.

Solution: introduce twist

$$ \theta_{12}^i = \frac{\partial^2}{\partial \xi_1 \partial \xi_2} {}^{e}u_3^h(\boldsymbol{\xi}^i, t) $$

as an extra degree of freedom.

Shape functions matrix for the CR element

The approximate displacement in local coordinate is defined as:

$$ {}^{e}u_3^h(\boldsymbol{\xi},t) = \sum_{i=1}^{4} {}^{a}\mathbf{h}_i(\boldsymbol{\xi})\,{}^{e}\mathbf{q}^i(t) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{q}(t) $$

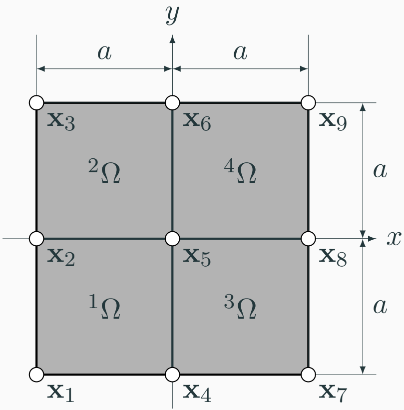
- 4 DOFs per node: ${}^{e}\mathbf{q}^i(t) = \begin{bmatrix} {}^{e}d^i(t) \\ {}^{e}\theta_1^i(t) \\ {}^{e}\theta_2^i(t) \\ {}^{e}\theta_{12}^i(t) \end{bmatrix} = \begin{bmatrix} {}^{e}u_3^h(\boldsymbol{\xi}^i,t) \\ \partial_{\xi_2}\,{}^{e}u_3^h(\boldsymbol{\xi}^i,t)/b \\ -\partial_{\xi_1}\,{}^{e}u_3^h(\boldsymbol{\xi}^i,t)/a \\ \partial^2_{\xi_1\xi_2}\,{}^{e}u_3^h(\boldsymbol{\xi}^i,t)/(ab) \end{bmatrix}$
- The shape function matrix is ${}^{a}\mathbf{H} = \begin{bmatrix} {}^{a}\mathbf{h}_1 & {}^{a}\mathbf{h}_2 & {}^{a}\mathbf{h}_3 & {}^{a}\mathbf{h}_4 \end{bmatrix}$ where

$$ {}^{a}\mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} f_i(\xi_1)f_i(\xi_2) \\ b f_i(\xi_1)g_i(\xi_2) \\ -a g_i(\xi_1)f_i(\xi_2) \\ ab\, g_i(\xi_1)g_i(\xi_2) \end{bmatrix}^T $$

Example: isotropic square plate in free vibrations

Frequencies estimate for square plate with CR element

A mesh of 2×2 CR elements is used to estimate the fundamental frequencies of a square plate ($\ell \times \ell$) which is simply supported on all four edges (SSSS).



- $2a = 1$ length $[m]$
- $2a = 1$ height $[m]$
- $h = 0.01$ thickness $[m]$
- $E = 210e9$ Young's modulus $[Pa]$
- $\nu = 0.3$ Poisson's ratio
- $\rho = 7800$ material density $[kg/m^3]$

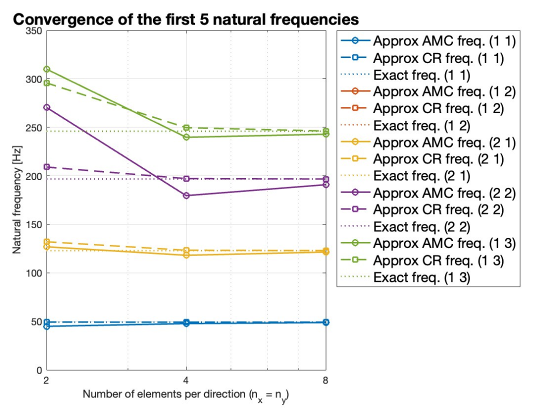
Frequencies approximation for square plate with CR elements

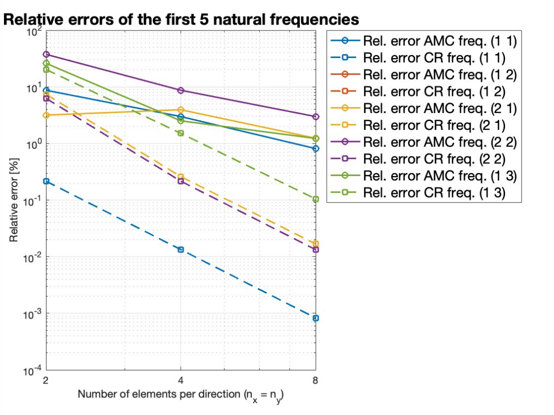
Comparison of exact and approximate natural frequencies with a mesh of 2×2 CR elements for a thin SSSS square plate.

m	n	Exact Freq. [Hz]	Approx. Freq. [Hz]	Rel. Error (%)
1	1	49.171	49.277	0.21558
1	2	122.93	132	7.3816
2	1	122.93	132	7.3816
2	2	196.69	209.01	6.2666
1	3	245.86	295.69	20.269
3	1	245.86	295.69	20.27
2	3	319.61	366.03	14.523

Comparison: AMC vs CR plate bending element

In these figures, we present the convergence and relative errors of the first five natural frequencies of a square plate, obtained using AMC and CR elements for different mesh refinements.





Comparison: AMC vs CR plate bending element

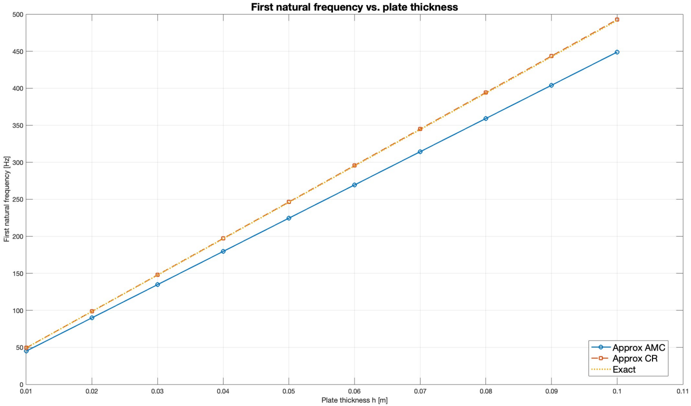


Figure 3: Evolution of the approximated and exact fundamental frequencies with thickness for a SSSS square plate.

Comparison: AMC vs CR plate bending element

AMC element (*Non conforming*)

- 3 degrees of freedom per node: u_3, θ_1, θ_2.
- Only displacement u_3 is continuous across elements; rotations may have jumps.
- Nonconforming element: does not fully satisfy C^1 continuity.
- Simpler and computationally cheaper; suitable for practical applications.

CR element (*Conforming*)

- 4 degrees of freedom per node: u_3, θ_1, θ_2, and θ_{12}.
- Displacement u_3 and rotations θ_1, θ_2 are continuous across element boundaries.
- Fully conforming to the C^1 continuity required by Kirchhoff plate theory.
- Higher computational cost and complexity.

Why would you use AMC element if it's nonconforming?

- Computationally cheaper than full C^1 elements (fewer DOFs).
- It still converges (theoretical results for nonconforming FEMs show convergence *under certain conditions*).
- Suitable when small slope discontinuities are acceptable (e.g., dynamic problems, large meshes).
- Tends to underestimate the natural frequencies, making it a useful benchmark for detecting overstiffness in numerical model.