Dynamic analysis of planar beams

Special structural elements

ME473 Dynamic finite element analysis of structures

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Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	
7		Planar beams	Project 1 submission

Summary

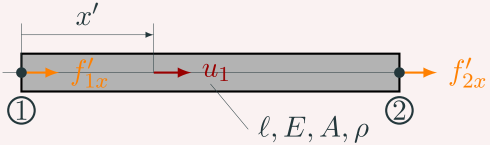
- Recap week 6
- Euler-Bernoulli beams
- Geometric stiffness and buckling
- Timoshenko beams

Recommended readings

1. Logan, A first course in the finite element method, 6th ed. (chap. 4)
2. Paz and Leigh, Structural dynamics, 6th ed. (chap. 10)
3. Ferreira and Fantuzzi, MATLAB Codes for Finite Element Analysis, 2nd ed. (chap. 6 and 10)

Recap week 6
Vibrations of bars and trusses

Non-oriented bar element



Differential equation governing the dynamics:

$$EA\partial^2_{x'x'}u_1(x',t) = \rho A\ddot{u}_1(x',t)$$

- Displacements approximation:

$$u_1^h(x',t) = \mathbf{H}(x')\mathbf{q}_{loc}(t) = \begin{bmatrix} h_1(x') & h_2(x') \end{bmatrix} \begin{bmatrix} q'_{1x}(t) \\ q'_{2x}(t) \end{bmatrix}$$

- Linear local shape functions:

$$h_1(x') = 1 - \frac{x'}{\ell} \qquad \text{and} \qquad h_2(x') = \frac{x'}{\ell}$$

- Semi-discrete weak form:

$$\boldsymbol{\delta}\mathbf{q}_{loc}^T\big(\mathbf{M}_{loc}\ddot{\mathbf{q}}_{loc}(t) + \mathbf{K}_{loc}\mathbf{q}_{loc}(t) - \mathbf{f}_{loc}(t)\big) = 0$$

Non-oriented bar element discretization

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \int_0^\ell EA \,\frac{d\mathbf{H}}{dx'}^T \frac{d\mathbf{H}}{dx'}\, dx' = \int_0^\ell EA \begin{bmatrix} (h_1')^2 & h_1' h_2' \\ h_2' h_1' & (h_2')^2 \end{bmatrix} dx' = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \int_0^\ell \rho A \,\mathbf{H}^T \mathbf{H}\, dx' = \int_0^\ell \rho A \begin{bmatrix} (h_1)^2 & h_1 h_2 \\ h_2 h_1 & (h_2)^2 \end{bmatrix} dx' = \frac{\rho A \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Element applied loads vector in local coordinates:

$$\mathbf{f}_{loc}(t) = \begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} f'_{1x}(t) + \begin{bmatrix} h_1(\ell) \\ h_2(\ell) \end{bmatrix} f'_{2x}(t) = \begin{bmatrix} f'_{1x}(t) \\ f'_{2x}(t) \end{bmatrix}$$

Arbitrarly oriented bar element

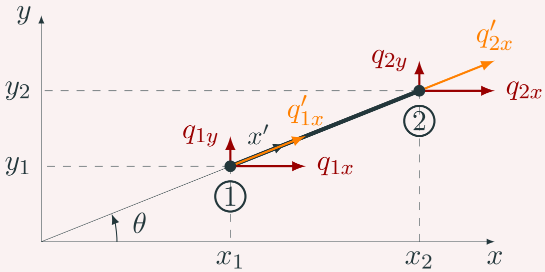
- Displacements in local coordinates:

$$\mathbf{q}_{loc} = [q'_{1x}, q'_{2x}]^T$$

- Displacements in global coordinates:

$$\mathbf{q} = [q_{1x}, q_{1y}, q_{2x}, q_{2y}]^T$$

- **Relation between local and global displacements:**



$$\underbrace{\begin{bmatrix} q'_{1x} \\ q'_{2x} \end{bmatrix}}_{\mathbf{q}_{loc}} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \end{bmatrix}}_{\mathbf{q}}$$

Discretization of arbitrarly oriented bar

- Element stiffness matrix in global coordinates:

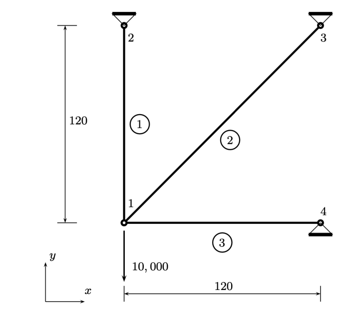
$$\mathbf{K} = \mathbf{T}^T \mathbf{K}_{loc} \mathbf{T} = \frac{EA}{\ell} \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) & -\cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ & \sin^2(\theta) & -\sin(\theta)\cos(\theta) & -\sin^2(\theta) \\ & & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \text{Symm.} & & & \sin^2(\theta) \end{bmatrix}$$

- Element consistent mass matrix in global coordinates:

$$\mathbf{M} = \mathbf{T}^T \mathbf{M}_{loc} \mathbf{T} = \frac{\rho A \ell}{6} \begin{bmatrix} 2\cos^2(\theta) & 2\sin(\theta)\cos(\theta) & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ & 2\sin^2(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta) \\ & & 2\cos^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \text{Symm.} & & & 2\sin^2(\theta) \end{bmatrix}$$

Illustrative example

Plane truss: structure composed of oriented bar elements that all lies in a common plane and are connected by frictionless pins.



Elements	Nodes	${}^e\theta$	${}^e\ell$
1	1, 2	90°	120 mm
2	1, 3	45°	$120\sqrt{2}$ mm
3	1, 4	0°	120 mm

Elementary stiffness and mass matrices:

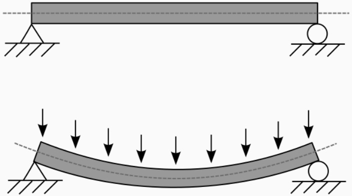
$$
{}^e\mathbf{K} = {}^e\mathbf{T}\mathbf{K}_{loc}{}^e\mathbf{T} \qquad e = 1, 2, 3
$$
$$
{}^e\mathbf{M} = {}^e\mathbf{T}\mathbf{M}_{loc}{}^e\mathbf{T}
$$

Global stiffness and mass matrices: $\mathbf{K} = \mathop{\mathsf{A}}_{e=1}^{3} {}^e\mathbf{K}$ and $\mathbf{M} = \mathop{\mathsf{A}}_{e=1}^{3} {}^e\mathbf{M}$.

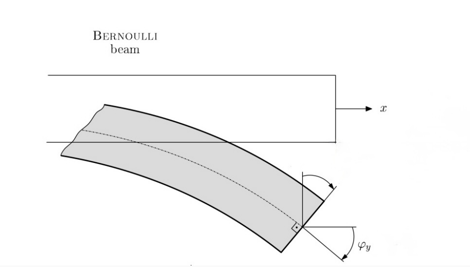
Euler-Bernoulli planar beams

What is a beam?

- Considered to be a uniaxial (slender) element: the longitudinal direction is sufficiently larger than the other two.
- Cross-section does not change along the element's length.
- Subjected to **transversal loading** that produced significant bending effects.
- Very common type of structures used in steel buildings, bridges, towers, etc...



Euler-Bernoulli beam assumptions

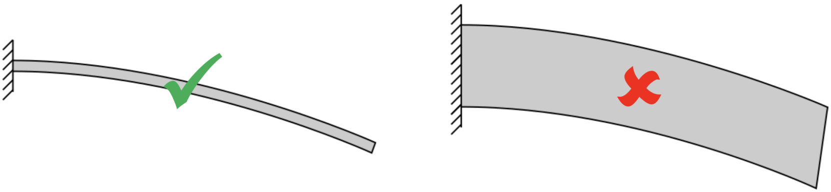


Thin beam theory

- Planar cross section perpendicular to the longitudinal centroidal axis of the beam remains **planar** and **perpendicular** to the longitudinal axis after bending occurs.
- Shear deformations ε_{12} of the planar cross section are neglected.
- The beam cross-section is infinitely rigid in its own plane.
- Reasonable model for slender structures made of isotropic materials.

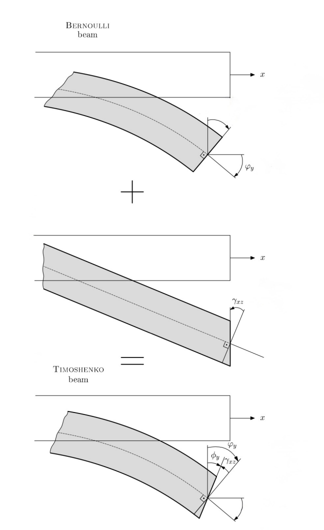
Euler-Bernoulli beam assumptions

Valid for: slender beams ($h/\ell < 1/100$).



(Credit: Chatzi and Egger)

Timoshenko beam assumptions

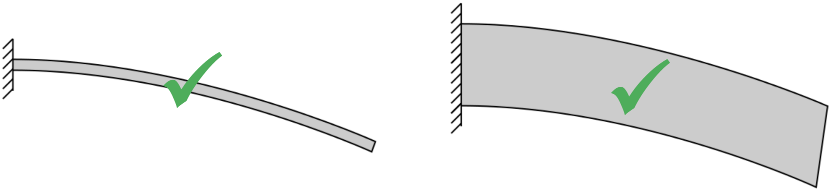


Thick beam theory

- Planar cross section perpendicular to the longitudinal centroidal axis of the beam remains **planar** but **not necessarily perpendicular** to the longitudinal beam axis after bending occurs.
- Shear deformations ε_{12} of the planar cross section are considered.
- Reasonable model for beams made of composite material that require the shear effect account.

Timoshenko beam assumptions

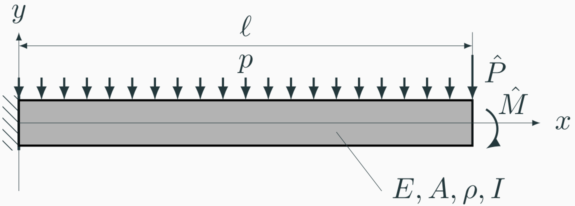
Valid for: slender beams ($h/\ell < 1/100$) and thick beams ($h/\ell > 1/10$).



(Credit: Chatzi and Egger)

Kinematic assumptions for planar Euler-Bernoulli beam

1. The beam dynamics will be restricted to the $O(x, y)$ plane.
2. The beam cross-section remains planar after deformation.
3. Lines that are straight and perpendicular to the beam axis remain straight and perpendicular during deformation.



Model parameters:

- A cross-sectional area
- E Young's modulus
- ρ material density
- I moment of inertia
- ℓ length

Loads:

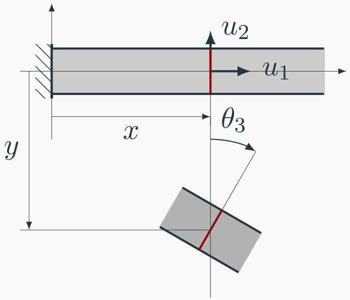
- $\hat{M}$ bending moment at free end
- $\hat{P}$ load at free end
- p distributed transversal load

Variables:

- $u_1(x, t)$ axial displacement
- $u_2(x, t)$ transversal displacement

Displacements field

Introduce an auxiliary variable $\theta_3(x,t)$ representing the total rotation of the section around the Oz axis. Rotations are positive in clockwise direction.



- The first Euler-Bernoulli assumption implies

$$u_3 = 0.$$

- The second Euler-Bernoulli assumption implies

$$u_1 = -y\theta_3 \qquad \textit{(rotation-axial displ.)}$$

- The third Euler-Bernoulli assumption implies

$$\theta_3 = \partial_x u_2 \qquad \textit{(rotation-transversal displ.)}$$

Only one unknown: transversal displacement $u_2(x,t)$.

Stress, strain and displacement fields

Strain-displacement relationships

Substituting the displacement field $\mathbf{u}$ into $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ yield to :

$$\varepsilon_{11} = \partial_x u_1 = -y\partial^2_{xx} u_2 \qquad \varepsilon_{22} = \partial_y u_2 = 0$$
$$\varepsilon_{12} = \partial_x u_2 + \partial_y u_1 = \partial_x u_2 - \theta_3 = 0 \qquad \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$$

Generalized Hooke's law

Using the linear relationship for homogeneous and isotropic material, $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, is possible to write the stress field in term of the strain field as :

$$\sigma_{11} = (\lambda + 2G)\varepsilon_{11} \qquad \sigma_{22} = \lambda\varepsilon_{11}$$
$$\sigma_{33} = \lambda\varepsilon_{11} \qquad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

where $\lambda = \frac{E}{(1+\nu)(1-2\nu)}$ and the shear modulus $G = \frac{E}{2(1+\nu)}$ are Lamé constants.

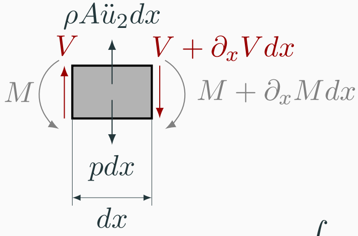
Dynamic equilibrium equations

The theoretical stresses do not agree well with the experimental measurements.

Additional assumptions regarding the stress field

$$\sigma_{11} = E\varepsilon_{11}, \qquad \text{and} \qquad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$

Justification : the dimension of the cross-section is much smaller that the beam length.



- Equilibrium of transverse forces:

$$\partial_x V + p = \rho A \ddot{u}_2$$

- Equilibrium of moments:

$$\partial_x M + V = 0$$

$$M = -\int_A y\sigma_{11}\, dA = -\int_A yE\varepsilon_{11}\, dA = EI\partial^2_{xx} u_2$$

Strong form for transversal vibrations for Euler-Bernoulli beam

The strong form consists of finding the transversal displacement $u_2 \in C^4([0,\ell] \times [0,T])$ such that the following equilibrium equation is satisfied:

$$\partial^2_{xx}\left(EI\partial^2_{xx}u_2\right) + \rho A\ddot{u}_2 = p.$$

Coupled with four boundary conditions:

$$u_2(0,t) = 0 \qquad \partial_x\left(-EI\partial^2_{xx}u_2\right)(\ell,t) = \hat{P}$$

$$\partial_x u_2(0,t) = 0 \qquad EI\partial^2_{xx}u_2(\ell,t) = \hat{M}$$

and two initial conditions:

$$u_2(x,0) = u_0(x) \qquad \dot{u}_2(x,0) = v_0(x)$$

Weak form for transversal vibrations for Euler-Bernoulli beam

Find the transversal displacement $u_2 \in \mathcal{U}$ such that the following equation is satisfied for every virtual transversal displacement $\delta u_2 \in \mathcal{V}$:

$$\int_0^\ell EI\,\partial^2_{xx}u_2\,\partial^2_{xx}\delta u_2\,dx + \int_0^\ell \rho A\,\ddot{u}_2\,\delta u_2\,dx = \int_0^\ell p\,\delta u_2\,dx + \hat{M}\,\delta u_2'(\ell) + \hat{P}\,\delta u_2(\ell)$$

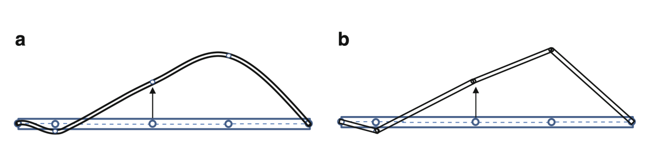
$$\mathcal{U} = \left\{u_2(\cdot,t) \in H^2(]0,\ell[) \mid u_2(0,t) = \partial_x u_2(0,t) = 0\ \forall t \in]0,T[\right\}$$
$$\mathcal{V} = \left\{\delta u_2 \in H^2(]0,\ell[) \mid \delta u_2(0) = \delta u_2'(0) = 0\right\}$$

The Sobolev space $H^2(]0,\ell[)$ is the defined as:

$$H^2(]0,\ell[) = \left\{f \in L^2(]0,\ell[) \mid \int_0^l \left(\partial_x f(x)\right)^2\,dx < \infty,\ \int_0^l \left(\partial^2_{xx} f(x)\right)^2\,dx < \infty\right\}.$$

Approximated transversal displacement

- Euler-Bernoulli theory states that both transversal displacement u_2 and rotation $\theta_3 = \partial_x u_2$ must be differentiable within finite elements and continuous between elements.
- Shape functions that meet this requirement are said to have **$\mathbf{C^1}$ continuity**.

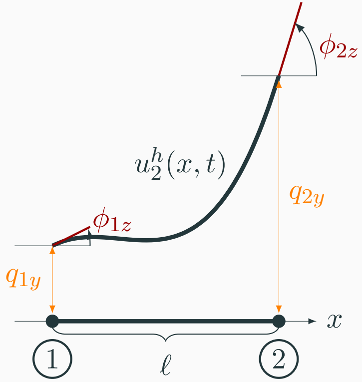


Cubic vs. linear elements. *Credit: [N]*

Assume the approximated transversal displacement to be

$$u_2^h(x,t) = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)$$

Nodal degrees of freedom within a beam element



Parameters shift:

$$a_0,\ a_1,\ a_2,\ a_3 \quad \Rightarrow \quad q_{1y},\ q_{2y},\ \phi_{1z},\ \phi_{2z}$$

A planar beam element has two DOFs per node:

- q_{1y} et q_{2y}: **nodal displacements** in the transverse direction.
- ϕ_{1z} et ϕ_{2z}: **nodal rotations** around the axis normal to the beam plane.

Hermite C^1 shape functions

- Expressing the approximated displacement u^h as a function of the nodal four DOFs yield to:

$$u_2^h(x,t) = \mathbf{H}(x)\mathbf{q}(t) = [h_1(x),\ h_2(x),\ h_3(x),\ h_4(x)] \begin{bmatrix} q_{1y}(t) \\ \phi_{1z}(t) \\ q_{2y}(t) \\ \phi_{2z}(t) \end{bmatrix}$$

- **Cubic local shape functions:**

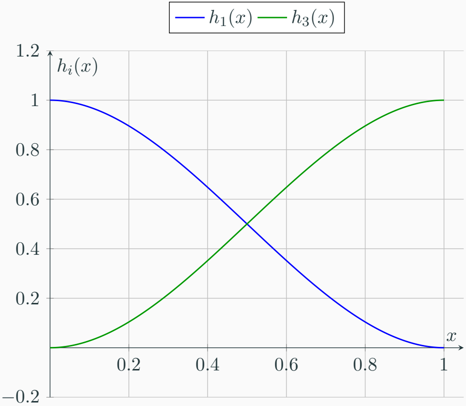
$$h_1(x) = 2(x/\ell)^3 - 3(x/\ell)^2 + 1 \qquad h_3(x) = 3(x/\ell)^2 - 2(x/\ell)^3$$

$$h_2(x) = x(1 - x/\ell)^2 \qquad h_4(x) = x(x/\ell)(x/\ell - 1)$$

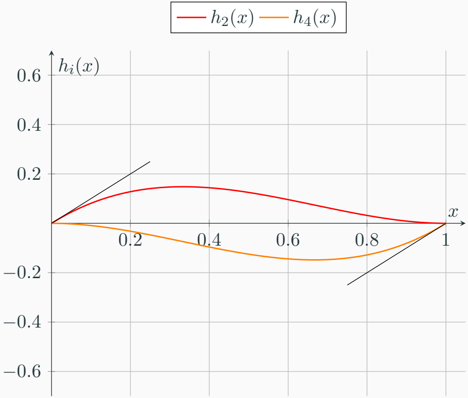
- Virtual displacement approximation follows the same logic:

$$\delta u^h(x) = \mathbf{H}(x)\boldsymbol{\delta}\mathbf{q} \qquad \text{where} \qquad \boldsymbol{\delta}\mathbf{q} = [\delta u_1,\ \delta\phi_{1z},\ \delta q_{2y},\ \delta\phi_{2z}]^T$$

Hermite C^1 shape functions visualizations



Translational shape functions



Rotational shape functions

Element stiffness matrix

$$\mathbf{K} = \int_0^\ell EI\ \mathbf{B}^T\mathbf{B}\ dx$$

$$= \int_0^\ell EI \begin{bmatrix} (h_1'')^2 & h_1''h_2'' & h_1''h_3'' & h_1''h_4'' \\ & (h_2'')^2 & h_2''h_3'' & h_2''h_4'' \\ & & (h_3'')^2 & h_3''h_4'' \\ Symm. & & & (h_4'')^2 \end{bmatrix} dx$$

$$= \frac{EI}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ & 4\ell^2 & -6\ell & 2\ell^2 \\ & & 12 & -6\ell \\ Symm. & & & 4\ell^2 \end{bmatrix}$$

The deformation matrix is

$$\mathbf{B} = \frac{d^2\mathbf{H}}{dx^2} = \begin{bmatrix} h_1''(x) & h_2''(x) & h_3''(x) & h_4''(x) \end{bmatrix}$$

Element consistent mass matrix

$$\mathbf{M} = \int_0^\ell \rho A \; \mathbf{H}^T \mathbf{H} \; dx$$

$$= \int_0^\ell \rho A \begin{bmatrix} (h_1)^2 & h_1 h_2 & h_1 h_3 & h_1 h_4 \\ & (h_2)^2 & h_2 h_3 & h_2 h_4 \\ & & (h_3)^2 & h_3 h_4 \\ Symm. & & & (h_4)^2 \end{bmatrix} dx$$

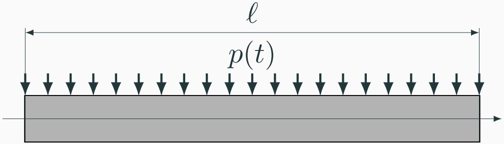
$$= \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ & 4\ell^2 & 13\ell & -3\ell^2 \\ & & 156 & -22\ell \\ Symm. & & & 4\ell^2 \end{bmatrix}$$

Element lumped mass matrix

- Assumption that the mass of the structure is lumped at the nodal coordinates where translational displacements are defined.
- Inertial effect associated with any rotational degree of freedom is usually assumed to be zero.
- Recall that $\rho\,A$ is the mass per unit length along the beam, then the lumped mass matrix is defined as

$$\mathbf{M} = \frac{\rho A \ell}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

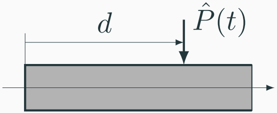
- **Uniformly distributed load:**

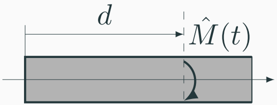


$$\mathbf{r} = \int_0^\ell p(t)\,\mathbf{H}^T\,dx = \frac{p(t)\ell}{2}\begin{bmatrix} 1 \\ \ell/6 \\ 1 \\ -\ell/6 \end{bmatrix}$$

An uniform transverse load can be replaced by two equivalent transverse nodes loads of value $(p\ell)/2$ and by two equivalent nodal moments of values $\pm p\ell^2/12$.

- **Concentrated loads:**





$$\mathbf{r}(t) = \hat{P}(t)\,\mathbf{H}^T(d)$$

$$\mathbf{r}(t) = \hat{M}(t)\,\frac{d\mathbf{H}}{dx}^T(d)$$

Post-processing: approximation of bending moment and shear force

- The approximated bending moment along the beam element is

$$M^h(x,t) = EI\partial_{xx}^2 u_2^h(x,t) = EI\,\mathbf{B}(x)\mathbf{q}(t).$$

- The approximated shear force along the beam element is

$$V^h(x,t) = -\partial_x M^h(x,t) = -EI\,\frac{d\mathbf{B}}{dx}(x)\mathbf{q}(t).$$

MATLAB examples

- A Cantilever beam subjected to a downward force
- A clamped-clamped beam in free vibration.

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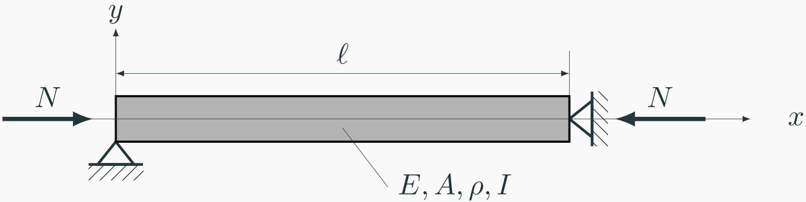
Geometric stiffness

Beam subjected to axial loads

- Consider a beam subjected to axial forces N applied at both ends.

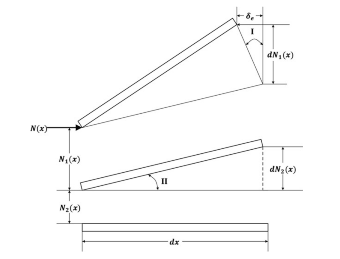


- Beam carrying large axial loads or undergoing large displacements have nonlinear behavior arising from the internal moments that are the product of the axial loads and the displacements transverse to the loads.
- The stiffness coefficients k_{ij} must be modified by the presence of the axial force to the corresponding **geometric stiffness coefficients** k_{ij}^g.

Geometric stiffness matrix

k_{ij}^{g} is defined as the force corresponding to the nodal coordinate i due to a unit displacement at coordinate j and resulting for the axial force N.

Calculation of k_{12}^{g}: vertical force at node 1 due to a unit rotation $\theta_2 = 1$ caused by the axial force N. May be evaluated by the principle of virtual work:



$$dW = N\delta_e$$

By similar triangles we have

$$\frac{\delta_e}{dh_1(x)} = \frac{dh_2(x)}{dx}$$

$$\delta_e = \frac{dh_1}{dx}(x)\frac{dh_2}{dx}(x)\,dx$$

$$k_{12}^{g} = N\int_0^{\ell} h_1'(x)h_2'(x)\,dx$$

Geometric stiffness matrix

- In general, any geometric stiffness coefficient may be expressed as

$$k_{ij}^g = N \int_0^\ell h_i'(x) h_i'(x)\, dx$$

- We define the **geometric stiffness matrix** as:

$$\mathbf{K}^g = N \int_0^\ell \frac{d\mathbf{H}}{dx}^T \frac{d\mathbf{H}}{dx}\, dx$$

- When N is constant along the beam length we obtain

$$\mathbf{K}^g = \frac{N}{30\ell} \begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}$$

Example: stability of Bernoulli beam

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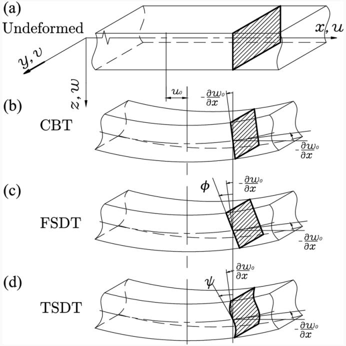
Timoshenko planar beam

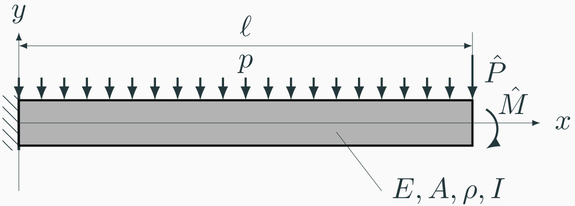
Higher-order deformation theories for beams



- **Classical beam theory** (Euler-Bernoulli) is a zero-order shear deformation theory.
- **Timoshenko theory** is a first-order shear deformation theory. Cross-sectional planes remain straight but not necessarily perpendicular to the mid-surface.
  - Displacements are linear functions through thickness.
  - In-plane strains are linear in z.
  - Shear strains are constant in z.
- **Third-order shear deformation theory**: normal cross-sectional planes to the mid-surface can rotate and deform.
  - Displacements are cubic functions through thickness.
  - In-plane strains are cubic in z.
  - Shear strains are quadratic in z.

Kinematic assumptions for planar Timoshenko beam

1. The beam dynamics will be restricted to the $O(x, y)$ plane.
2. The beam cross-section remains planar after deformation.
3. Lines that are straight and perpendicular to the geometrical beam axis remain straight but **not necessarily perpendicular** during deformation.



Model parameters:

- A cross-sectional area
- E Young's modulus
- ρ material density
- I moment of inertia
- ℓ length

Loads:

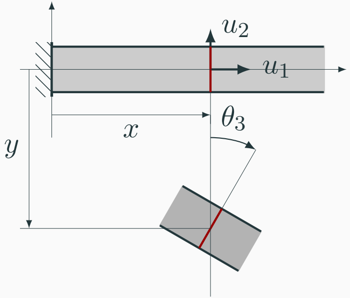
- $\hat{M}$ bending moment at free end
- $\hat{P}$ load at free end
- p distributed transversal load

Variables:

- $u_1(x, t)$ axial displacement
- $u_2(x, t)$ transversal displacement

Displacements field

Introduce an auxiliary variable $\theta_3(x,t)$ representing the total rotation of the section around the Oz axis. Rotations are positive in clockwise direction.



- The first Timoshenko assumption implies

$$u_3 = 0.$$

- The second Timoshenko assumption implies

$$u_1 = -y\theta_3 \qquad \textit{(rotation-axial displ.)}$$

Two unknowns: $u_2(x,t)$ transversal displacement and $\theta_3(x,t)$ rotation of the section around the Oz axis.

Stress, strain and displacement fields

Strain-displacement relationships

Substituting the displacement field $\mathbf{u}$ into $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ yield to :

$$\varepsilon_{11} = \partial_x u_1 = -y\partial^2_{xx} u_2 \qquad \varepsilon_{22} = \partial_y u_2$$

$$\varepsilon_{12} = \partial_x u_2 + \partial_y u_1 = \partial_x u_2 - \theta_3 \qquad \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$$

Generalized Hooke's law

Using the linear relationship for homogeneous and isotropic material, $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, is possible to write the stress field in term of the strain field as :

$$\sigma_{11} = (\lambda + 2G)\varepsilon_{11} \qquad \sigma_{12} = 2G\varepsilon_{12}$$

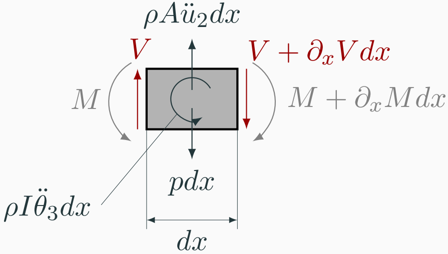
$$\sigma_{22} = \sigma_{33} = \lambda\varepsilon_{11} \qquad \sigma_{13} = \sigma_{23} = 0$$

where $\lambda = \frac{E}{(1+\nu)(1-2\nu)}$ and the shear modulus $G = \frac{E}{2(1+\nu)}$ are Lamé constants.

Dynamic equilibrium equations

Additional assumptions regarding the stress field

$$\sigma_{11} = E\varepsilon_{11}, \quad \sigma_{12} = 2G\varepsilon_{12} \qquad \text{and} \qquad \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0.$$



- Equilibrium of transverse forces:

$$\partial_x V + p = \rho A\ddot{u}_2$$

- Equilibrium of moments:

$$\partial_x M + V = \rho I\ddot{\theta}_3$$

$$V = \int_A \sigma_{12}\,dA = kGA\varepsilon_{12} = kGA(\partial_x u_2 - \theta_3)$$

$$M = -\int_A y\sigma_{11}\,dA = -\int_A yE\varepsilon_{11}\,dA = EI\partial_x\theta_3$$

Strong form for transversal vibrations for Timoshenko beam

The weak formulation of the problem consists of finding the transversal displacement $u_2 \in C^2([0,\ell] \times [0,T])$ and the rotation $\theta_3 \in C^2([0,\ell] \times [0,T])$ such that the following equilibrium equations, boundary and initial conditions are satisfied.

$$\partial_x \big(kGA(\partial_x u_2 - \theta_3) \big) + p = \rho A \ddot{u}_2$$

$$\partial_x \big(EI \partial_x \theta_3 \big) + kGA(\partial_x u_2 - \theta_3) = \rho I \ddot{\theta}_3$$

In matrix form: $\nabla_\sigma^T \mathbf{C} \nabla_u \mathbf{u} + \mathbf{f} = \mathbf{M}\ddot{\mathbf{u}}$, where

$$\underbrace{\begin{bmatrix} \partial_x & 0 \\ 1 & \partial_x \end{bmatrix}}_{\nabla_\sigma^T} \underbrace{\begin{bmatrix} kGA & 0 \\ 0 & EI \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \partial_x & -1 \\ 0 & \partial_x \end{bmatrix}}_{\nabla_u} \underbrace{\begin{bmatrix} u_2 \\ \theta_3 \end{bmatrix}}_{\mathbf{u}} + \underbrace{\begin{bmatrix} p \\ 0 \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{u}_2 \\ \ddot{\theta}_3 \end{bmatrix}}_{\ddot{\mathbf{u}}}$$

Boundary conditions

$$u_2(0,t) = 0 \qquad kGA\big(\partial_x u_2(\ell,t) - \theta_3(\ell,t)\big) = \hat{P}$$

$$\theta_3(0,t) = 0 \qquad EI\partial_x\theta_3(\ell,t) = \hat{M}$$

In matrix form: let $\hat{\mathbf{f}} = [\hat{P},\ \hat{M}]^T$ then

$$\mathbf{u}(0,t) = \mathbf{0}$$

$$\mathbf{C}\nabla_u\mathbf{u}(\ell,t) = \hat{\mathbf{f}}$$

Initial conditions

$$u_2(x,0) = u_0(x) \qquad \dot{u}_2(x,0) = v_0(x)$$

$$\theta_3(x,0) = \theta_0(x) \qquad \dot{\theta}_3(x,0) = \phi_0(x)$$

In matrix form: $\mathbf{u_0} = [u_0,\ \theta_0]^T$ and $\mathbf{v_0} = [v_0,\ \phi_0]^T$ then

$$\mathbf{u}(x,0) = \mathbf{u_0}$$

$$\dot{\mathbf{u}}(x,0) = \mathbf{v_0}$$

Weak form for transversal vibrations for Timoshenko beam

The weak formulation of the problem consists in finding the unknown function $\mathbf{u} \in \mathcal{U}$ which satisfies the equation $\forall \boldsymbol{\delta}\mathbf{u} \in \mathcal{V}$:

$$\int_0^\ell (\nabla_u \boldsymbol{\delta}\mathbf{u})^{\mathrm{T}} \mathbf{C} (\nabla_u \mathbf{u}) \, \mathrm{d}x + \int_0^\ell \boldsymbol{\delta}\mathbf{u}^{\mathrm{T}} \mathbf{M} \ddot{\mathbf{u}} \, dx = \int_0^\ell \boldsymbol{\delta}\mathbf{u}^{\mathrm{T}} \mathbf{f} dx + \boldsymbol{\delta}\mathbf{u}^{\mathrm{T}}(\ell) \hat{\mathbf{f}}.$$

The function spaces $\mathcal{U}$ and $\mathcal{V}$ are defined as follows

$$\mathcal{U} = \left\{ \mathbf{u} = \{u_2, \theta_3\}^{\mathrm{T}} \mid u_2(\cdot, t) \in H^1(]0, \ell[); \theta_3(\cdot, t) \in H^1(]0, \ell[); u_2(0, t) = \theta_3(0, t) = 0 \right\},$$

$$\mathcal{V} = \left\{ \boldsymbol{\delta}\mathbf{u} = \{\delta u_2, \delta\theta_3\}^{\mathrm{T}} \mid \delta u_2 \in H^1(]0, \ell[); \delta\theta_3 \in H^1(]0, \ell[); \delta u_2(0) = \delta\theta_3(0) = 0 \right\}.$$

Displacement approximation

- We approximate the generalized displacement using the ansatz:

$$\mathbf{u}^h(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t) \qquad \text{and} \qquad \delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q}$$

- **Cubic shape functions** are used to approximate the transversal displacement and **linear shape functions** for the rotation field:

$$\mathbf{H} = \begin{bmatrix} h_1^c & h_2^c & h_3^c & h_4^c \\ 0 & h_1^l & 0 & h_2^l \end{bmatrix}$$

$$h_1^c(x) = 2(x/\ell)^3 - 3(x/\ell)^2 + 1 \qquad h_3^c(x) = 3(x/\ell)^2 - 2(x/\ell)^3$$

$$h_2^c(x) = x(1 - x/\ell)^2 \qquad h_4^c(x) = x(x/\ell)(x/\ell - 1)$$

$$h_1^l(x) = 1 - x/\ell \qquad h_2^l(x) = x/\ell$$

- $\mathbf{q}(t)$ is a vector of (*unknown*) nodal displacements and $\boldsymbol{\delta}\mathbf{q}$ is a vector of constants:

$$\mathbf{q}(t) = \begin{bmatrix} q_{1y}(t) & \phi_{1z}(t) & q_{2y}(t) & \phi_{2z}(t) \end{bmatrix}^T \qquad \text{and} \qquad \boldsymbol{\delta}\mathbf{q} = \begin{bmatrix} \delta q_{1y} & \delta\phi_{1z} & \delta q_{2y} & \delta\phi_{2z} \end{bmatrix}^T$$

Deformation, stiffness, mass matrices and loads vector

- $\mathbf{B} = \nabla_u \mathbf{H}$ is the deformation matrix:

$$\mathbf{B} = \begin{bmatrix} \partial_x h_1^c & \partial_x h_2^c - h_1^l & \partial_x h_3^c & \partial_x h_4^c - h_2^l \\ 0 & \partial_x h_1^l & 0 & \partial_x h_2^l \end{bmatrix}$$

- The stiffness $\mathbf{K}$ and mass $\mathbf{Ma}$ matrices and load vector $\mathbf{r}$ are defined as

$$\mathbf{K} = \int_0^\ell \mathbf{B}^T \mathbf{C} \mathbf{B} \, dx,$$

$$\mathbf{Ma} = \int_0^\ell \mathbf{H}^T \mathbf{M} \mathbf{H} \, dx,$$

$$\mathbf{r}(t) = \int_0^\ell \mathbf{H}^T \mathbf{f} \, dx + \mathbf{H}^T(\ell) \hat{\mathbf{f}}.$$

- The semi-discrete equations of motion are given by

$$\mathbf{Ma} \, \ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t).$$

MATLAB example - modal analysis of a Timoshenko beam

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