Dynamic analysis of trusses

Special structural elements

ME473 Dynamic finite element analysis of structures

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Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	

Summary

- Eigenvalues and eigenvectors errors bounds
- Trusses in 2d
- Matlab example of a 2d truss in free vibrations
- Trusses in 3d
- Matlab example of a 3d truss in free vibrations

Recommended readings

1. Logan, A first course in the finite element method, 6th ed. (chap. 3)
2. Paz and Leigh, Structural dynamics, 6th ed. (chap. 14)
3. Ferreira and Fantuzzi, MATLAB Codes for Finite Element Analysis, 2nd ed. (chap. 4 and 5)

Eigenvalues and eigenvectors errors bounds

A priori error estimates for eigenvalues and eigenvectors

$$\lambda_i \leq \lambda_i^h \leq \lambda_i + c\, h^{2m}\, \lambda_i^{m+1}$$

$$\|\boldsymbol{\phi}_i^h - \boldsymbol{\phi}_i\|_0 \leq c\, h^{\min(m+1,2m)}\, \lambda_i^{(m+1)/2}$$

- λ_i^h are the *approximated* eigenvalues and $\boldsymbol{\phi}_i^h$ the corresponding eigenvectors,
- λ_i are the *exact* eigenvalues and $\boldsymbol{\phi}_i$ the corresponding eigenvectors,
- h represents the characteristic mesh size,
- m is the degree of the highest complete polynomial used,
- c is a constant independent of h,
- $\|\cdot\|_0$ Euclidean L^2 norm.

A priori error estimates for frequencies

From the relationship between eigenvalues and frequencies, $\omega_i^h = \sqrt{\lambda_i^h}$, we deduce the bound on the approximate frequencies.

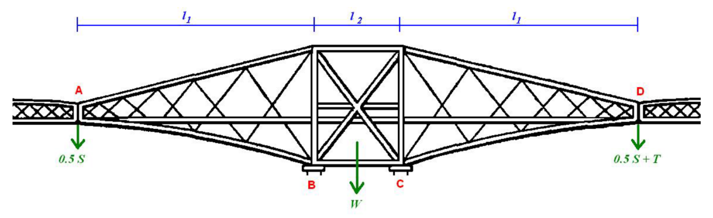
$$\omega_i \leq \omega_i^h \leq \omega_i + c\, h^{2m}\, \omega_i^{2m+1}$$

- Approximate frequencies ω_i^h always overestimate the exact frequencies ω_i.
- The quality of approximated frequencies degrades for higher frequencies.
- The convergence rates for eigenvectors and frequencies are both of order h^{2m}.

Structural 1d elements : general ideas

Example: analysis of the forth bridge

(Credit: theforthbridges.org)



(Credit: Chatzi and Egger)

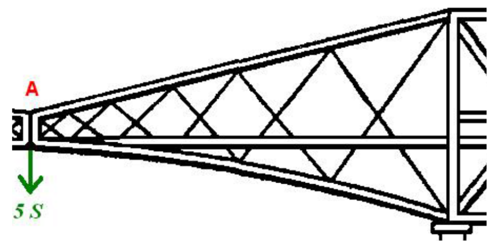
Abstract model

Abstraction

Modelling

Discretization

Solution

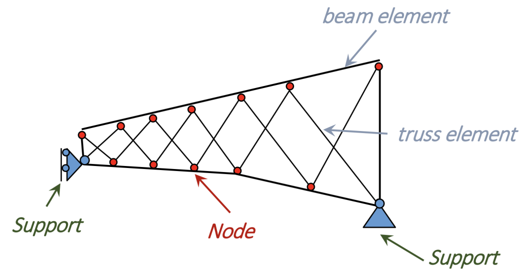


(Credit: Chatzi and Egger)

Conceptual framework: develop a simplified idealisation of the structure, or part of the structure, representing the geometry and loads.

Mathematical model

Abstraction
Modelling
Discretization
Solution



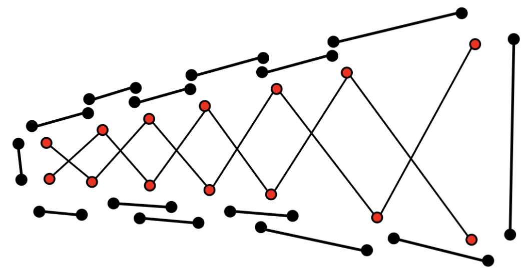
(Credit: Chatzi and Egger)

Connect the structure to a known theoretical model capable of describing its functioning with sufficient accuracy.

Assumption: decompose the system into 1D components, such as trusses and beams.

Discrete model

Abstraction → **Modelling** → **Discretization** → **Solution**



(Credit: Chatzi and Egger)

Direct stiffness method framework: the continuum is disassembled using a mesh of finite elements that are connected at nodes located on the element boundaries.

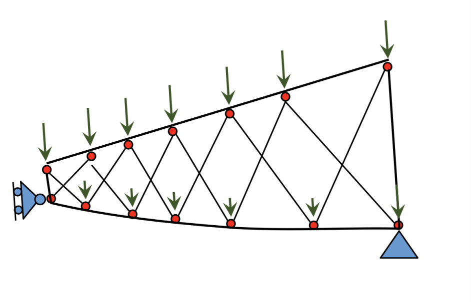
Deformed model

Abstraction

Modelling

Discretization

Solution



(Credit: Chatzi and Egger)

Solve for the structure degrees of freedom:

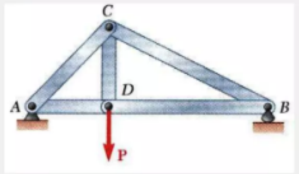
- **assembly** of element quantities and application of boundary conditions,
- **post-processing:** computation of strains, stresses, reaction forces, and moments.

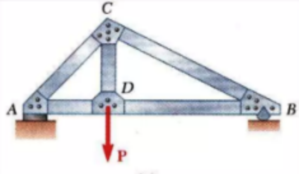
Trusses in 2d

What is a truss structure?

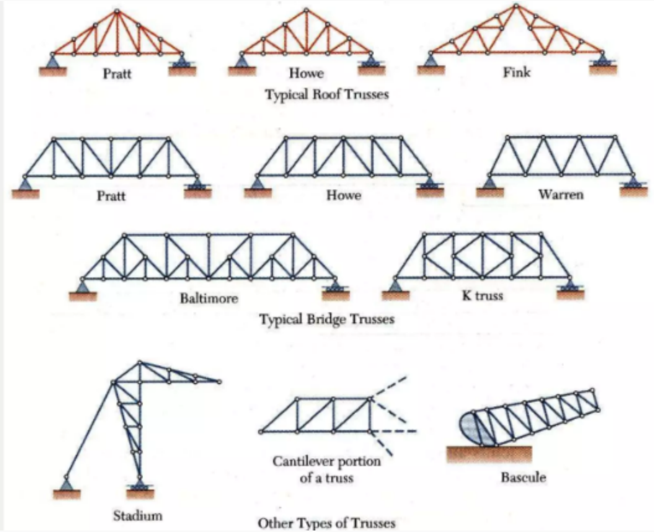
Plane truss

- Structure composed of *oriented bar (rod) elements* that all lies in a common plane and are connected by *frictionless pins.*
- Loads are acting only in the common plane and they must be applied at the nodes or joints.
- Very common type of structures used in steel buildings, bridges, towers, etc...





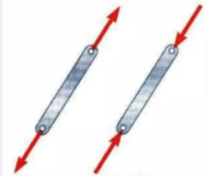
Examples of 2d trusses

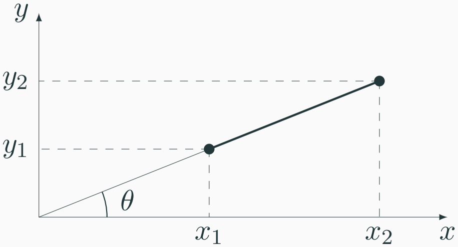


Kinematic assumptions

Trusses are assumed to exhibit the following characteristics:

- they experience either **compressive** or **tensile forces**,
- their weight is considered negligible in comparison to the loads they support,
- they have varying orientations with respect to a fixed global coordinate system, which serves as a stationary reference framework that remains unchanged regardless of the orientation of individual elements.





Equation of motion for a single non-oriented bar element

- A cross-sectional area
- E Young's modulus (isotropic bar)
- ρ material density
- ℓ length
- u axial displacement
- x' (local) axial coordinate

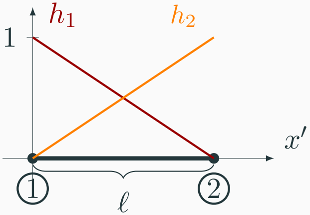
- Strong form:

$$\begin{cases} \partial_{x'}\left(EA\partial_{x'}u(x',t)\right) = \rho A\ddot{u}(x',t) \\ EA\partial_{x'}u(0,t) = -f_1(t) \\ EA\partial_{x'}u(\ell,t) = f_2(t) \end{cases} \qquad \begin{cases} u(x',0) = u_0(x') \\ \dot{u}(x',0) = v_0(x') \end{cases}$$

- Semi-discrete weak form:

$$\boldsymbol{\delta}\mathbf{q}_{loc}^T\big(\mathbf{M}_{loc}\ddot{\mathbf{q}}_{loc}(t) + \mathbf{K}_{loc}\mathbf{q}_{loc}(t) - \mathbf{f}_{loc}(t)\big) = 0$$

Approximated displacements in local coordinates



Linear local shape functions:

$$h_1(x') = 1 - \frac{x'}{\ell}$$

$$h_2(x') = \frac{x'}{\ell}$$

- Displacements approximation local coordinates:

$$u^h(x', t) = h_1(x')q_1(t) + h_2(x')q_2(t) = \begin{bmatrix} h_1(x') & h_2(x') \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

- Virtual displacements approximation local coordinates:

$$\delta u^h(x') = h_1(x')\delta q_1 + h_2(x')\delta q_2 = \begin{bmatrix} h_1(x') & h_2(x') \end{bmatrix} \begin{bmatrix} \delta q_1 \\ \delta q_2 \end{bmatrix}$$

Elementary quantities in local coordinates

- Element stiffness matrix in local coordinates:

$$\mathbf{K}_{loc} = \int_0^{\ell} EA \begin{bmatrix} (h_1')^2 & h_1' h_2' \\ h_2' h_1' & (h_2')^2 \end{bmatrix} dx' = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \int_0^{\ell} \rho A \begin{bmatrix} (h_1)^2 & h_1 h_2 \\ h_2 h_1 & (h_2)^2 \end{bmatrix} dx' = \frac{\rho A \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Element applied loads vector in local coordinates:

$$\mathbf{f}_{loc}(t) = \begin{bmatrix} h_1(0) \\ h_2(0) \end{bmatrix} f_1(t) + \begin{bmatrix} h_1(\ell) \\ h_2(\ell) \end{bmatrix} f_2(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

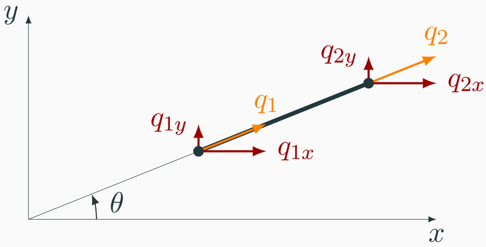
Displacements for a single oriented bar

- Displacement vector in local coordinates:

$$\mathbf{q}_{loc} = [q_1, q_2]^T.$$

- Displacement vector in global coordinates:

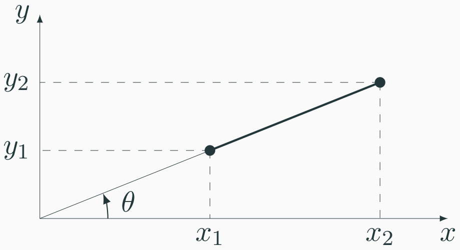
$$\mathbf{q} = [q_{1x}, q_{1y}, q_{2x}, q_{2y}]^T.$$



Relation between local and global displacements:

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{q}_{loc}} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \end{bmatrix}}_{\mathbf{q}}$$

Calculation of direction sines and cosines



The direction sines and cosines can be calculated from the element geometry:

$$\sin(\theta) = \frac{y_2 - y_1}{\ell}$$

$$\cos(\theta) = \frac{x_2 - x_1}{\ell}$$

$$\ell = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Elementary stiffness and consistent mass matrices in global coordinates

- Element stiffness matrix in global coordinates:

$$\begin{aligned}\mathbf{K} &= \mathbf{T}^T\mathbf{K}_{loc}\mathbf{T}\\ &= \frac{EA}{\ell}\begin{bmatrix}\cos^2(\theta) & \sin(\theta)\cos(\theta) & -\cos^2(\theta) & -\sin(\theta)\cos(\theta)\\ & \sin^2(\theta) & -\sin(\theta)\cos(\theta) & -\sin^2(\theta)\\ & & \cos^2(\theta) & \sin(\theta)\cos(\theta)\\ \text{Symm.} & & & \sin^2(\theta)\end{bmatrix}\end{aligned}$$

- Element consistent mass matrix in global coordinates:

$$\begin{aligned}\mathbf{M} &= \mathbf{T}^T\mathbf{M}_{loc}\mathbf{T}\\ &= \frac{\rho A\ell}{6}\begin{bmatrix}2\cos^2(\theta) & 2\sin(\theta)\cos(\theta) & \cos^2(\theta) & \sin(\theta)\cos(\theta)\\ & 2\sin^2(\theta) & \sin(\theta)\cos(\theta) & \sin^2(\theta)\\ & & 2\cos^2(\theta) & 2\sin(\theta)\cos(\theta)\\ \text{Symm.} & & & 2\sin^2(\theta)\end{bmatrix}\end{aligned}$$

Lumped mass matrix

- Element consistent mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \frac{\rho A \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Element lumped mass matrix in local coordinates:

$$\mathbf{M}_{loc} = \frac{\rho A \ell}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Element lumped mass matrix in global coordinates:

$$\begin{aligned} \mathbf{M} &= \mathbf{T}^T \mathbf{M}_{loc} \mathbf{T} \\ &= \frac{\rho A \ell}{2} \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) & 0 & 0 \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) & 0 & 0 \\ 0 & 0 & \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ 0 & 0 & \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix} \end{aligned}$$

Benefits of using lumped mass matrix

- ✓ **Computational efficiency**

  Band matrix for faster computations and lower memory usage.
- ✓ **Improved numerical stability**

  Help avoid non-physical coupling between DOFs, which can cause instabilities, in explicit time integration (Newmark or central difference methods).
- ✓ **Physical realism for trusses**

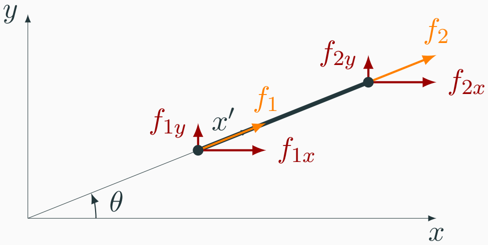
  Truss mass is mostly at joints so lumped mass better reflects reality.
- ✓ **Good approximation in practice**

  Accurate enough for natural frequency and mode shape estimation.

Band mass matrix = faster, simpler, and often accurate enough!

Applied loads for a single oriented bar

- Nodal force vector in local coordinates:
$$\mathbf{f}_{loc} = [f_1, f_2]^T.$$
- Nodal force vector in global coordinates:
$$\mathbf{f} = [f_{1x}, f_{1y}, f_{2x}, f_{2y}]^T.$$



Forces undergo transformation in the same manner as displacements:

$$\underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\mathbf{f}_{loc}} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{bmatrix}}_{\mathbf{f}}$$

Elementary loads vector in global coordinates

- Element applied loads vector in global coordinates:

$$\mathbf{f} = \mathbf{T}^T \mathbf{f}_{loc} = \begin{bmatrix} \cos(\theta) f_1 \\ \sin(\theta) f_1 \\ \cos(\theta) f_2 \\ \sin(\theta) f_2 \end{bmatrix}$$

- Loads are only applied at pins and are given in the global coordinates system, the assembled loads vector can be computed directly.
- Distributed/self-weight loads are transformed to equivalent nodal loads.

Assembly of stiffness and mass matrices and loads vector

Given a 2d truss structure made of m oriented bars, n nodes.

1. **Element quantities:**
   - For each bar element e, compute the element quantities global coordinates:

$$
\begin{aligned}
{}^{e}\mathbf{K} &= {}^{e}\mathbf{T}^{T}\,{}^{e}\mathbf{K}_{loc}\,{}^{e}\mathbf{T} \\
{}^{e}\mathbf{M} &= {}^{e}\mathbf{T}^{T}\,{}^{e}\mathbf{M}_{loc}\,{}^{e}\mathbf{T} \\
{}^{e}\mathbf{f} &= {}^{e}\mathbf{T}^{T}\,{}^{e}\mathbf{f}_{loc}
\end{aligned}
$$

2. **Global assembly:**
   - Initialize global stiffness matrix $\mathbf{K}$ and global mass matrix $\mathbf{M}$ of size $2n \times 2n$.
   - Initialize global loads vector $\mathbf{f}$ of size $2n \times 1$.
   - Assemble each ${}^{e}\mathbf{K}$, ${}^{e}\mathbf{M}$ and ${}^{e}\mathbf{f}$ for $e = 1, \ldots, m$, into $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{f}$ respectively using element connectivity.

Reduced stiffness and mass matrices and loads vector

3. **Boundary conditions:**
    - Identify constrained (*fixed or supported*) degrees of freedom,
    - Partition global matrices and vectors to separate free and constrained DOFs:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fc} \\ \mathbf{K}_{cf} & \mathbf{K}_{cc} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_{ff} & \mathbf{M}_{fc} \\ \mathbf{M}_{cf} & \mathbf{M}_{cc} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_{f} \\ \mathbf{f}_{c} \end{bmatrix}.$$

    - Apply constraints by removing or modifying rows and columns corresponding to constrained DOFs.

Modal analysis

- Solve the free vibrations (homogeneous) problem without external loads:

$$\mathbf{M}_{ff}\ddot{\mathbf{q}}_f(t) + \mathbf{K}_{ff}\mathbf{q}_f(t) = \mathbf{0}.$$

- Assume harmonic motion $\mathbf{q}_f(t) = \boldsymbol{\phi}\, e^{i\omega t}$ and derive the eigenvalue problem:

$$\left(\mathbf{K}_{ff} - \omega^2\mathbf{M}_{ff}\right)\boldsymbol{\phi} = \mathbf{0}.$$

- Solve for eigenvalues $\lambda = \omega^2$ (squared natural frequencies):

$$\det\left(\mathbf{K}_{ff} - \lambda\mathbf{M}_{ff}\right) = 0.$$

- Solve for corresponding eigenvectors $\boldsymbol{\phi}$ (mode shapes). Normalize eigenvectors with respect to $\mathbf{M}_{ff}$ or $\mathbf{K}_{ff}$.

Transient analysis

- The dynamic equilibrium equation for the free DOFs becomes:

$$\mathbf{M}_{ff}\ddot{\mathbf{q}}_f(t) + \mathbf{K}_{ff}\mathbf{q}_f(t) = \mathbf{f}_f(t).$$

- This coupled system of equations can be uncoupled by transforming to modal coordinates using the normal mode matrix $\mathbf{\Phi}$, where $\mathbf{q}_f(t) = \mathbf{\Phi}\mathbf{z}(t)$.
- Substituting into the equation and pre-multiplying by $\mathbf{\Phi}^T$, we obtain a decoupled system:

$$\ddot{\mathbf{z}}(t) + \mathbf{\Lambda}\mathbf{z}(t) = \mathbf{\Phi}^T\mathbf{f}_f(t),$$

  where $\mathbf{\Lambda}$ is a diagonal matrix of squared natural frequencies.
- The decoupled equations in modal space can be solved independently using time integration methods, greatly simplifying the dynamic analysis.

Post-processing: stress computation

- In the local coordinate system, the approximated axial stress in element e is

$$^{e}\sigma^{h}_{loc} = E^{e}\varepsilon^{h}_{loc}$$

- Stain-displacement relationship:

$$^{e}\varepsilon^{h}_{loc} = \frac{d}{dx'}u^{h} = \begin{bmatrix} \frac{dh_1}{dx'} & \frac{dh_2}{dx'} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{^{e}\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

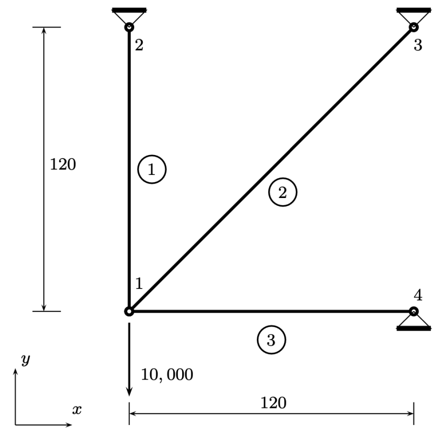
- Using the coordinate transformation $^{e}\mathbf{q}_{loc} = {}^{e}\mathbf{T}\mathbf{q}$ it leads to the approximated stress computed in global coordinates:

$$^{e}\sigma^{h} = \frac{^{e}E}{^{e}\ell} \begin{bmatrix} -\cos(^{e}\theta) & -\sin(^{e}\theta) & \cos(^{e}\theta) & \sin(^{e}\theta) \end{bmatrix} \begin{bmatrix} q_{ix} \\ q_{iy} \\ q_{jx} \\ q_{jy} \end{bmatrix}$$

$$= \frac{^{e}E}{^{e}\ell} \left[\cos(^{e}\theta)(q_{jx} - q_{ix}) + \sin(^{e}\theta)(q_{jy} - q_{iy}) \right]$$

Note that element e is connected to nodes numbered as i and j.

Illustrative example - problem description



Magnesium alloy material properties:

- cross-sectional area $A = 78.5 \text{ mm}^2$
- Young's modulus $E = 40 \cdot 10^3$ MPa
- Density $\rho = 1.810 \text{ ton/m}^3$

Parameters:

Elements	Nodes	${}^e\theta$	${}^e\ell$
1	1, 2	90°	120 mm
2	1, 3	45°	$120\sqrt{2}$ mm
3	1, 4	0°	120 mm

Objective: Compute the equation of motion in semi-discrete weak form.

Credit: Ferreira and Fantuzzi, MATLAB Codes for Finite Element Analysis

Illustrative example - element stiffness matrices

$$
\begin{aligned}
{}^1\mathbf{K} &= {}^1\mathbf{T}^T \mathbf{K}_{loc} {}^1\mathbf{T} \\
&= \frac{EA}{{}^1\ell} \begin{bmatrix} \cos(90°) & 0 \\ \sin(90°) & 0 \\ 0 & \cos(90°) \\ 0 & \sin(90°) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos(90°) & \sin(90°) & 0 & 0 \\ 0 & 0 & \cos(90°) & \sin(90°) \end{bmatrix} \\
&= \frac{78500}{3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

By analogy:

$$
{}^2\mathbf{K} = \frac{78500}{6\sqrt{2}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad {}^3\mathbf{K} = \frac{78500}{3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Illustrative example - element consistent mass matrices

$$
\begin{aligned}
{}^1\mathbf{M} &= {}^1\mathbf{T}^T \mathbf{M}_{loc} {}^1\mathbf{T} \\
&= \frac{\rho A^1 \ell}{6} \begin{bmatrix} \cos(90^\circ) & 0 \\ \sin(90^\circ) & 0 \\ 0 & \cos(90^\circ) \\ 0 & \sin(90^\circ) \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \cos(90^\circ) & \sin(90^\circ) & 0 & 0 \\ 0 & 0 & \cos(90^\circ) & \sin(90^\circ) \end{bmatrix} \\
&= \frac{28417}{10} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

By analogy:

$$
{}^2\mathbf{M} = \frac{28417}{20}\sqrt{2} \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \qquad {}^3\mathbf{M} = \frac{28417}{10} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Illustrative example - assembly of the global stiffness matrix

Local index	bar 1	bar 2	bar 3
1	1	1	1
2	2	3	4

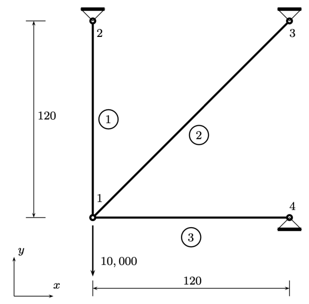
$$\mathbf{K} = \frac{78500}{6} \begin{bmatrix} 2+1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} & -2 & 0 \\ 1/\sqrt{2} & 2+1/\sqrt{2} & 0 & -2 & -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

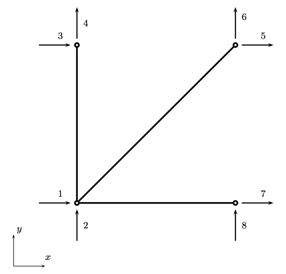
Illustrative example - assembly of the global mass matrix

Local index	bar 1	bar 2	bar 3
1	1	1	1
2	2	3	4

$$\mathbf{M} = \frac{28417}{20} \begin{bmatrix} 4+2\sqrt{2} & 2\sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} & 2 & 0 \\ 2\sqrt{2} & 4+2\sqrt{2} & 0 & 2 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & -\sqrt{2} & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 & 0 & 2\sqrt{2} & 2\sqrt{2} & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Illustrative example - applied loads and boundary conditions





$$\mathbf{f} = \begin{bmatrix} 0 \\ -10000 \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Credit: Ferreira and Fantuzzi, MATLAB Codes for Finite Element Analysis

Example - equations of motions for free nodes

$$\frac{28417}{20}\begin{bmatrix} 4+2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2}4+2\sqrt{2} & \end{bmatrix}\begin{bmatrix} \ddot{q}_{1x} \\ \ddot{q}_{1y} \end{bmatrix}+\frac{78500}{6}\begin{bmatrix} 2+1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 2+1/\sqrt{2} \end{bmatrix}\begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}=\begin{bmatrix} 0 \\ -10000 \end{bmatrix}$$

An example of a 2d truss in free vibration

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Trusses in 3d

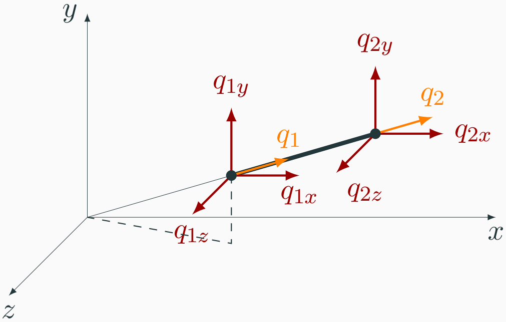
Displacements for oriented bar in 3d

- Displacements in local coord. does not change w.r.t the one in 2d:

$$\mathbf{q}_{loc} = [q_1, q_2]^T.$$

- Displacements in global coordinates projected from nodes 1 and 2 have now 6 components:

$$\mathbf{q} = [q_{1x}, q_{1y}, q_{1z}, q_{2x}, q_{2y}, q_{2z}]^T.$$



Relation between local and global displacements

The relationship between local and global displacements is given by the *direction cosines matrix*:

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{q}_{loc}} = \underbrace{\begin{bmatrix} l_x & l_y & l_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & l_y & l_z \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{1z} \\ q_{2x} \\ q_{2y} \\ q_{2z} \end{bmatrix}}_{\mathbf{q}}$$

where

$$l_x = \frac{x_2 - x_1}{{}^e\ell}, \quad l_y = \frac{y_2 - y_1}{{}^e\ell}, \quad l_z = \frac{z_2 - z_1}{{}^e\ell}$$

$${}^e\ell = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Element stiffness matrix in global coordinates

$$
\begin{aligned}
{}^e\mathbf{K} &= {}^e\mathbf{T}^T \mathbf{K}_{loc} {}^e\mathbf{T} \\
&= \frac{{}^e(EA)}{{}^e\ell} \begin{bmatrix} l_x & l_y & l_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & l_y & l_z \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l_x & l_y & l_z & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x & l_y & l_z \end{bmatrix} \\
&= \frac{{}^e(EA)}{{}^e\ell} \begin{bmatrix} l_x^2 & l_x l_y & l_x l_z & -l_x^2 & -l_x l_y & -l_x l_z \\ & l_y^2 & l_y l_z & -l_x l_y & -l_y^2 & -l_y l_z \\ & & l_z^2 & -l_x l_z & -l_y l_z & -l_z^2 \\ & & & l_x^2 & l_x l_y & l_x l_z \\ & \text{Symm.} & & & l_y^2 & l_y l_z \\ & & & & & l_z^2 \end{bmatrix}
\end{aligned}
$$

Element mass matrices in global coordinates:

- Consistent mass matrix:

$$^{e}\mathbf{M} = {^{e}\mathbf{T}}^{T}\mathbf{M}_{loc}\,{^{e}\mathbf{T}} = \frac{^{e}(\rho A \ell)}{6}\begin{bmatrix} 2l_x^2 & 2l_xl_y & 2l_xl_z & l_x^2 & l_xl_y & l_xl_z \\ 2l_xl_y & 2l_y^2 & 2l_yl_z & l_xl_y & l_y^2 & l_yl_z \\ 2l_xl_z & 2l_yl_z & 2l_z^2 & l_xl_z & l_yl_z & l_z^2 \\ l_x^2 & l_xl_y & l_xl_z & 2l_x^2 & 2l_xl_y & 2l_xl_z \\ l_xl_y & l_y^2 & l_yl_z & 2l_xl_y & 2l_y^2 & 2l_yl_z \\ l_xl_z & l_yl_z & l_z^2 & 2l_xl_z & 2l_yl_z & 2l_z^2 \end{bmatrix}$$

- Lumped mass matrix:

$$^{e}\mathbf{M} = {^{e}\mathbf{T}}^{T}\mathbf{M}_{loc}\,{^{e}\mathbf{T}} = \frac{^{e}(\rho A \ell)}{2}\begin{bmatrix} l_x^2 & l_xl_y & l_xl_z & 0 & 0 & 0 \\ l_xl_y & l_y^2 & l_yl_z & 0 & 0 & 0 \\ l_xl_z & l_yl_z & l_z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_x^2 & l_xl_y & l_xl_z \\ 0 & 0 & 0 & l_xl_y & l_y^2 & l_yl_z \\ 0 & 0 & 0 & l_xl_z & l_yl_z & l_z^2 \end{bmatrix}$$

An example of a 3d truss in free vibration

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