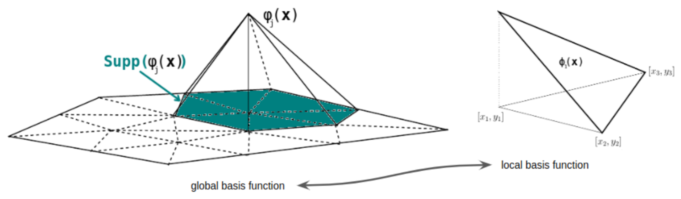
Finite element method in local coordinates

Linear elastodynamics

ME473 Dynamic finite element analysis of structures

Stefano Burzio

2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement

Summary

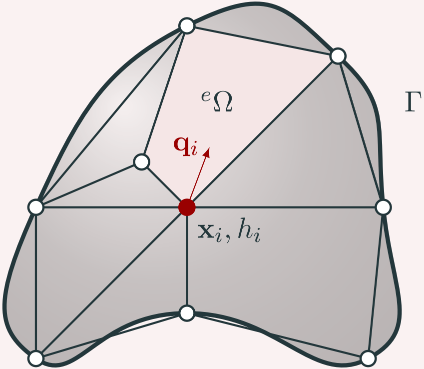
- Recap week 3
- Localization and elementary quantities
- Automating integration and archetypal shape functions
- Example: dynamic analysis of a clamped beam

Recommended readings

1. Gmür, Dynamique des structures (§3.3)
2. Neto et al., Engineering Computation of Structures (§2.3 and chap. 7)

Recap week 3

Displacements approximation in finte element method



Let p the number of nodes of the mesh.

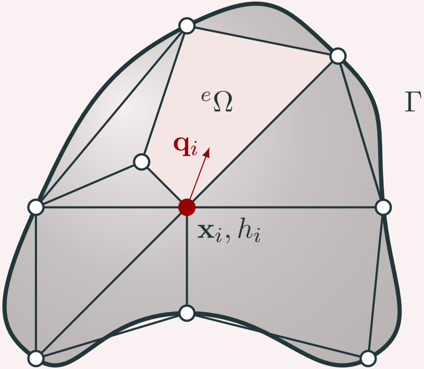
$$\mathbf{u}^h(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t) = \sum_{i=1}^{p} h_i(\mathbf{x})\mathbf{q}_i(t)$$

- $\mathbf{H}(\mathbf{x})$ is a $3 \times 3p$ matrix of **shape functions**:

  $$\mathbf{H} = \left[\begin{array}{c:c:c:c:c:c} h_1\mathbf{I} & h_2\mathbf{I} & \dots & h_i\mathbf{I} & \dots & h_p\mathbf{I} \end{array}\right]$$

  $\mathbf{I}$ is the 3×3 identity matrix.
- $\mathbf{q}(t)$ is a $3p \times 1$ vector of (*unknown*) nodal displacements.

Global nodal shape functions requirements



Propretries of h_i:

- Linearly independent polynomial basis.
- Satisfy Kronecker delta property:

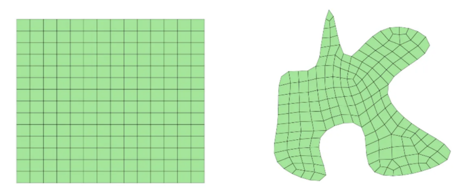
$$h_i(\mathbf{x}_i) = 1 \quad \text{and} \quad h_i(\mathbf{x}_j) = 0.$$

- Vanish on non-adjacent elements.
- Continuous at interfaces.
- Differentiable inside elements.
- Ensure rigid body motion & constant deformations.

Drawbacks of the global approach

- ✗ Limited capability in handling complex (unstructured) mesh topologies.
- ✗ Computationally expensive: it requires defining one shape function per node.
- ✗ Limited utilization of the compact support of nodal shape functions.

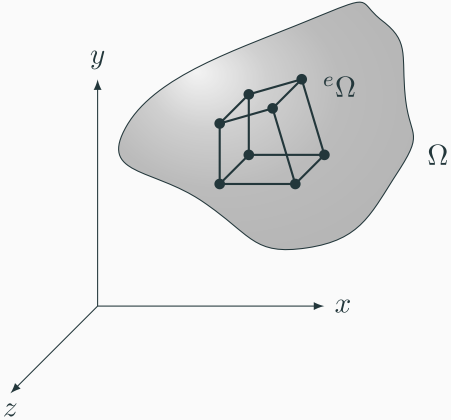
Local approach: provides a quicker and more sistematic way to compute the stiffenss and mass matrices and the applied forces vector:



(Credit: Onscale - structured vs unstructured meshes)

Localization and elementary quantities

Localization



- Let p be the number of nodes.
- Let m be the number of finite elements.
- Let $^e\Omega$ a finite element.
- Let ^ep the number of nodes in the element $^e\Omega$.

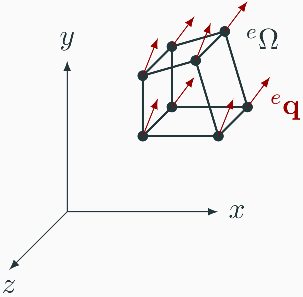
Localization of displacements

Restriction of displacements $\mathbf{u}^h$ and $\delta\mathbf{u}^h$ on the finite element ${}^e\Omega$:

$$ {}^e\mathbf{u}^h(\mathbf{x},t) = {}^e\mathbf{H}(\mathbf{x})\,{}^e\mathbf{q}(t) \qquad {}^e\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}) = {}^e\mathbf{H}(\mathbf{x})\,{}^e\boldsymbol{\delta}\mathbf{q} $$

- ${}^e\mathbf{u}^h$ *restriction* (3×1) of the displacement vector $\mathbf{u}^h$ on the finite element ${}^e\Omega$.
- ${}^e\boldsymbol{\delta}\mathbf{u}^h$ *restriction* (3×1) of the virtual displacement vector $\boldsymbol{\delta}\mathbf{u}^h$ on the finite element ${}^e\Omega$.
- ${}^e\mathbf{H}$ matrix $(3 \times 3\,{}^ep)$ of elementary shape functions of the finite element ${}^e\Omega$.
- ${}^e\mathbf{q}$ vector $(3\,{}^ep \times 1)$ of unknown nodal displacements in the finite element ${}^e\Omega$.
- ${}^e\boldsymbol{\delta}\mathbf{q}$ vector $(3\,{}^ep \times 1)$ of nodal displacements in the finite element ${}^e\Omega$.

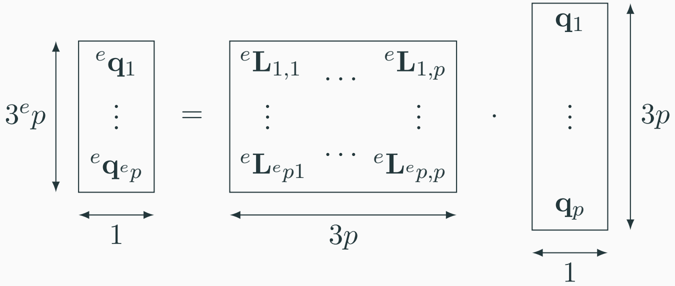
Local displacements approximation



$$ {}^e\mathbf{u}^h = \left[\ {}^eh_1\mathbf{I} \ \vdots\ {}^eh_2\mathbf{I} \ \vdots\ \dots \ \vdots\ {}^eh_{{}^ep}\mathbf{I}\ \right] \begin{pmatrix} {}^e\mathbf{q}_1 \\ {}^e\mathbf{q}_2 \\ \vdots \\ {}^e\mathbf{q}_{{}^ep} \end{pmatrix} $$

$$ {}^e\boldsymbol{\delta}\mathbf{u}^h = \left[\ {}^eh_1\mathbf{I} \ \vdots\ {}^eh_2\mathbf{I} \ \vdots\ \dots \ \vdots\ {}^eh_{{}^ep}\mathbf{I}\ \right] \begin{pmatrix} {}^e\boldsymbol{\delta}\mathbf{q}_1 \\ {}^e\boldsymbol{\delta}\mathbf{q}_2 \\ \vdots \\ {}^e\boldsymbol{\delta}\mathbf{q}_{{}^ep} \end{pmatrix} $$

Localisation matrices



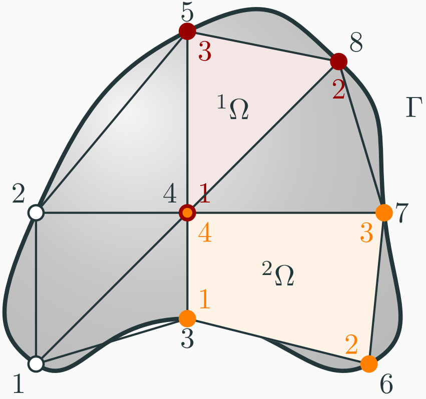
$$\begin{bmatrix} {}^e\mathbf{q}_1 \\ \vdots \\ {}^e\mathbf{q}_{^ep} \end{bmatrix} = \begin{bmatrix} {}^e\mathbf{L}_{1,1} & \dots & {}^e\mathbf{L}_{1,p} \\ \vdots & & \vdots \\ {}^e\mathbf{L}_{^ep1} & \dots & {}^e\mathbf{L}_{^ep,p} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_p \end{bmatrix}$$

$$ {}^e\mathbf{q} = {}^e\mathbf{L}\mathbf{q} $$

${}^e\mathbf{L}$ is a Boolean location matrix:

- ${}^e\mathbf{L}_{ij} = \mathbf{I}$ (3 × 3 identity matrix) if global node j corresponds to local node i,
- ${}^e\mathbf{L}_{ij} = \mathbf{0}$ (3 × 3 null matrix) otherwise.

Localization matrices - example



- Number of nodes in the mesh: $p = 8$.
- Number of elements in the mesh: $m = 6$.
- Number of nodes in the element $^1\Omega$: $^1p = 3$.

$$^1\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

- Number of nodes in the element $^2\Omega$: $^2p = 4$.

$$^2\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Localisation matrices - memory usage

Practical example:

Consider a mesh made of

- $p = 10'000$ nodes,
- trilinear hexahedral finite elements with ${}^{e}p = 8$ nodes each.

Every localization matrix ${}^{e}\mathbf{L}$ contains

- $720'000$ entries,
- of which $3\,{}^{e}p = 24$ are 1s,
- the remaining $719'976$ are 0s.

Connectivity table: local to global node numbering

- Elements and their connectivity are defined using a table.
- Example of connectivity table:

${}^{e}\Omega$	${}^{1}\Omega$	${}^{2}\Omega$	${}^{3}\Omega$	${}^{4}\Omega$
1	1	2	4	5
2	2	3	5	6
3	5	6	8	9
4	4	5	7	8

- The connectivity table provides the global numbering for each node in each element, corresponding to a column in the table above.
- The localization matrices ${}^{e}\mathbf{L}$ can be constructed from the connectivity table.

Additivity of integrals

- Approximated weak form:

$$\int_{\Omega} (\nabla \boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla \mathbf{u}^h \, d\Omega + \int_{\Omega} \rho (\boldsymbol{\delta}\mathbf{u}^h)^T \ddot{\mathbf{u}}^h \, d\Omega = \int_{\Gamma_\sigma} (\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_{\Omega} (\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega$$

- We localize the integrals using their additivity property:

$$\begin{aligned} &\sum_{e=1}^{m} \left(\int_{{}^e\Omega} (\nabla {}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla {}^e\mathbf{u}^h \, d\Omega + \int_{{}^e\Omega} \rho ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T {}^e\ddot{\mathbf{u}}^h \, d\Omega \right) \\ =&\sum_{e=1}^{m} \left(\int_{{}^e\Gamma_\sigma} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega \right) \end{aligned}$$

and consider the local quantities ${}^e\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\mathbf{q}$ and ${}^e\boldsymbol{\delta}\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\boldsymbol{\delta}\mathbf{q}$.

Additivity of integrals - inertial forces

Recall ${}^e\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\mathbf{q}$ and ${}^e\boldsymbol{\delta}\mathbf{u}^h = {}^e\mathbf{H}{}^e\mathbf{L}\boldsymbol{\delta}\mathbf{q}$.

① Consider only the term related to the virtual work of inertial forces (acceleration):

$$\sum_{e=1}^{m} \int_{{}^e\Omega} \rho({}^e\boldsymbol{\delta}\mathbf{u}^h)^T {}^e\ddot{\mathbf{u}}^h \, d\Omega = \boldsymbol{\delta}\mathbf{q}^T \Big[\sum_{e=1}^{m} {}^e\mathbf{L}^T \underbrace{\Big(\int_{{}^e\Omega} \rho {}^e\mathbf{H}^T {}^e\mathbf{H} \, d\Omega \Big)}_{{}^e\mathbf{M}} {}^e\mathbf{L} \Big] \ddot{\mathbf{q}}$$

Additivity of integrals - internal and external forces

Recall ${}^e\mathbf{u}^h = {}^e\mathbf{H}\,{}^e\mathbf{L}\mathbf{q}$ and ${}^e\boldsymbol{\delta}\mathbf{u}^h = {}^e\mathbf{H}\,{}^e\mathbf{L}\boldsymbol{\delta}\mathbf{q}$.

② Analogously for the term related to the virtual work of internal forces:

$$\sum_{e=1}^{m} \int_{{}^e\Omega} (\nabla\, {}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{C} \nabla\, {}^e\mathbf{u}^h \, d\Omega = \boldsymbol{\delta}\mathbf{q}^T \Big[\sum_{e=1}^{m} {}^e\mathbf{L}^T \underbrace{\Big(\int_{{}^e\Omega} \nabla\, {}^e\mathbf{H}^T \mathbf{C} \nabla\, {}^e\mathbf{H} \, d\Omega \Big)}_{{}^e\mathbf{K}} {}^e\mathbf{L} \Big] \mathbf{q}$$

③ Term realted to the virtual work of external forces:

$$\begin{aligned} &\sum_{e=1}^{m} \int_{{}^e\Gamma_\sigma} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} ({}^e\boldsymbol{\delta}\mathbf{u}^h)^T \mathbf{f} \, d\Omega \\ &= \boldsymbol{\delta}\mathbf{q}^T \sum_{e=1}^{m} {}^e\mathbf{L}^T \underbrace{\Big(\int_{{}^e\Gamma_\sigma} {}^e\mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{f} \, d\Omega \Big)}_{{}^e\mathbf{r}(t)} \end{aligned}$$

Elementary matrices and vectors

- **Elementary stiffness matrix** $(3^e p \times 3^e p)$:

$$ {}^e\mathbf{K} = \int_{{}^e\Omega} {}^e\mathbf{B}^T \mathbf{C} {}^e\mathbf{B} \, d\Omega $$

$$ {}^e\mathbf{B} = \nabla {}^e\mathbf{H} = \left[\begin{array}{c:c:c} \nabla {}^e h_1 & \ldots & \nabla {}^e h_{{}^e p} \end{array} \right] $$ elementary deformation matrix $(6 \times 3^e p)$.

- **Elementary mass matrix** $(3^e p \times 3^e p)$:

$$ {}^e\mathbf{M} = \int_{{}^e\Omega} \rho \, {}^e\mathbf{H}^T {}^e\mathbf{H} \, d\Omega. $$

- **Elementary applied forces vector** $(3^e p \times 1)$:

$$ {}^e\mathbf{r}(t) = \int_{{}^e\Gamma_\sigma} {}^e\mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{f} \, d\Omega. $$

Assembly operator

We define the assembly operator as follows:

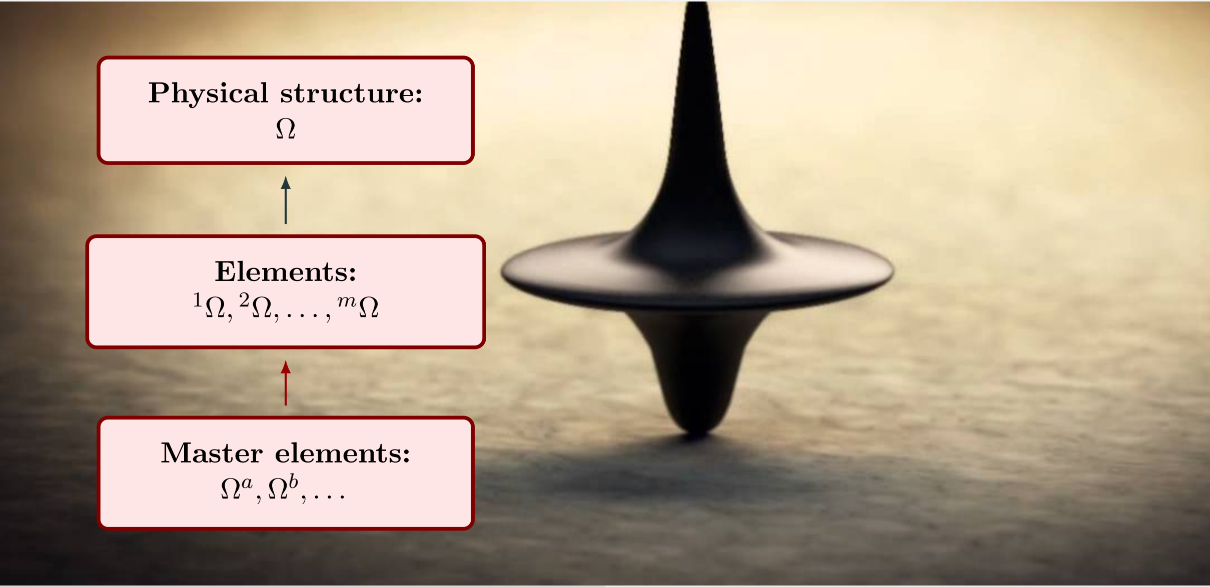
$$\mathbf{K} = \mathop{\mathrm{A}}_{e=1}^{m} {}^{e}\mathbf{K} = \sum_{e=1}^{m} {}^{e}\mathbf{L}^{T\,e}\mathbf{K}^{e}\mathbf{L}$$

$$\mathbf{M} = \mathop{\mathrm{A}}_{e=1}^{m} {}^{e}\mathbf{M} = \sum_{e=1}^{m} {}^{e}\mathbf{L}^{T\,e}\mathbf{M}^{e}\mathbf{L}$$

$$\mathbf{r} = \mathop{\mathrm{A}}_{e=1}^{m} {}^{e}\mathbf{r} = \sum_{e=1}^{m} {}^{e}\mathbf{L}^{T\,e}\mathbf{r}$$

Automating integration and archetypal shape functions

Elements



Physical structure:
Ω

Elements:
${}^1\Omega, {}^2\Omega, \ldots, {}^m\Omega$

Master elements:
$\Omega^a, \Omega^b, \ldots$

Coordinate transform

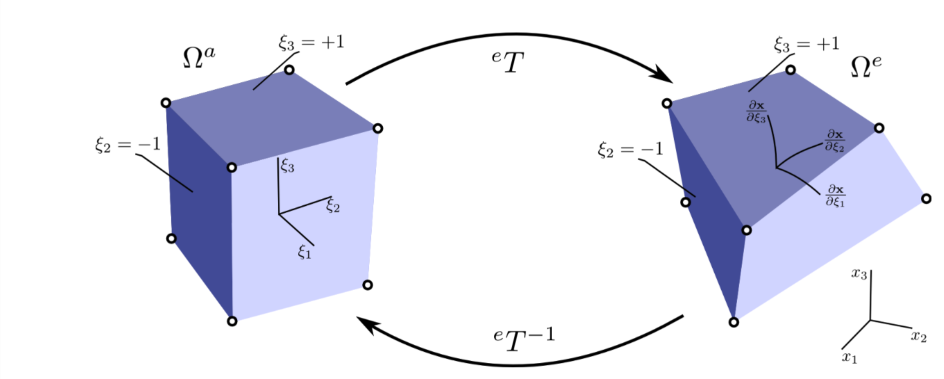
To automate the integration and simplify the definition of shape functions, we transform each *distorted* finite element ${}^{e}\Omega$ into a **reference (archetypal or master) element** Ω^a where we can apply standard numerical integration schemes.

- The coordinate transformation:

$$\begin{aligned} {}^{e}T : \Omega^a &\to {}^{e}\Omega \\ \boldsymbol{\xi} &\mapsto \mathbf{x}(\boldsymbol{\xi}) \end{aligned}$$

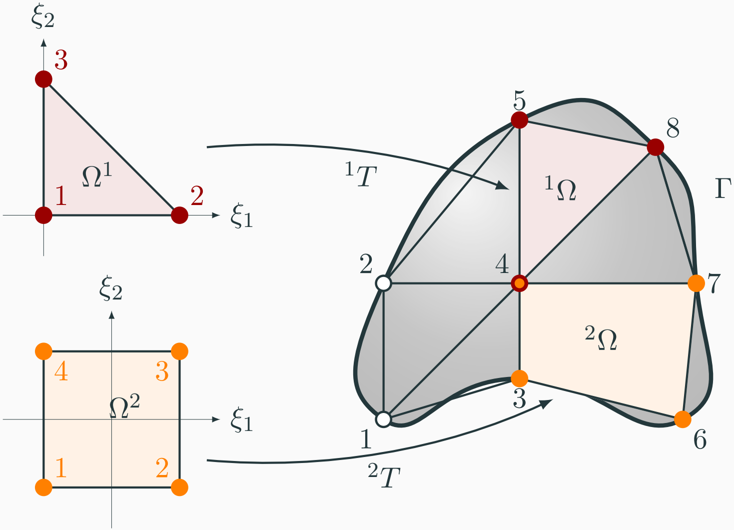
  maps any point $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}^T$ in Ω^a to its corresponding point of coordinate $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) = \{x_1(\boldsymbol{\xi}), x_2(\boldsymbol{\xi}), x_3(\boldsymbol{\xi})\}^T$ in ${}^{e}\Omega$.
- ${}^{e}T$ is a bijective application.

An illustration of a coordinate transform ${}^eT$ in 3D



(Credit: Joel Cugnoni - Finite Element Method applied to linear statics of deformable solids)

An illustration of coordinate transforms in 2D



Biunivocity of coordinate transformation ${}^eT$

$$
{}^e\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} \qquad {}^e\mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \frac{\partial \xi_1}{\partial x_3} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} & \frac{\partial \xi_3}{\partial x_3} \\ \frac{\partial \xi_3}{\partial x_1} & \frac{\partial \xi_3}{\partial x_2} & \frac{\partial \xi_3}{\partial x_3} \end{bmatrix}
$$

- ${}^e\mathbf{J}$ is the Jacobian matrix associated with ${}^eT$: ${}^e\mathbf{J}_{ij} = \frac{\partial x_i}{\partial \xi_j}$ $(i, j = 1, 2, 3)$,
- ${}^e\mathbf{J}^{-1}$ is the inverse Jacobian matrix,
- ${}^ej = \det({}^e\mathbf{J})$ is the determinant of the Jacobian matrix ${}^e\mathbf{J}$.

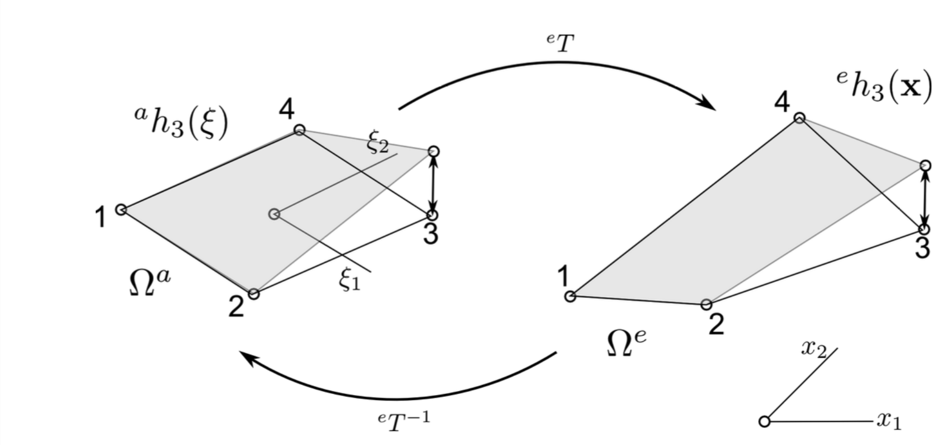
Sufficient condition for invertibility: if ${}^ej > 0$ everywhere in ${}^e\Omega$, then ${}^eT$ is invertible in ${}^e\Omega$ and ${}^e\mathbf{J}^{-1}$ exists.

Master elements and master shape functions

- The coordinate transformation ${}^{e}T^{-1}$ maps shape functions on ${}^{e}\Omega$ to the master element space Ω^{a}:
$$ {}^{a}\mathbf{H}(\boldsymbol{\xi}) = {}^{e}\mathbf{H}(\mathbf{x}(\boldsymbol{\xi})) $$
- Inside each element ${}^{e}\Omega$:
$$ {}^{e}\mathbf{u}^{h}[\mathbf{x}(\boldsymbol{\xi})] = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{q} $$
- This allows shape function ${}^{a}\mathbf{H}(\boldsymbol{\xi})$ to be defined only *once* on each master element ${}^{a}\Omega$.

☞ **Result: master shape functions!**

An illustration of a master shape function



(Credit: Joel Cugnoni - Finite Element Method applied to linear statics of deformable solids)

Choosing a simple coordinate transformation

☞ Coordinate transformations are defined using the shape functions:

$$ {}^{e}T : \mathbf{x}(\boldsymbol{\xi}) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{x} = \sum_{i=1}^{{}^{e}p} {}^{a}h_i(\boldsymbol{\xi})\,{}^{e}\mathbf{x}_i $$

- Inside each element ${}^{e}\Omega$, the local coordinates are interpolated as a linear combination of master shape functions ${}^{a}h_i$ and nodal coordinates ${}^{e}\mathbf{x}_i$.
- Kronecker property ensures node correspondence:

$$ {}^{e}h_i(\mathbf{x}_j) = \delta_{ij} \quad \Rightarrow \quad {}^{a}h_i(\boldsymbol{\xi}_j) = \delta_{ij}. $$

- This guarantees that each node of the master element Ω^{a} maps to a corresponding node in the deformed element ${}^{e}\Omega$.

Integration by substitution formulas

- Given ${}^eT : \Omega^a \to {}^e\Omega$, an integral over ${}^e\Omega$ of a function $F : {}^e\Omega \to \mathbb{R}$ can be rewritten as an integral over Ω^a:

$$\int_{{}^e\Omega} F(\mathbf{x})\, d\mathbf{x} = \int_{\Omega^a} F(\mathbf{x}(\boldsymbol{\xi}))\, {}^e j\, d\boldsymbol{\xi}$$

- When the integrand involves the operator ∇, then:

$$\int_{{}^e\Omega} \nabla_{\mathbf{x}} F(\mathbf{x})\, d\mathbf{x} = \int_{\Omega^a} \nabla_{\boldsymbol{\xi}} F(\mathbf{x}(\boldsymbol{\xi}))^e\mathbf{J}^{-1}\, {}^e j\, d\boldsymbol{\xi}$$

since

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} = \sum_{j=1}^{3} {}^e\mathbf{J}^{-1}_{ij} \frac{\partial}{\partial \xi_j}.$$

Master elements and derivatives

- The spatial derivative operator $\nabla_{\mathbf{x}}$ is defined in the global coordinate system (x_1, x_2, x_3). Applying the coordinate transform ${}^eT^{-1}$, we can then extend it to be applied on the master element Ω^a:

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} = \sum_{j=1}^{3} {}^e\mathbf{J}_{ij}^{-1} \frac{\partial}{\partial \xi_j}.$$

- The elementary strain-displacement matrix ${}^e\mathbf{B}$ can be directly derived from the master shape functions ${}^a\mathbf{H}$:

$${}^e\mathbf{B} = \left[\begin{array}{c:c:c} \nabla_{\mathbf{x}} {}^e h_1 & \ldots & \nabla_{\mathbf{x}} {}^e h_{e_p} \end{array} \right] = \left[\begin{array}{c:c:c} \nabla_{\boldsymbol{\xi}} {}^a h_1 {}^e\mathbf{J}^{-1} & \ldots & \nabla_{\boldsymbol{\xi}} {}^a h_{e_p} {}^e\mathbf{J}^{-1} \end{array} \right].$$

Automating the integration

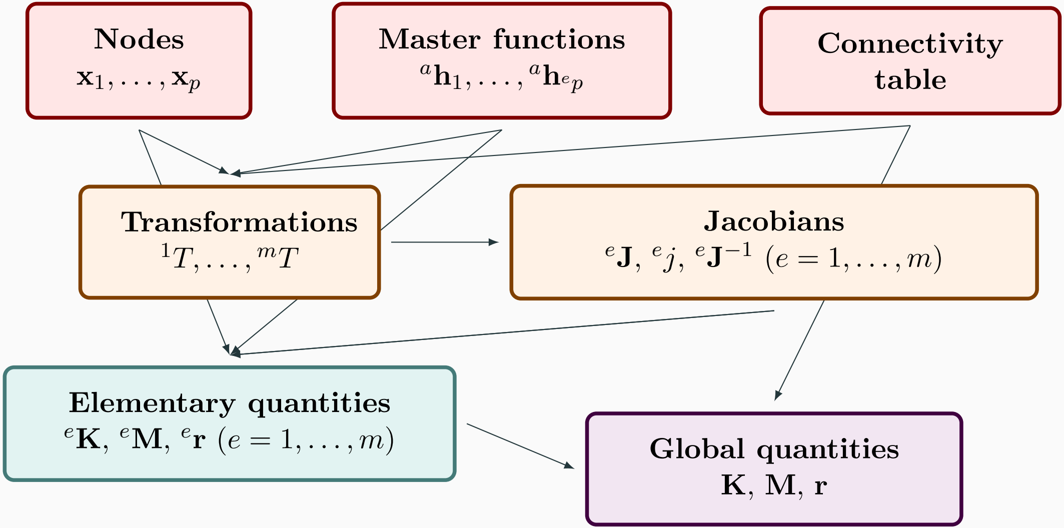
Using the coordinate transform ${}^{e}T$ and master shape functions, the integrals in the definitions of ${}^{e}\mathbf{K}$, ${}^{e}\mathbf{M}$ and ${}^{e}\mathbf{r}$, can be carried out directly on a standard domain Ω^{a}:

$$ {}^{e}\mathbf{K} = \int_{\Omega^{a}} \left(\nabla_{\boldsymbol{\xi}} {}^{a}\mathbf{H} {}^{e}\mathbf{J}^{-1}\right)^{T} \mathbf{C} \left(\nabla_{\boldsymbol{\xi}} {}^{a}\mathbf{H} {}^{e}\mathbf{J}^{-1}\right) {}^{e}j \, d\boldsymbol{\xi} $$

$$ {}^{e}\mathbf{M} = \int_{\Omega^{a}} \rho \, {}^{a}\mathbf{H}^{T} \, {}^{a}\mathbf{H} \, {}^{e}j \, d\boldsymbol{\xi} $$

$$ {}^{e}\mathbf{r}(t) = \int_{\Gamma_{\sigma}^{a}} {}^{a}\mathbf{H}^{T} \hat{\mathbf{f}} \, {}^{e}j \Big|_{\Gamma_{\sigma}^{a}} d\Gamma + \int_{\Omega^{a}} {}^{a}\mathbf{H}^{T} \mathbf{f} \, {}^{e}j \, d\boldsymbol{\xi} $$

Systematization of the finite elements algorithm



Nodes
$\mathbf{x}_1, \ldots, \mathbf{x}_p$

Master functions
${}^a\mathbf{h}_1, \ldots, {}^a\mathbf{h}_{e_p}$

Connectivity table

Transformations
${}^1T, \ldots, {}^mT$

Jacobians
${}^e\mathbf{J},\ {}^ej,\ {}^e\mathbf{J}^{-1}\ (e = 1, \ldots, m)$

Elementary quantities
${}^e\mathbf{K},\ {}^e\mathbf{M},\ {}^e\mathbf{r}\ (e = 1, \ldots, m)$

Global quantities
$\mathbf{K},\ \mathbf{M},\ \mathbf{r}$

Isoparametric, subparametric, superparametric elements

Displacement field as well as the geometrical representation of the finite elements could be approximated using different sets of shape functions.

- **Geometrical representation**

$$ {}^{e}T : \mathbf{x}(\boldsymbol{\xi}) = {}^{a}\mathbf{H}(\boldsymbol{\xi})\,{}^{e}\mathbf{x} $$

- **Displacement field approximation**

$$ {}^{e}\mathbf{u}^{h}(\mathbf{x}, t) = {}^{a}\mathbf{H}(\boldsymbol{\xi}(\mathbf{x}))\,{}^{e}\mathbf{q}(t) $$

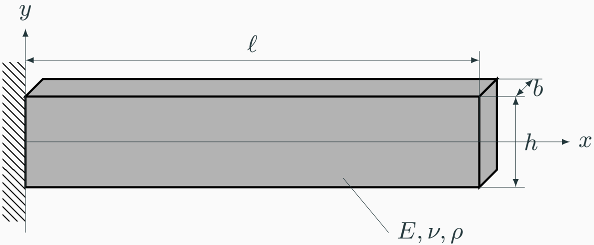
1. Subparametric: less nodes for geometric than for displacement,
2. Isoparametric: same nodes for both geometry and displacement,
3. Superparametric: more nodes for geometric than for displacement.

Example: modal analysis of a clamped beam

Modal analysis of a clamped beam

Kinematic assumptions:

- The beam is made of elastic material which is homogeneous and isotropic (E, ν and ρ).
- Assume a plane stress state (very small thickness). The structure can be modeled in the (x, y) plane using two-dimensional finite elements.



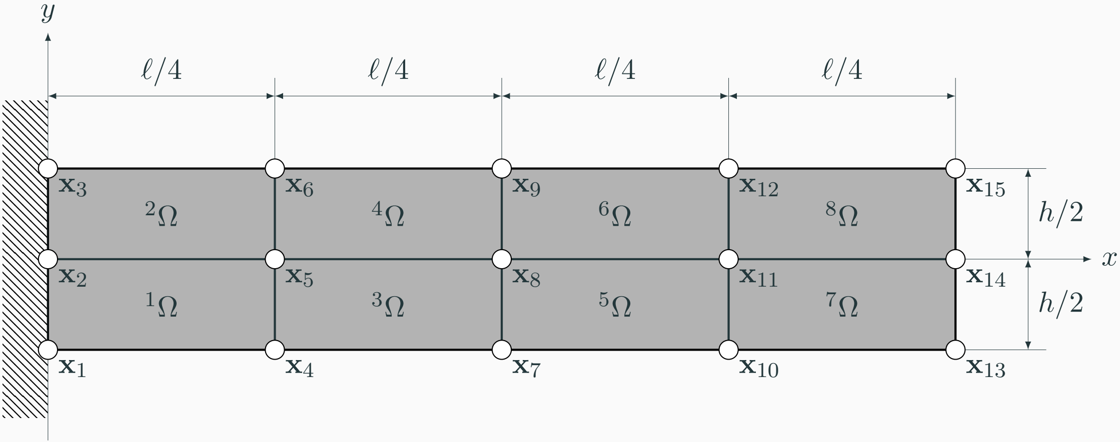
- $E = 210\ GPa$ Young's modulus
- $\nu = 0.3$ Poisson's ratio
- $\rho = 7800\ kg/m^3$ material density
- $\ell = 2\ m$ length
- $h = 0.5\ m$ height
- $b = 0.01\ m$ thickness
- x axial coordinate
- y transversal coordinate

Variables:

- $u_1(x, y, t)$ axial displacement
- $u_2(x, y, t)$ transversal displacement

Modal analysis of a clamped beam

Discretization into 8 bilinear quadrilateral finite elements (4 nodes each)



Modal analysis of a clamped beam

Objective: determine the first natural frequencies of the beam.

▸ Go to Matlab Drive