Finite element method

Linear elastodynamics

ME473 Dynamic finite element analysis of structures

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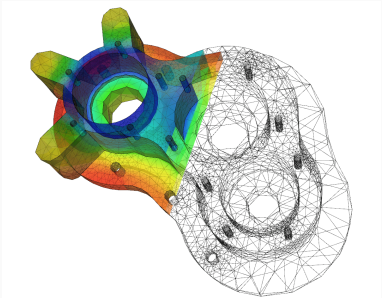


Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation

Summary

- Recap week 2
- Finite element method in global coordinate system
- 1D and 2D elements and shape functions
- Example: longitudinal vibration of a bar
- Localization and elementary quantities

Recommended readings

1. Gmür, Dynamique des structures (§3.1 and §3.2)
2. Gmür, Méthode des éléments finis (§5.3)
3. Neto et al., Engineering Computation of Structures (§2.3)

Recap week 2

Formulations of elastodynamics

Strong form:

$$\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

Weak form:

$$\int_\Omega (\nabla \boldsymbol{\delta u})^T \mathbf{C} \nabla \mathbf{u} \, d\Omega + \int_\Omega \rho \, \boldsymbol{\delta u}^T \ddot{\mathbf{u}} \, d\Omega = \int_{\Gamma_\sigma} \boldsymbol{\delta u}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \boldsymbol{\delta u}^T \mathbf{f} \, d\Omega$$

Semi-discrete weak form:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

Galerkin method

Shape functions:

$$\mathbf{u}^h(\mathbf{x},t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t) \qquad \text{and} \qquad \delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q}.$$

$$\begin{pmatrix} u_1^h(\mathbf{x},t) \\ u_2^h(\mathbf{x},t) \\ u_3^h(\mathbf{x},t) \end{pmatrix} = \begin{bmatrix} h_{11}(\mathbf{x}) & h_{12}(\mathbf{x}) & \dots & h_{1n}(\mathbf{x}) \\ h_{21}(\mathbf{x}) & h_{22}(\mathbf{x}) & \dots & h_{2n}(\mathbf{x}) \\ h_{31}(\mathbf{x}) & h_{32}(\mathbf{x}) & \dots & h_{3n}(\mathbf{x}) \end{bmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{pmatrix}$$

$$\begin{pmatrix} \delta u_1^h(\mathbf{x}) \\ \delta u_2^h(\mathbf{x}) \\ \delta u_3^h(\mathbf{x}) \end{pmatrix} = \begin{bmatrix} h_{11}(\mathbf{x}) & h_{12}(\mathbf{x}) & \dots & h_{1n}(\mathbf{x}) \\ h_{21}(\mathbf{x}) & h_{22}(\mathbf{x}) & \dots & h_{2n}(\mathbf{x}) \\ h_{31}(\mathbf{x}) & h_{32}(\mathbf{x}) & \dots & h_{3n}(\mathbf{x}) \end{bmatrix} \begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{pmatrix}$$

Definitions

- **Stiffness matrix** $(n \times n)$:

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega$$

  where $\mathbf{B}$ is the $(6 \times n)$ deformation matrix defined by $\mathbf{B} = \nabla \mathbf{H}$.

- **Mass matrix** $(n \times n)$:

$$\mathbf{M} = \int_{\Omega} \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega.$$

- **Applied forces vector** $(n \times 1)$:

$$\mathbf{r}(t) = \int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_{\Omega} \mathbf{H}^T \mathbf{f} \, d\Omega.$$

Advantages and drawbacks of Galerkin method

Advantages:

- ✓ Converges quickly with appropriate shape functions.
- ✓ Provides a systematic and structured approach for approximating solutions.
- ✓ The same set of functions is used to express real and virtual variables.

Drawbacks:

- ✗ Accuracy heavily dependent on choice of basis functions.
- ✗ No physical interpretation of the unknown variable $\mathbf{q}(t)$.
- ✗ The formulation of initial and boundary conditions in the discretized form is cumbersome.

Static, modal, and transient analysis

- **Static analysis**: determines deformations $\mathbf{q}$ due to constant loads.

$$\mathbf{K}\mathbf{q} = \mathbf{r}$$

- **Modal analysis**: studies dynamic properties in the frequency domain.

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$

  Natural frequency: key structural property essential in structural design.
  - Lower frequencies → higher displacement amplitudes → more dangerous.
  - Resonance when external excitation frequency matches a natural frequency.
  - Prolonged resonance → catastrophic failure.

- **Transient analysis**: examines time-dependent structural responses $\mathbf{q}(t)$ to time-dependent excitations $\mathbf{r}(t)$.

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

Modal analysis

- Free vibrations: $\mathbf{r}(t) = \mathbf{0}$.
- Assume harmonic solution: $\mathbf{q}(t) = \mathbf{p}\alpha\cos(\omega t - \varphi)$
  - $\mathbf{p}$ is the mode shape (eigenvector).
  - ω is the natural frequency (eigenvalue).
  - α (amplitude) and φ (phase) depend on initial conditions.
- Leads to the eigenvalue problem:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{p} = \mathbf{0}.$$

- Non-trivial solutions exist only if:

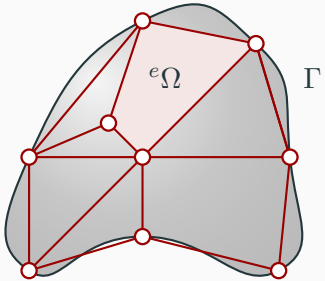
$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0.$$

The eigenvalues $\lambda_i = \omega_i^2$ are the squares of the natural frequencies.
The eigenvectors $\mathbf{p}_i$ are the corresponding mode shapes.

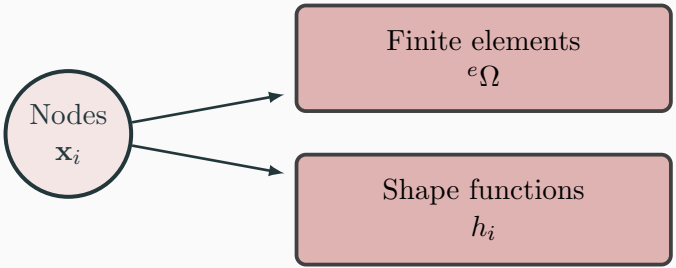
Finite element method in global coordinate system

Divide and conquer mentality

- The geometry of a structure is discretised when it is split into a mesh of finite elements ${}^1\Omega, \ldots, {}^m\Omega$ (smaller components or regular shapes).
- The discretisation introduces an approximation error which can be reduced by using a finer mesh (i.e. more elements), or by increasing the accuracy of the finite elements chosen.
- Discretisation starts with the definition of a **set of nodes**: $\mathbf{x}_i$, $i = 1, \ldots, p$.



Nodes, elements and shape functions



- **Finite element method:** a special case of the Galerkin method, where shape functions are systematically defined in terms of nodes.
- Each shape function is non-zero only on a very limited number of finite elements (*compact support*).
- The computation of the mass and stiffness matrix is enhanced by the local nature of the shape functions.

Displacements approximation

Let p the number of nodes of the mesh.

$$\mathbf{u}^h(\mathbf{x}, t) = \mathbf{H}(\mathbf{x})\mathbf{q}(t)$$
$$\delta\mathbf{u}^h(\mathbf{x}) = \mathbf{H}(\mathbf{x})\delta\mathbf{q}$$

- $\mathbf{H}(\mathbf{x})$ is a $3 \times 3p$ matrix of **shape functions**.

$$\mathbf{H} = \left[\; h_1\mathbf{I} \;\vdots\; h_2\mathbf{I} \;\vdots\; \ldots \;\vdots\; h_i\mathbf{I} \;\vdots\; \ldots \;\vdots\; h_p\mathbf{I} \; \right]$$

  $\mathbf{I}$ is the 3×3 identity matrix.
- $\mathbf{q}(t)$ is a $3p \times 1$ vector of (*unknown*) time-dependent functions.
- $\delta\mathbf{q}$ is a $3p \times 1$ vector of constants.

Displacements approximation - matrix notation

$$\begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \end{pmatrix} = \left[\begin{array}{ccc:ccc:c:ccc} h_1 & 0 & 0 & h_2 & 0 & 0 & & h_p & 0 & 0 \\ 0 & h_1 & 0 & 0 & h_2 & 0 & \dots & 0 & h_p & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 & & 0 & 0 & h_p \end{array}\right] \begin{pmatrix} q_{1,1} \\ q_{1,2} \\ q_{1,3} \\ \vdots \\ q_{p,1} \\ q_{p,2} \\ q_{p,3} \end{pmatrix}$$

$$\begin{pmatrix} \delta u_1^h \\ \delta u_2^h \\ \delta u_3^h \end{pmatrix} = \left[\begin{array}{ccc:ccc:c:ccc} h_1 & 0 & 0 & h_2 & 0 & 0 & & h_p & 0 & 0 \\ 0 & h_1 & 0 & 0 & h_2 & 0 & \dots & 0 & h_p & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 & & 0 & 0 & h_p \end{array}\right] \begin{pmatrix} \delta q_{1,1} \\ \delta q_{1,2} \\ \delta q_{1,3} \\ \vdots \\ \delta q_{p,1} \\ \delta q_{p,2} \\ \delta q_{p,3} \end{pmatrix}$$

Displacements approximation - index notation

$$\mathbf{u}^h(\mathbf{x},t) = \sum_{i=1}^{p} h_i(\mathbf{x})\mathbf{q}_i(t)$$

$$\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x},t) = \sum_{i=1}^{p} h_i(\mathbf{x})\boldsymbol{\delta}\mathbf{q}_i$$

$$\mathbf{q}_i(t) = \begin{pmatrix} q_{i,1}(t) \\ q_{i,2}(t) \\ q_{i,3}(t) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\delta}\mathbf{q}_i = \begin{pmatrix} \delta q_{i,1} \\ \delta q_{i,2} \\ \delta q_{i,3} \end{pmatrix}$$

Deformation, stiffness, mass matrices and loads vector

- $\mathbf{B} = \nabla \mathbf{H}$ is a $(6 \times 3p)$ matrix:

$$\mathbf{B} = \left[\begin{array}{ccc:c:ccc} \partial_x h_1 & 0 & 0 & & \partial_x h_p & 0 & 0 \\ 0 & \partial_y h_1 & 0 & \dots & 0 & \partial_y h_p & 0 \\ 0 & 0 & \partial_z h_1 & & 0 & 0 & \partial_z h_p \\ 0 & \partial_z h_1 & \partial_y h_1 & & 0 & \partial_z h_p & \partial_y h_p \\ \partial_z h_1 & 0 & \partial_x h_1 & \dots & \partial_z h_p & 0 & \partial_x h_p \\ \partial_y h_1 & \partial_x h_1 & 0 & & \partial_y h_p & \partial_x h_p & 0 \end{array}\right] = \left[\begin{array}{c:c:c} \nabla h_1 & \dots & \nabla h_p \end{array}\right]$$

- $\mathbf{K}$ and $\mathbf{M}$ are $(3p \times 3p)$ matrices and $\mathbf{r}$ is a $(3p \times 1)$ vector:

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{C} \mathbf{B} \, d\Omega, \qquad \mathbf{M} = \int_\Omega \rho \, \mathbf{H}^T \mathbf{H} \, d\Omega, \qquad \mathbf{r}(t) = \int_{\Gamma_\sigma} \mathbf{H}^T \hat{\mathbf{f}} \, d\Gamma + \int_\Omega \mathbf{H}^T \mathbf{f} \, d\Omega.$$

Global nodal shape functions - construction guidelines

Global nodal shape functions $h_i : \Omega \to \mathbb{R}$ are characterized by the following properties:

- They form a linearly independent basis of polynomials of a given degree.
- Their values lie in the interval $[0, 1]$.
- They satisfy the Kronecker delta property:

$$h_i(\mathbf{x}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- The functions h_i vanish on all finite elements that are not adjacent to the node $\mathbf{x}_i$.

Properties of shape functions - convergence criteria

- Continuity of h_i at nodes and interfaces of finite elements.
- Differentiability of h_i inside each finite element.
- Completeness (rigid body motion and constant deformation):

$$\sum_{i=1}^{p} h_i(\mathbf{x}) = 1 \quad \text{and} \quad \sum_{i=1}^{p} \nabla h_i(\mathbf{x}) = \mathbf{0}.$$

- Approximate real and virtual diplacements at the nodes:

$$\mathbf{u}^h(\mathbf{x}_j, t) = \sum_{i=1}^{p} h_i(\mathbf{x}_j)\mathbf{q}_i(t) = \mathbf{q}_j(t),$$

$$\boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}_j) = \sum_{i=1}^{p} h_i(\mathbf{x}_j)\boldsymbol{\delta}\mathbf{q}_i = \boldsymbol{\delta}\mathbf{q}_j.$$

Treatment of initial conditions

Since $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$, enforcing $\mathbf{u}^h(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ ensures that the approximate displacement field is consistent with the initial condition. This leads to

$$\mathbf{q}_j(0) = \mathbf{u}^h(\mathbf{x}_j, 0) = \mathbf{u}_0(\mathbf{x}_j).$$

Similarly, imposing the initial velocity condition, $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$, gives

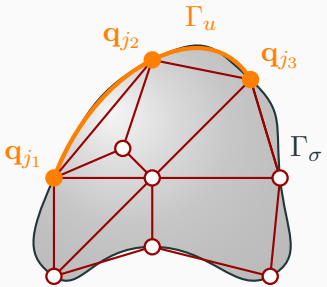
$$\dot{\mathbf{q}}_j(0) = \dot{\mathbf{u}}^h(\mathbf{x}_j, 0) = \mathbf{v}_0(\mathbf{x}_j).$$

✓ These expressions provide the initial values of the unknown vector $\mathbf{q}$ directly in terms of the given initial data $\mathbf{u}_0$ and $\mathbf{v}_0$.

Treatment of boundary conditions

On Γ_u we impose $\mathbf{u}^h = \hat{\mathbf{u}}$ and $\boldsymbol{\delta}\mathbf{u}^h = \mathbf{0}$. Consequently for every node $\mathbf{x}_j \in \Gamma_u$:

$$\mathbf{q}_j(t) = \mathbf{u}^h(\mathbf{x}_j, t) = \hat{\mathbf{u}}(\mathbf{x}_j, t) \quad \text{and} \quad \boldsymbol{\delta}\mathbf{q}_j = \boldsymbol{\delta}\mathbf{u}^h(\mathbf{x}_j, t) = \mathbf{0}.$$



✓ The j-th component of the approximated nodal displacement $\mathbf{q}$ is know for every node $\mathbf{x}_j \in \Gamma_u$.

Semi-discrete weak form of elastodynamics

Given Ω, Γ, $\mathbf{C}$, ρ, $\mathbf{f}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{f}}$, $\mathbf{u}_0$, $\mathbf{v}_0$, and p nodes $\mathbf{x}_1, \dots \mathbf{x}_p$, find the approximated nodal displacements vector $\mathbf{q} \in C^2([0,T], \mathbb{R}^n)$ such that for every vector $\boldsymbol{\delta}\mathbf{q}$:

$$\boldsymbol{\delta}\mathbf{q}^T \left[\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) - \mathbf{r}(t) \right] = 0$$

coupled with initial conditions

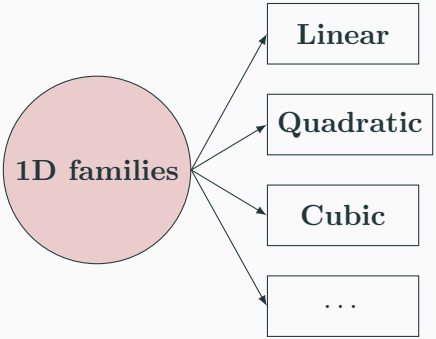
$$\mathbf{q}(0) = \mathbf{u}_0 \qquad \text{and} \qquad \dot{\mathbf{q}}(0) = \mathbf{v}_0.$$

Moreover for every node $\mathbf{x}_j \in \Gamma_u$ we have

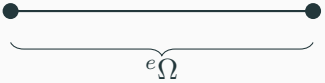
$$\mathbf{q}_j(t) = \hat{\mathbf{u}}(\mathbf{x}_j, t) \qquad \text{and} \qquad \boldsymbol{\delta}\mathbf{q}_j = \mathbf{0}.$$

1D and 2D elements and shape functions

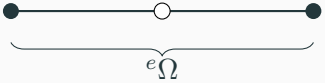
Shape functions for one dimensional structures



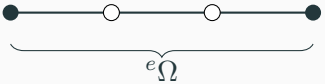
1D finite elements



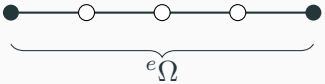
(a) Linear element



(b) Quadratic element

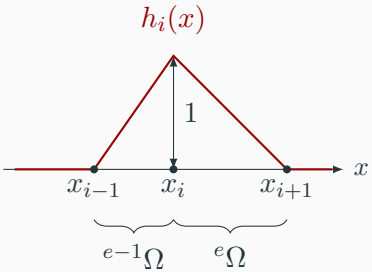


(c) Cubic element



(d) Quartic element

Linear shape functions



$$h_i(x) = \begin{cases} l_2[x_{i-1}, x_i](x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in {}^{e-1}\Omega \\ l_1[x_i, x_{i+1}](x) = \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & x \in {}^{e}\Omega \\ 0 & \text{otherwise} \end{cases}$$

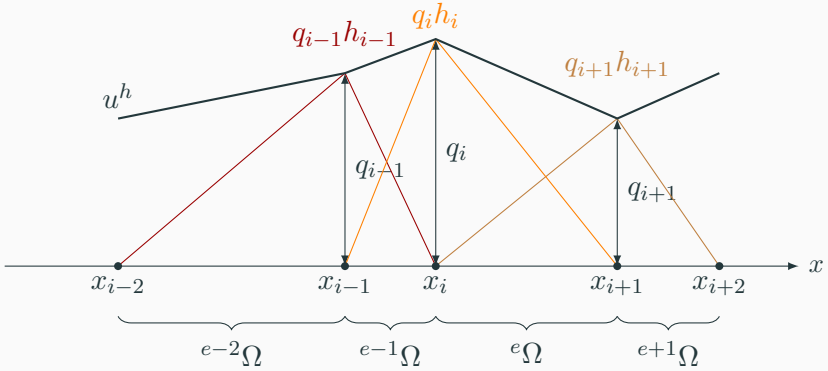
- h_i are piecewise linear functions that, within each finite element, corresponds to a first-degree Lagrange polynomial:

$$l_j[x_1, \ldots, x_n](x) = \prod_{\substack{m=1 \\ m \neq j}}^{n} \frac{x - x_m}{x_j - x_m}$$

- Any linear piecewise function can be expressed as a linear combination of the h_i.

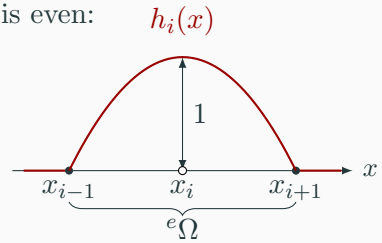
1D displacement approximation via linear shape functions

$$u^h(x,t) = \sum_{i=1}^{p} h_i(x) q_i(t)$$



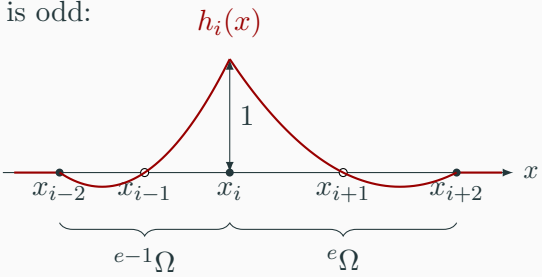
Quadratic shape functions

If i is even:



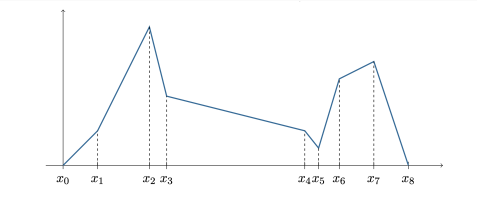
$$h_i(x) = \begin{cases} l_2[x_{i-1}, x_i, x_{i+1}](x) & x \in {}^e\Omega \\ 0 & \text{otherwise} \end{cases}$$

If i is odd:

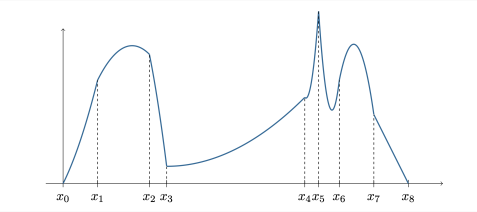


$$h_i(x) = \begin{cases} l_3[x_{i-2}, x_{i-1}, x_i](x) & x \in {}^{e-1}\Omega \\ l_1[x_i, x_{i+1}, x_{i+2}](x) & x \in {}^e\Omega \\ 0 & \text{otherwise} \end{cases}$$

Examples of 1D displacement approximation

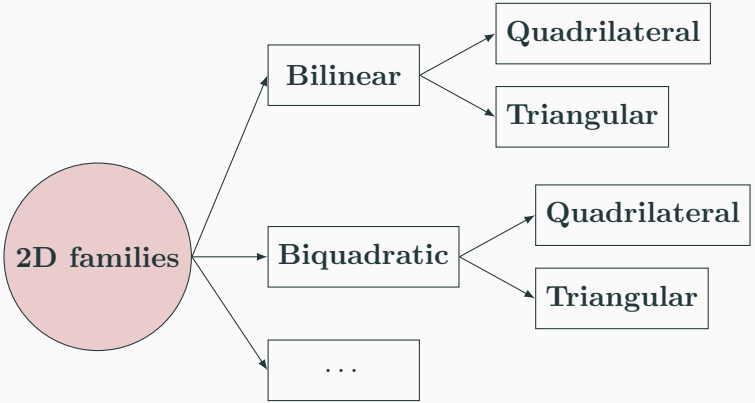


Displacement approximation via linear shape functions

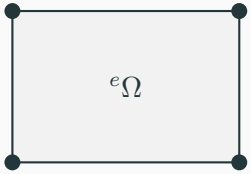


Displacement approximation via quadratic shape functions

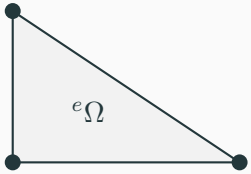
Shape functions for two-dimensional structures



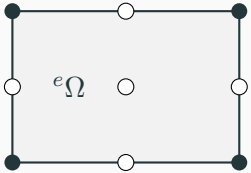
2D finite elements



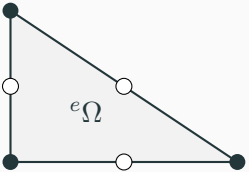
(a) Bilinear Quadrilateral



(b) Bilinear Triangular

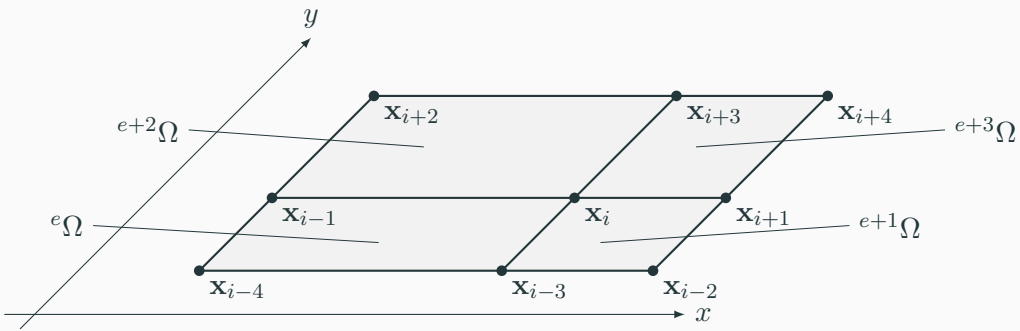


(c) Biquadratic Quadrilateral

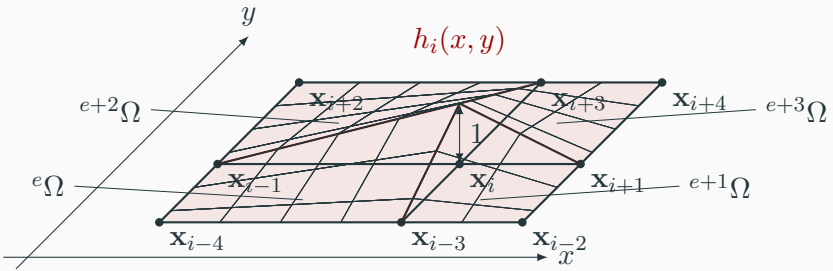


(d) Biquadratic Triangular

Bilinear quadrilateral finite elements mesh

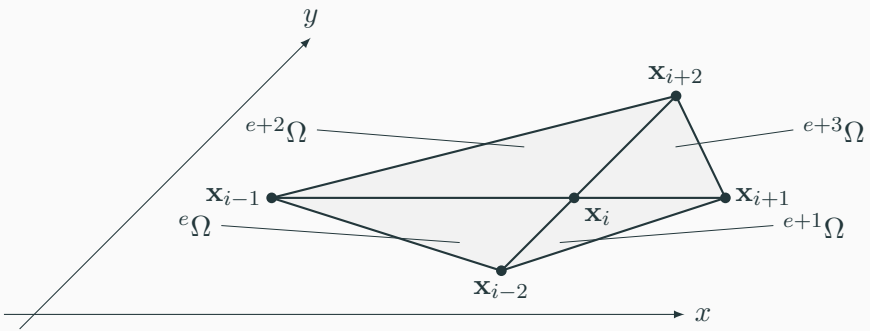


Bilinear quadrilateral shape functions

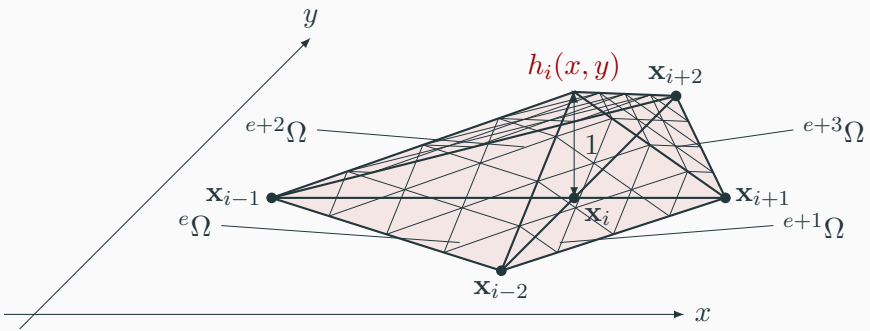


$$h_i(x,y) = \begin{cases} l_2[x_{i-1}, x_i](x) l_2[y_{i-3}, y_i](y) & x \in {}^{e}\Omega \\ l_1[x_i, x_{i+1}](x) l_2[y_{i-3}, y_i](y) & x \in {}^{e+1}\Omega \\ l_2[x_{i-1}, x_i](x) l_1[y_i, y_{i+3}](y) & x \in {}^{e+2}\Omega \\ l_1[x_i, x_{i+1}](x) l_1[y_i, y_{i+3}](y) & x \in {}^{e+3}\Omega \\ 0 & \text{otherwise} \end{cases}$$

Bilinear triangular finite elements mesh



Bilinear triangular shape functions



Advantages and drawbacks of FEM in global coordinate system

Advantages:

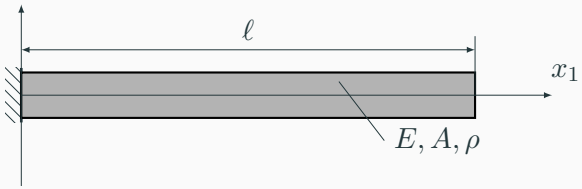
- ✓ The unknowns $\mathbf{q}_i$ have a well-defined physical interpretation, representing the approximate displacement at $\mathbf{x}_i$.
- ✓ Shape functions are systematically defined, ensuring a structured approach to the algorithm.
- ✓ The implementation of boundary and initial conditions is simplified.

Drawbacks:

- ✗ Limited capability in handling complex mesh topologies.
- ✗ Algebraic expressions for shape functions can be computationally cumbersome.
- ✗ The computation of stiffness and mass matrices and loads vector is not optimal.

Example: longitudinal vibration of a bar

Example - Finite elements approximation of longitudinal vibrations of a bar



- A (constant) cross-sectional area
- E (constant) Young's modulus (isotropic)
- ρ material density
- ℓ length
- u_1 axial displacement
- x_1 axial coordinate

Objective: determine the first two natural frequencies of the bar using n linear shape functions and compare the results with those obtained from Galerkin's approximation.

Example - Finite elements approximation of longitudinal vibrations of a bar

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