Problem set 2

Problem 1

Consider the longitudinal vibrations of a free-free (not clamped at both ends) bar of length ℓ, with a constant cross-sectional area A, constant Young's modulus E, and constant material density ρ. The longitudinal vibrations of the bar are governed by the one-dimensional wave equation:

$$\begin{cases} EA\,\partial^2_{xx} u_1(x,t) = \rho A\,\ddot{u}_1(x,t) & \forall (x,t) \in]0,\ell[\times]0,T[\\ EA\,\partial_x u_1(0,t) = 0 & \forall t \in]0,T[\\ EA\,\partial_x u_1(\ell,t) = 0 & \forall t \in]0,T[. \end{cases}$$

These equations are coupled with the initial conditions:

$$u_1(x,0) = u_0(x), \quad \dot{u}_1(x,0) = v_0(x), \quad \forall x \in]0,\ell[.$$

Write the weak form and the semi-discrete weak form of the problem. Then approximate the first and second natural frequencies using a two-term Galerkin approximation based on the shape functions[1]:

$$h_1(x) = 1, \quad h_2(x) = \frac{x}{\ell}.$$

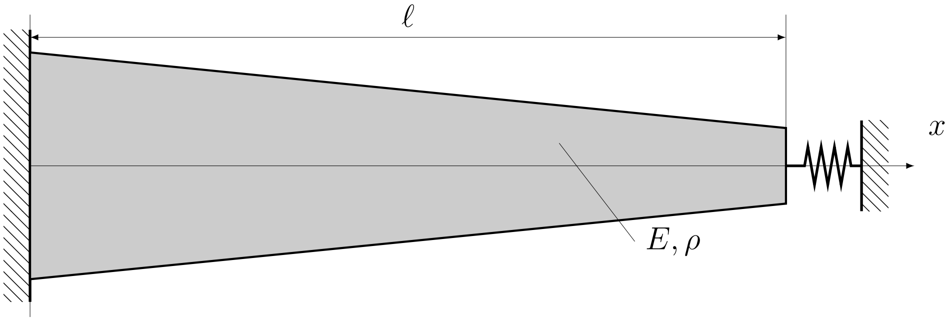
Compare the relative error in the second fundamental frequency with the exact one, obtained by the analytical solution:

$$\omega_2^{\text{exact}} = \pi\sqrt{\frac{E}{\rho\ell^2}}.$$

Finally, provide a physical interpretation the first fundamental frequency.

Problem 2

Write a `MATLAB` script to approximate the natural frequencies of a tapered bar of length $l = 2$ m which is fixed at the left end and spring-supported at the right end, as shown in the figure below.



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[1] Readers familiar with the finite element method will readily recognize that the chosen functions are linear combinations of the two basis functions of a one-dimensional two-node finite element. Furthermore, observe that $h_1(0) \neq 0$. What is the reason for this?

The governing equations are:

$$\begin{cases} \partial_x\big(EA(x)\partial_x u_1(x,t)\big) = \rho A(x)\,\ddot{u}_1(x,t) & \forall (x,t) \in]0,\ell[\times]0,T[\\ u_1(0,t) = 0 & \forall t \in]0,T[\\ EA(\ell)\,\partial_x u_1(\ell,t) + k u_1(\ell,t) = 0 & \forall t \in]0,T[. \end{cases}$$

The homogenous and isotropic bar is made of steel S235 and has Young's modulus $E = 210$ GPa and mass density $\rho = 7850$ kg/m^3. The cross-sectional area of the bar follows a decreasing linear function:

$$A(x) = A_0\big(1 - \beta\frac{x}{\ell}\big)$$

where $A_0 = 150\,\mathrm{mm}^2$ is the cross-section at $x = 0$, and $\beta = 4/15$ is the tapering coefficient. Moreover the spring characteristic constant has value $k = 12$ kN/m.
Apply the Galerkin method using two distinct sets of shape functions:

1. *Harmonic approximation:* a two-terms approximation based on harmonic functions.
2. *Polynomial approximation:* a two-terms approximation based on polynomial functions.

For both approximations, compute the first two approximate natural frequencies. The units used in `MATLAB` are Newton (N), Pascal (Pa) and meter (m).

Problem 1 is taken from [G] example 2.3.3
[G] Gmür, Dynamique des structures: analyse modale numérique. EPFL Press, 1997