

# Transient analysis

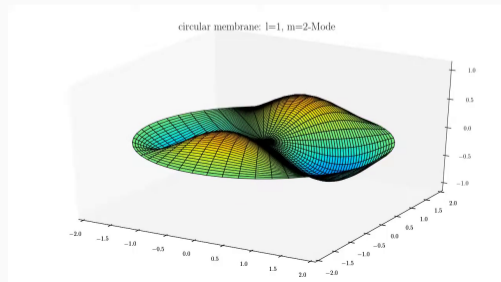
Analysis of free and forced vibrations

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ME473 Dynamic finite element analysis of structures

Stefano Burzio

2025



## Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	
7		Planar beams	Project 1 submission
8		Frames and grids	Project 2 statement
9		Kirchhoff-Love plates	
10		Reissner-Mindlin plates	
11		Shells	Project 2 submission
12	Free and forced vibrations	Analysis of free vibrations	Project 3 statement
13		Analysis of free vibrations	
14		Transient analysis	

## Summary

- Recap week 13
- Analysis of forced vibrations
- Direct integration methods

## Recommended readings

- (B) Bathe, Finite Element Procedures in Engineering Analysis (chap. 9)
- (P) Petyt, Introduction to finite element vibration analysis (chap. 12)
- (G) Gmür, Dynamique des structures (§5.1 and 5.2)

## General end-of-course information

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# Final examination

- **Date:** January 14th
- **Location:** Room CM1221
- **Duration:** 20 min individual presentation followed by a 10 min discussion
- **Report:** Submit the project report via Moodle by January 9th



**Avoid unnecessary stress!**

## Preparation recommendations:


- Review the lecture notes and relevant textbook chapters.
- Thoroughly review the assigned problem sets.
- Use Ed-discussion forum or drop by my office (ME A2 390) if you have any questions.

# CAPE evaluation survey

**MODÈLE**

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Standard Questionnaire in English

 **EVASY5**

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**Codeur:**     Veuillez utiliser un style ou un marqueur fin. Ce questionnaire sera traité automatiquement.

**Conteur:**     Remplissez complètement le cas échéamment applicable, puis cochez votre réponse finale.

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**1. This questionnaire will allow the teacher or teaching team, and the section, to know your opinion on the course.**

1.1 Please indicate your section:

<input type="checkbox"/> AN	<input type="checkbox"/> CSC	<input type="checkbox"/> DH
<input type="checkbox"/> AL	<input type="checkbox"/> DC	<input type="checkbox"/> JPA
<input type="checkbox"/> AT	<input type="checkbox"/> IN	<input type="checkbox"/> MX
<input type="checkbox"/> CL	<input type="checkbox"/> MTS	<input type="checkbox"/> CSC
<input type="checkbox"/> EC	<input type="checkbox"/> PH	<input type="checkbox"/> SH
<input type="checkbox"/> ENL	<input type="checkbox"/> RSC	<input type="checkbox"/> Other

**Merci d'évaluer les affirmations suivantes. / Please use the following statements.**

Tout à fait d'accord - I completely agree - Tout à fait d'accord - I completely agree - Tout à fait d'accord - I completely agree

Strongly agree - Agree - Disagree - Strongly disagree

	Strongly agree	Agree	Disagree	Strongly disagree	No answer
1.2 I find this course interesting.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
1.3 I think the course is well organized.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
1.4 It was clear to me during the course what I should know and be able to do by the end of it.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
1.5 The course activities (exercises, tests, projects, readings, etc.) helped me learn.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
1.6 I could get advice and useful feedback on my work during the semester.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
1.7 I feel that the class climate enabled me to participate, contribute, or ask questions.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
1.8 I feel the workload adequately given the course's weighting in credits/courses.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

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**2. Please, give your general appreciation and comments on the course.**

2.1 Overall, I think this course is good.

	Strongly agree	Agree	Disagree	Strongly disagree	No answer
2.2 Your comments on this course:	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

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**MODÈLE**

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**Please complete the in-depth evaluation survey!**



## Student assistant (AE) needed !

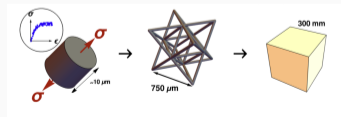
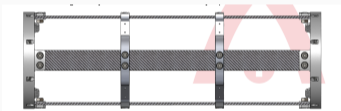
- We are seeking one, possibly two, teaching assistants (AEs) to support the course through:
  - Assistance with exercise sessions
  - Supervision of student mini-projects

## Semester projects

- Structural and dynamic analysis via FEM
- Neural networks meet finite elements

## Master projects

- In collaboration with D-Orbit (topic to be finalized)
- Experimental modal analysis



## Recap week 13

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Free undamped  
discrete vibration  
problem:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$



Ansatz:

$$\mathbf{q}(t) = \alpha \mathbf{p} \cos(\omega t + \varphi)$$



Generalized  
eigenvalue problem:

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{p} = \mathbf{0}$$

Solving the eigenvalue problem:

$$n = \dim(\mathbf{K}) = \dim(\mathbf{M})$$

There are  $n$  eigenvalues  $\lambda_j$  and  $n$  corresponding eigenvectors  $\mathbf{p}_j$  such that:

**Eigenvalues** (*natural frequencies squared*):  
 $\lambda_j = \omega_j^2$  are the roots of the characteristic polynomial:

$$\det(\mathbf{K} - \lambda \mathbf{M}) = 0.$$

**Eigenvectors** (*modal shapes*):  $\mathbf{p}_j$  are the solution of the equation

$$(\mathbf{K} - \lambda_j \mathbf{M})\mathbf{p}_j = \mathbf{0}.$$

## Rigid body modes

In the semi-discrete weak form obtained via finite element discretization:

- The **mass matrix**  $\mathbf{M}$  is symmetric and strictly positive definite.
- The **stiffness matrix**  $\mathbf{K}$  is symmetric and positive semi-definite:

$$\mathbf{K}\mathbf{p} = \mathbf{0} \quad \text{for certain nonzero vectors } \mathbf{p}.$$

Consequently, the eigenvalues  $\omega_j^2$  of the generalized eigenvalue problem are all *real* and *non-negative*:

$$0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n.$$

**Rigid body modes:** zero eigenvalues (i.e.,  $\omega_j = 0$ ) correspond to *rigid body motions*, where the system undergoes displacement without internal deformation.

## Orthonormalization of mode shapes

The eigenvectors (mode shapes)  $\mathbf{p}_j$  are *orthonormal* with respect to the mass matrix  $\mathbf{M}$  and the stiffness matrix  $\mathbf{K}$ .

Modal matrix  $\mathbf{P} = [ \mathbf{p}_1 \mid \mathbf{p}_2 \mid \dots \mid \mathbf{p}_n ]$  then

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \mathbf{I} \quad \text{and} \quad \mathbf{P}^T \mathbf{K} \mathbf{P} = \mathbf{\Lambda}$$

where  $\mathbf{I}$  is the identity matrix of order  $n$  and  $\mathbf{\Lambda}$  the spectral matrix:

$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n) = \text{diag}(\omega_1^2, \dots, \omega_n^2).$$

## Rayleigh's quotient

- Rayleigh's quotient of a generic vector  $\mathbf{w}$ :

$$\mathcal{R}(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{K} \mathbf{w}}{\mathbf{w}^T \mathbf{M} \mathbf{w}}$$

- For a given eigenvector  $\mathbf{p}_i$ , the Rayleigh quotient provides the corresponding eigenvalue  $\omega_i^2$ :

$$\mathcal{R}(\mathbf{p}_i) = \omega_i^2.$$

- The first variation of Rayleigh's quotient is zero at the eigenvectors:

$$\delta \mathcal{R}(\mathbf{w}) = 0 \quad \text{if and only if} \quad \mathbf{w} = \mathbf{p}_i,$$

where  $\mathbf{p}_i$  is an eigenvector.

- **Bounding theorem:**  $\lambda_1 \leq \mathcal{R}(\mathbf{w}) \leq \lambda_n$

The Rayleigh quotient can be used to approximate the smallest eigenvalue  $\omega_1^2$  by minimizing  $\mathcal{R}(\mathbf{w})$  over all non-zero vectors  $\mathbf{w}$ .

# Inverse iteration algorithm

**Goal:** Compute **one** eigenvector using iterative refinement.

**Algorithm:**

- ① Choose initial guess  $\mathbf{p}^{(0)}$ , spectral shift  $\sigma$ , tolerance  $\varepsilon$ .
- ② Define shifted stiffness:  $\mathbf{K}_\sigma = \mathbf{K} + \sigma\mathbf{M}$
- ③ For  $k = 1, 2, \dots$  until convergence:
  - ① Solve:  $\mathbf{K}_\sigma \bar{\mathbf{p}}^{(k)} = \mathbf{M} \mathbf{p}^{(k-1)}$
  - ② Normalize:  $\mathbf{p}^{(k)} = \bar{\mathbf{p}}^{(k)} / \sqrt{(\bar{\mathbf{p}}^{(k)})^T \mathbf{M} \bar{\mathbf{p}}^{(k)}}$
  - ③ Check:  $\|\mathbf{p}^{(k)} - \mathbf{p}^{(k-1)}\| < \varepsilon$
- ④ Output: eigenvector  $\mathbf{p}^{(k)}$ , eigenvalue  $\lambda^{(k)} = \mathcal{R}(\mathbf{p}^{(k)})$

**Key points:**

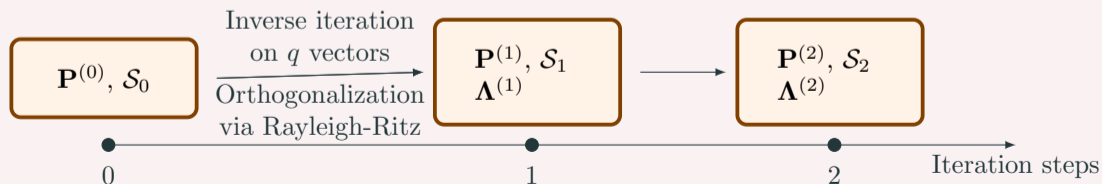
- Converges to eigenvector closest to  $\sigma$  if  $\mathbf{p}^{(0)}$  not  $\mathbf{M}$ -orthogonal to it.
- Without shift ( $\sigma = 0$ )  $\rightarrow$  eigenvector of smallest eigenvalue.
- Convergence rate:  $\rho = \frac{|\lambda_j - \sigma|}{|\lambda_{j+1} - \sigma|}$

# Subspace iteration method

**Goal:** Compute the lowest  $n_{modes} \ll n$  eigenpairs  $(\mathbf{p}_i, \lambda_i)$  of using a  $q$ -dimensional subspace iteration.

**Key ideas and advantages:**

- Iteration is performed on a subspace  $\mathcal{S}_k = \text{span}(\mathbf{p}_1^{(k)}, \dots, \mathbf{p}_q^{(k)})$ , not individual vectors.
- At each iteration, inverse iteration is applied to all  $q$  vectors in the subspace and then orthogonalized via Rayleigh-Ritz.
- Easier to initialize a subspace close to the span of required eigenvectors than individual vectors.



## Subspace iteration algorithm

**Inputs:**  $\mathbf{K}, \mathbf{M}, n_{modes}, q > n_{modes}, \mathbf{P}^{(0)} \in \mathbb{R}^{n \times q}, \sigma, \varepsilon$

**Iteration:** For  $k = 1, 2, \dots$  until convergence:

- 1 Solve  $\mathbf{K}_\sigma \overline{\mathbf{P}^{(k)}} = \mathbf{M} \mathbf{P}^{(k-1)}$  (inverse iteration on  $q$  vectors)
- 2 Project and solve small eigenproblem:

$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{K} \overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{M} \overline{\mathbf{P}^{(k)}}$$
$$\mathbf{K}^{(k)} \mathbf{Z}^{(k)} = \mathbf{M}^{(k)} \mathbf{Z}^{(k)} \mathbf{\Lambda}^{(k)}$$

- 3 Update modal matrix:  $\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}} \mathbf{Z}^{(k)}$ , ensure  $(\mathbf{P}^{(k)})^T \mathbf{M} \mathbf{P}^{(k)} = \mathbf{I}$
- 4 Check convergence:  $\frac{|\lambda_i^{(k)} - \lambda_i^{(k-1)}|}{\lambda_i^{(k)}} < \varepsilon$

**Notes:**

- Orthogonalization is critical to prevent collinearity of iteration vectors.
- Starting vectors should be linearly independent; larger  $q$  increases convergence rate but also computational cost.
- After convergence, use Sturm sequence to verify the number of eigenvalues computed.

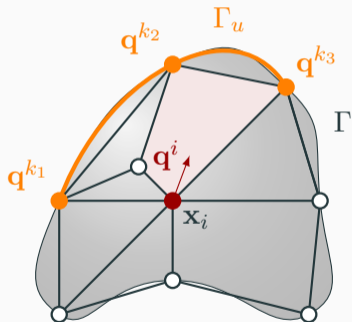
# Analysis of forced vibrations

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## Forced vibrations of non-rotating conservative systems

The discretization of linear three-dimensional elastodynamics, as well as the analysis of vibrations in beams and plates via FEM, all lead to a system of ODE:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t), \quad \forall t \in [0, T]$$



- Generalized nodal displacements:

$$\mathbf{q}(t) = [\mathbf{q}^1(t), \dots, \mathbf{q}^n(t)]^T.$$

- Excitation (force) vector:  $\mathbf{r}(t) \neq \mathbf{0}$
- **Boundary conditions:**  $\mathbf{q}^k = \hat{\mathbf{q}}^k$  for all  $k$  such that  $\mathbf{x}_k \in \Gamma_u$ .
- **Initial conditions:**  $\mathbf{q}(0) = \mathbf{u}_0$  and  $\dot{\mathbf{q}}(0) = \mathbf{v}_0$

Interest in finding the **temporal response**  $\mathbf{q}(t)$  of the structure.



$$\mathbf{q}(t) = \mathbf{P}\mathbf{z}(t) = \sum_{i=1}^n \mathbf{p}_i z_i(t)$$

- $\mathbf{z}$ : vector of modal coordinates
- $\mathbf{P}$ : modal matrix
- $\mathbf{p}_i$ : eigenvector of order  $i$
- $z_i$ : modal coordinate of order  $i$

Insertion of the change of basis in the forced regime equation:

$$\begin{aligned} \mathbf{M}\mathbf{P}\ddot{\mathbf{z}}(t) + \mathbf{K}\mathbf{P}\mathbf{z}(t) &= \mathbf{r}(t) \\ \mathbf{P}^T\mathbf{M}\mathbf{P}\ddot{\mathbf{z}}(t) + \mathbf{P}^T\mathbf{K}\mathbf{P}\mathbf{z}(t) &= \mathbf{P}^T\mathbf{r}(t) \end{aligned}$$

## Accounting for the orthogonality of eigenmodes

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \mathbf{I} \quad \text{and} \quad \mathbf{P}^T \mathbf{K} \mathbf{P} = \mathbf{\Lambda}$$

- $\mathbf{I}$ : identity matrix
- $\mathbf{\Lambda} = \text{diag}(\omega_1^2, \dots, \omega_i^2, \dots, \omega_n^2)$ : diagonal matrix of eigenvalues,

## Decoupling of the forced regime system:

$$\begin{aligned} \mathbf{P}^T \mathbf{M} \mathbf{P} \ddot{\mathbf{z}}(t) + \mathbf{P}^T \mathbf{K} \mathbf{P} \mathbf{z}(t) &= \mathbf{P}^T \mathbf{r}(t) \\ \ddot{\mathbf{z}}(t) + \mathbf{\Lambda} \mathbf{z}(t) &= \mathbf{s}(t) \end{aligned}$$

- Projected initial conditions:  $\mathbf{z}(0) = \mathbf{P}^T \mathbf{M} \mathbf{u}_0$  and  $\dot{\mathbf{z}}(0) = \mathbf{P}^T \mathbf{M} \mathbf{v}_0$ .
- $\mathbf{s}(t) = \mathbf{P}^T \mathbf{r}(t)$ : projection of  $\mathbf{r}(t)$  onto the modal basis  $\mathbf{P}$ .

## Exact solution

**Component-wise form of the decoupled forced regime:** ( $i = 1, 2, \dots, n$ )

$$\begin{aligned}\ddot{z}_i(t) + \omega_i^2 z_i(t) &= s_i(t) \\ z_i(0) &= \mathbf{p}_i^T \mathbf{M} \mathbf{u}_0 \\ \dot{z}_i(0) &= \mathbf{p}_i^T \mathbf{M} \mathbf{v}_0\end{aligned}$$

Exact solution by Laplace transform (convolution or Duhamel's integral):

$$z_i(t) = z_i(0) \cos(\omega_i t) + \frac{\dot{z}_i(0)}{\omega_i} \sin(\omega_i t) + \frac{1}{\omega_i} \int_0^t \sin(\omega_i(t - \tau)) s_i(\tau) d\tau$$

$$\begin{aligned}\mathbf{q}(t) &= \sum_{i=1}^n \mathbf{p}_i z_i(t) = \sum_{i=1}^n \mathbf{p}_i \left( \mathbf{p}_i^T \mathbf{M} \mathbf{u}_0 \cos(\omega_i t) + \frac{1}{\omega_i} \mathbf{p}_i^T \mathbf{M} \mathbf{v}_0 \sin(\omega_i t) \right) \\ &+ \sum_{i=1}^n \mathbf{p}_i \left( \frac{1}{\omega_i} \int_0^t s_i(t - \tau) \sin(\omega_i \tau) d\tau \right)\end{aligned}$$

## Direct integration methods

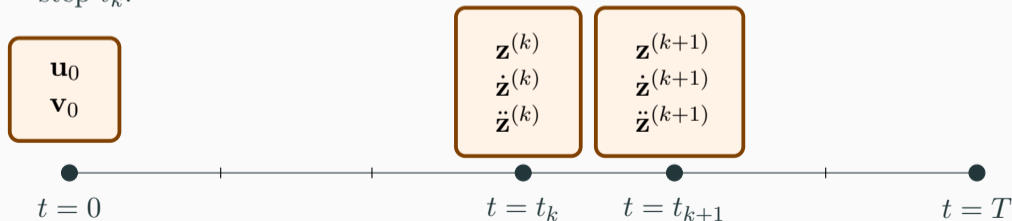
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## Finite difference method for time integration

- Solve dynamic equilibrium equations by replacing time derivatives with discrete approximations.
- Time domain  $[0, T]$  is discretized into  $N_t$  equal intervals  $\Delta t = T/N_t$ :

$$t_k = k\Delta t \quad k = 0, \dots, N_t.$$

- Displacement, velocity, and acceleration are approximated within each time step  $t_k$ .



General kinematic scheme (for second-order differential equation):

$$\begin{bmatrix} z_i^{(k)} \\ \dot{z}_i^{(k)} \end{bmatrix} = \sum_{j=1}^q \alpha_j \begin{bmatrix} z_i^{(k-j)} \\ \dot{z}_i^{(k-j)} \end{bmatrix} + \Delta t \sum_{j=0}^q \beta_j \begin{bmatrix} \dot{z}_i^{(k-j)} \\ \ddot{z}_i^{(k-j)} \end{bmatrix}, \quad k = 1, 2, \dots; \quad k \geq q$$

- $z_i^{(k)}, \dot{z}_i^{(k)}, \ddot{z}_i^{(k)}$ : component  $i$  of displacement, velocity, and acceleration at time step  $k$ ,
- $q, \alpha_j, \beta_j$  for  $j = 1, \dots, q$ : given constants,
- $\Delta t = t_k - t_{k-1}$ : time step,
- $k$ : time step index.

# Types of time integration schemes

## Single-step methods ( $q = 1$ ):

- Use only the state at the previous time step to compute the current time step.
- Widely used due to balance between accuracy and efficiency.
- *Example:* Newmark methods are commonly used in finite element software.

## Multi-step methods ( $q > 1$ ):

- Use multiple previous time steps to compute the current time step.
- *Examples:* Houbolt's method, Wilson- $\theta$  method, Park's algorithm,

## Explicit vs implicit ( $\beta_0$ ):

- Explicit:  $\beta_0 = 0$ , current values are computed directly from known past values.
- Implicit:  $\beta_0 \neq 0$ , require solving equations at each time step.

# Classification of finite difference methods

General kinematic scheme:

$$\begin{bmatrix} z_i^{(k)} \\ \dot{z}_i^{(k)} \end{bmatrix} = \sum_{j=1}^q \alpha_j \begin{bmatrix} z_i^{(k-j)} \\ \dot{z}_i^{(k-j)} \end{bmatrix} + \Delta t \sum_{j=0}^q \beta_j \begin{bmatrix} \dot{z}_i^{(k-j)} \\ \ddot{z}_i^{(k-j)} \end{bmatrix}, \quad k = 1, 2, \dots; \quad k \geq q$$

■  $q = 1$  : one-step scheme  $\rightarrow$  Newmark method(s)

- $\beta_0 = 0$  : *explicit scheme*
- $\beta_0 \neq 0$  : *implicit scheme*

$$\begin{array}{l} z_i^{(k)} \leftarrow z_i^{(k-1)}, \dot{z}_i^{(k-1)} \\ z_i^{(k)} \leftarrow z_i^{(k-1)}, \dot{z}_i^{(k-1)}, \ddot{z}_i^{(k)} \end{array}$$

■  $q > 1$  : multi-step scheme  $\rightarrow$  Park, Houbolt, Wilson, ...

- $\beta_0 = 0$  : *explicit scheme*
- $\beta_0 \neq 0$  : *implicit scheme*

## Newmark's one-step method ( $t = 0$ )

Resolution of dynamic equations using a one-step Newmark's methods:

- Initial conditions:

$$z_i^{(0)} = \mathbf{p}_i^T \mathbf{M} \mathbf{u}_0$$

$$\dot{z}_i^{(0)} = \mathbf{p}_i^T \mathbf{M} \mathbf{v}_0$$

- Dynamic equation:

$$\ddot{z}_i^{(k)} + \omega_i^2 z_i^{(k)} = s_i^{(k)}$$

Computation of initial acceleration:

$$\ddot{z}_i^{(0)} = -\omega_i^2 z_i^{(0)} + s_i^{(0)}$$

## Newmark's one-step method ( $t = t_{k-1}$ to $t = t_k$ )

Kinematic scheme for one-step Newmark's methods:

$$\begin{bmatrix} z_i^{(k)} \\ \dot{z}_i^{(k)} \end{bmatrix} = \alpha_1 \begin{bmatrix} z_i^{(k-1)} \\ \dot{z}_i^{(k-1)} \end{bmatrix} + \Delta t \left( \beta_0 \begin{bmatrix} \dot{z}_i^{(k)} \\ \ddot{z}_i^{(k)} \end{bmatrix} + \beta_1 \begin{bmatrix} \dot{z}_i^{(k-1)} \\ \ddot{z}_i^{(k-1)} \end{bmatrix} \right) \quad (k = 1, 2, \dots)$$

- $\alpha_1$ ,  $\beta_0$ , and  $\beta_1$ : parameters characterizing the different variants of the schemes.
- To compute  $\alpha_1$ ,  $\beta_0$ , and  $\beta_1$  we use Taylor expansion with integral remainder:

$$\underbrace{z_i(t_k)}_{z_i^{(k)}} = \underbrace{z_i(t_{k-1})}_{z_i^{(k-1)}} + \Delta t \underbrace{\dot{z}_i(t_{k-1})}_{\dot{z}_i^{(k-1)}} + \int_{t_{k-1}}^{t_k} (t_k - \tau) \ddot{z}_i(\tau) d\tau$$
$$\underbrace{\dot{z}_i(t_k)}_{\dot{z}_i^{(k)}} = \underbrace{\dot{z}_i(t_{k-1})}_{\dot{z}_i^{(k-1)}} + \int_{t_{k-1}}^{t_k} \ddot{z}_i(\tau) d\tau$$

# First-order integral remainder approximation

## Taylor series for acceleration:

let  $\beta \in ]0, 1/2[$  a constant, then

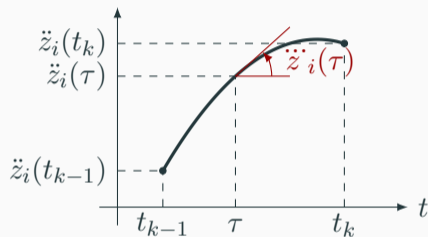
$$\times (1 - 2\beta) \quad \ddot{z}_i(t_{k-1}) = \ddot{z}_i(\tau) - (\tau - t_{k-1}) \dddot{z}_i(\tau) + \dots$$

$$\times 2\beta \quad \ddot{z}_i(t_k) = \ddot{z}_i(\tau) + (t_k - \tau) \dddot{z}_i(\tau) + \dots$$

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$$\Rightarrow (1 - 2\beta)\ddot{z}_i(t_{k-1}) + 2\beta\ddot{z}_i(t_k) \approx \ddot{z}_i(\tau)$$

Notice that terms in  $\dddot{z}_i(\tau)$  and higher are neglected because they are multiplied in displacement by  $\Delta t^3$ , and in velocity by  $\Delta t^2$ .



# Taylor formula with integral remainder for modal displacements

Integral quadrature in Taylor expansion:

$$\begin{aligned}\int_{t_{k-1}}^{t_k} (t_k - \tau) \ddot{z}_i(\tau) d\tau &= \int_{t_{k-1}}^{t_k} (t_k - \tau) \left[ (1 - 2\beta) \ddot{z}_i^{(k-1)} + 2\beta \ddot{z}_i^{(k)} \right] d\tau \\ &= \dots \\ &= \Delta t^2 \left[ \left( \frac{1}{2} - \beta \right) \ddot{z}_i^{(k-1)} + \beta \ddot{z}_i^{(k)} \right]\end{aligned}$$

**Kinematic scheme for displacements:**

$$z_i^{(k)} = z_i^{(k-1)} + \Delta t \dot{z}_i^{(k-1)} + \Delta t^2 \left[ \left( \frac{1}{2} - \beta \right) \ddot{z}_i^{(k-1)} \right] + \beta \Delta t^2 \ddot{z}_i^{(k)}$$

## First-order integral remainder approximation

**Taylor series for acceleration:** let  $\gamma \in ]0, 1[$  a second constant, then

$$\Rightarrow (1 - \gamma)\ddot{z}_i(t_{k-1}) + \gamma\ddot{z}_i(t_k) \approx \ddot{z}_i(\tau)$$

**Integral quadrature in Taylor expansion:**

$$\int_{t_{k-1}}^{t_k} \ddot{z}_i(\tau) d\tau = \int_{t_{k-1}}^{t_k} \left[ (1 - \gamma)\ddot{z}_i^{(k-1)} + \gamma\ddot{z}_i^{(k)} \right] d\tau = \Delta t \left[ (1 - \gamma)\ddot{z}_i^{(k-1)} + \gamma\ddot{z}_i^{(k)} \right]$$

**Kinematic scheme for velocity:**

$$\dot{z}_i^{(k)} = \dot{z}_i^{(k-1)} + \Delta t \left[ (1 - \gamma)\ddot{z}_i^{(k-1)} \right] + \gamma\Delta t\ddot{z}_i^{(k)}$$

## Newmark's one-step method ( $t = t_{k-1}$ to $t = t_k$ )

Stability control parameter  $\beta$ :  $0 \leq \beta \leq 1/4$

Numerical dissipation parameter  $\gamma$ :  $1/2 \leq \gamma \leq 3/4$ .

### ■ Kinematic scheme for displacement:

$$z_i^{(k)} = z_i^{(k-1)} + \Delta t \dot{z}_i^{(k-1)} + \Delta t^2 \left[ \left( \frac{1}{2} - \beta \right) \ddot{z}_i^{(k-1)} \right] + \beta \Delta t^2 \ddot{z}_i^{(k)}$$

### ■ Kinematic scheme for velocity:

$$\dot{z}_i^{(k)} = \dot{z}_i^{(k-1)} + \Delta t \left[ (1 - \gamma) \ddot{z}_i^{(k-1)} \right] + \gamma \Delta t \ddot{z}_i^{(k)}$$

### ■ Dynamic equation:

$$\ddot{z}_i^{(k)} + \omega_i^2 z_i^{(k)} = s_i^{(k)}$$

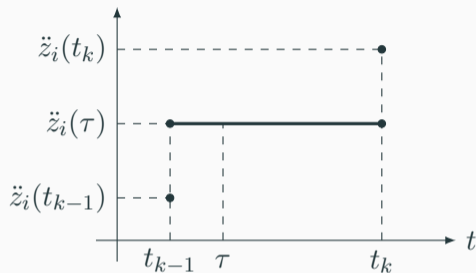
## Average acceleration method (Trapezoidal rule)

Special case:

$$\beta = \frac{1}{4} \quad \text{and} \quad \gamma = \frac{1}{2}$$

Then for  $t_{k-1} \leq \tau \leq t_k$  we have

$$\ddot{z}_i(\tau) = \frac{\ddot{z}_i(t_{k-1}) + \ddot{z}_i(t_k)}{2}$$



**Kinematic schemes for displacement and velocity:**

$$z_i^{(k)} = z_i^{(k-1)} + \Delta t \dot{z}_i^{(k-1)} + \frac{1}{4} \Delta t^2 \left( \ddot{z}_i^{(k-1)} + \ddot{z}_i^{(k)} \right)$$

$$\dot{z}_i^{(k)} = \dot{z}_i^{(k-1)} + \frac{1}{2} \Delta t \left( \ddot{z}_i^{(k-1)} + \ddot{z}_i^{(k)} \right)$$



*“The person who learns the most in any classroom is the teacher.”*

*— James Clear*

**Please complete the in-depth evaluation survey!**