Eigenvalue problems of vibrations and stability

Analysis of free and forced vibrations

ME473 Dynamic finite element analysis of structures

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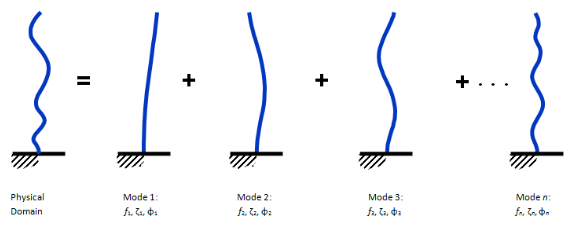
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	
7		Planar beams	Project 1 submission
8		Frames and grids	Project 2 statement
9		Kirchhoff-Love plates	
10		Reissner-Mindlin plates	
11		Shells	Project 2 submission
12	Free and forced vibrations	Analysis of free vibrations	Project 3 statement

Summary

- Recap week 11
- Modal properties of conservative systems
- Example: Rayleigh's quotient for axial vibrations of a free-free bar
- Numerical modal extraction algorithms for conservative systems

Recommended readings

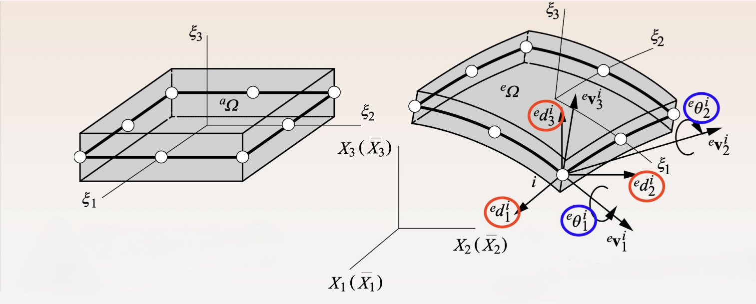
(N) Neto et al., Engineering Computation of Structures (chap. 2.5)

(P) Petyt, Introduction to finite element vibration analysis (chap. 11)

(G) Gmür, Dynamique des structures (§4.1 and §4.2)

Recap week 11
Vibrations of shells

Shell element



- have 5 DOFs per node, no rotation $^e\theta_3^i$.
- lead to huge computational time savings since allow modeling with fewer mesh elements.
- less prone to negative Jacobian errors which might occur when using extremely thin 3d solid elements.

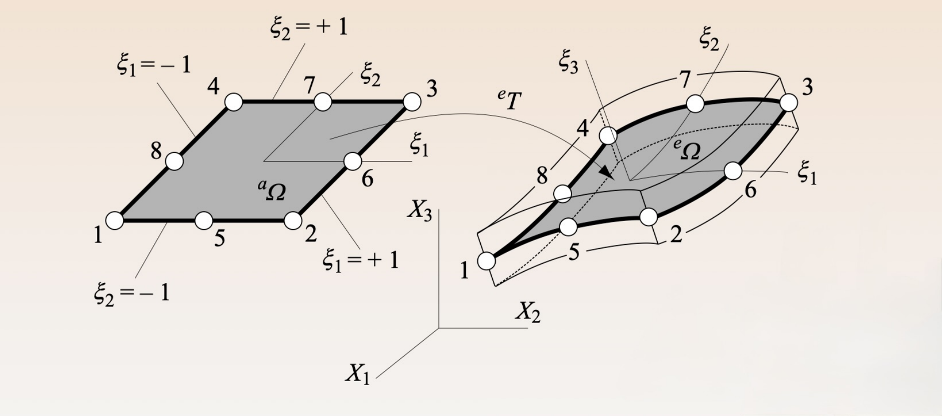
Shape functions matrix

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left[{}^e\mathbf{d}^i + \frac{1}{2}\xi_3 {}^e t_3^i \left(-{}^e\theta_1^i \, {}^e\mathbf{v}_2^i + {}^e\theta_2^i \, {}^e\mathbf{v}_1^i \right) \right]$$

$$= \sum_{i=1}^{{}^e p} {}^a\mathbf{H}_i(\boldsymbol{\xi}) {}^e\mathbf{q}^i(t)$$

$$= \sum_{i=1}^{{}^e p} \underbrace{\left[\begin{array}{ccc|cc} {}^a h_i & 0 & 0 & -\frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{21}^i & \frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{11}^i \\ 0 & {}^a h_i & 0 & -\frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, v_{22}^i & \frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{12}^i \\ 0 & 0 & {}^a h_i & -\frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{23}^i & \frac{1}{2}\xi_3 {}^e t_3^i \, {}^a h_i \, {}^e v_{13}^i \end{array} \right]}_{{}^a\mathbf{H}_i} \underbrace{\begin{bmatrix} {}^e d_1^i \\ {}^e d_2^i \\ {}^e d_3^i \\ {}^e\theta_1^i \\ {}^e\theta_2^i \end{bmatrix}}_{{}^e\mathbf{q}^i}$$

Example: 8 nodes quadrangular shell element

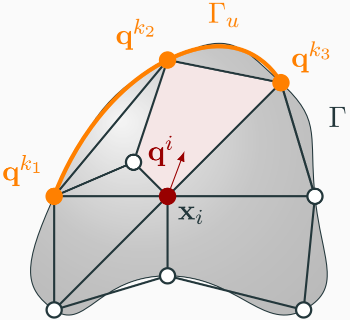


Analysis of free vibrations

Free vibrations of non-rotating conservative systems

The discretization of linear three-dimensional elastodynamics, as well as the analysis of vibrations in beams, plates and shells, all lead to a system of ODE:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t),$$



Free vibration: no external forcing is applied, i.e. $\mathbf{r}(t) = \mathbf{0}$.

- Generalized nodal displacements:

$$\mathbf{q}(t) = [\mathbf{q}^1(t), \ldots, \mathbf{q}^n(t)]^T.$$

- **Boundary conditions:** $\mathbf{q}^k = \hat{\mathbf{q}}^k$ for all k such that $\mathbf{x}_k \in \Gamma_u$.
- **Initial conditions:** $\mathbf{q}(0) = \mathbf{u}_0$ and $\dot{\mathbf{q}}(0) = \mathbf{v}_0$

Harmonic response

The solution for a free and undamped discrete vibration problem

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}$$

is sinusoidal, described via an harmonic function:

$$\mathbf{q}(t) = \alpha \mathbf{p} \cos(\omega t + \varphi)$$

- $\mathbf{p}$: mode shape (modal vector)
- ω: natural frequency
- φ: phase
- α: scaling factor

- **Solutions are defined up to a scalar factor: α.**
- Three scalars unknowns α, ω, φ and one unknown vector $\mathbf{p}$.

Generalized eigenvalue problem

Substituting the proposed ansatz into the equation of motion yields a generalized eigenvalue problem:

$$\alpha(\mathbf{K} - \omega^2\mathbf{M})\mathbf{p}\cos(\omega t - \varphi) = 0$$
$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{p} = 0$$

Solving the eigenvalue problem:

- *Eigenvalue*: $\lambda_j = \omega_j^2$ are the roots of the characteristic polynomial:

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0.$$

- *Eigenvector*: $\mathbf{p}_j$ are the solution of the equation

$$(\mathbf{K} - \lambda_j\mathbf{M})\mathbf{p}_j = \mathbf{0}.$$

General solution via modal superposition

- Once the set of eigenvalue-eigenvector pairs $(\omega_j^2, \mathbf{p}_j)$ have been determined, the linearity of the system allows the general solution to be expressed as a superposition of modal contributions:

$$\mathbf{q}(t) = \sum_{j=1}^{n} \alpha_j \, \mathbf{p}_j \cos(\omega_j t + \varphi_j)$$

- The constants α_j and φ_j are determined from the initial conditions: $\mathbf{q}(0) = \mathbf{u}_0$ and $\dot{\mathbf{q}}(0) = \mathbf{v}_0$ leading to the system:

$$\sum_{j=1}^{n} \alpha_j \mathbf{p}_j \cos(\varphi_j) = \mathbf{u}_0$$

$$\sum_{j=1}^{n} \alpha_j \omega_j \mathbf{p}_j \sin(\varphi_j) = -\mathbf{v}_0$$

Fundamental properties of generalized eigenvalue problem

Spectral property

- If both **K** and **M** are symmetric and strictly positive definite, then all eigenvalues λ_j of the generalized eigenvalue problem

$$\mathbf{K}\mathbf{p}_j = \lambda_j \mathbf{M}\mathbf{p}_j$$

  are real and positive.
- Thus we can define the angular frequencies

$$\omega_j = \sqrt{\lambda_j}$$

  and

$$0 < \omega_1 \leq \omega_2 \leq \omega_3 \leq \cdots \leq \omega_n.$$

Rigid body modes

In the semi-discrete weak form obtained via finite element discretization:

- The **mass matrix M** is symmetric and strictly positive definite.
- The **stiffness matrix K** is symmetric and positive semi-definite:

$$\mathbf{K}\mathbf{p} = \mathbf{0} \quad \text{for certain nonzero vectors } \mathbf{p}.$$

Consequently, the eigenvalues ω_j^2 of the generalized eigenvalue problem are all real and non-negative:

$$0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n.$$

Rigid body modes: zero eigenvalues (i.e., $\omega_j = 0$) correspond to *rigid body motions*, where the system undergoes displacement without internal deformation.

Spectral offset technique

To ensure that the stiffness matrix is *strictly* positive definite, a **spectral shift** strategy is employed. Given $\sigma > 0$, define the modified stiffness matrix as:

$$\mathbf{K}_\sigma = \mathbf{K} + \sigma \mathbf{M}$$

- Consider the offset eigenvalue problem: $(\mathbf{K}_\sigma - \lambda_\sigma \mathbf{M})\mathbf{p} = \mathbf{0}$
- Substituting the definition of $\mathbf{K}_\sigma$:

$$(\mathbf{K} + \sigma \mathbf{M} - \lambda_\sigma \mathbf{M})\mathbf{p} = (\mathbf{K} + (\sigma - \lambda_\sigma)\mathbf{M})\mathbf{p} = \mathbf{0}$$

- Comparing with $(\mathbf{K} - \lambda \mathbf{M})\mathbf{p} = \mathbf{0}$, we identify: $\lambda_\sigma = \lambda + \sigma > 0$.

Eigenvalues are shifted by σ, but the eigenvectors remain the same.

Empirical rule: $\sigma \approx \dfrac{1}{100} \dfrac{\mathrm{tr}(\mathbf{K})}{\mathrm{tr}(\mathbf{M})}$

Orthogonality of mode shapes

Let $\mathbf{p}_i$ and $\mathbf{p}_j$ two eigenvectors corresponding to two *distinct* eigenvalues λ_i and λ_j, then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = 0 \qquad \text{and} \qquad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = 0 \qquad (i \neq j).$$

Consequence: two different harmonic responses:

$$\mathbf{q}_i(t) = \alpha_i \mathbf{p}_i \cos(\omega_i t + \varphi_i) \qquad \text{and} \qquad \mathbf{q}_j(t) = \alpha_j \mathbf{p}_j \cos(\omega_j t + \varphi_j)$$

are **M**- and **K**-orthogonal:

$$\mathbf{q}_i^T \mathbf{M} \mathbf{q}_j = 0 \qquad \text{and} \qquad \mathbf{q}_i^T \mathbf{K} \mathbf{q}_j = 0.$$

The virtual work of inertial and elastic forces of a given mode, along the displacement given by a different mode, is zero.

Normalization of mode shapes

Let $\mathbf{p}_i$ and $\mathbf{p}_j$ two eigenvectors corresponding to the eigenvalues λ_i and λ_j, then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = \mu_i \delta_{ij} \qquad \text{and} \qquad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = \kappa_i \delta_{ij}$$

where

- μ_i = modal mass of mode i,
- κ_i = modal stiffness of mode i,
- δ_{ij} = Kronecker delta.

Usual normalization choice: orthonormalization

$$\mu_i = 1 \qquad \Rightarrow \qquad \lambda_i = \frac{\kappa_i}{\mu_i} = \omega_i^2.$$

Orthonormalization of mode shapes

Let $\mathbf{p}_i$ and $\mathbf{p}_j$ two eigenvectors corresponding to the eigenvalues λ_i and λ_j, then

$$\mathbf{p}_i^T \mathbf{M} \mathbf{p}_j = \delta_{ij} \qquad \text{and} \qquad \mathbf{p}_i^T \mathbf{K} \mathbf{p}_j = \omega_i^2 \delta_{ij}$$

where δ_{ij} represent Kronecker symbol.

We store the modal vectors $\mathbf{p}_i$ in a so-called *modal matrix* $\mathbf{P}$:

$$\mathbf{P} = \left[\begin{array}{c:c:c:c} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{array} \right]$$

then

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \mathbf{I} \qquad \text{and} \qquad \mathbf{P}^T \mathbf{K} \mathbf{P} = \mathbf{\Lambda}$$

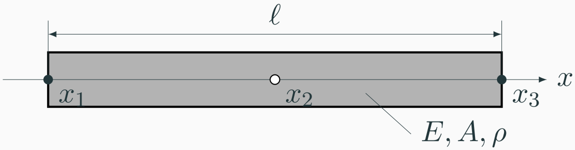
where $\mathbf{I}$ is the identity matrix of order n and $\mathbf{\Lambda}$ the spectral matrix:

$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n) = \text{diag}(\omega_1^2, \dots, \omega_n^2).$$

Example: longitudinal vibrations of free-free bar

Longitudinal vibrations of free-free bar

Consider a free-free bar discretized by one bilinear element.



- ℓ length
- A cross-sectional area
- E Young's modulus
- ρ material density

Objectives:

1. compute the approximate natural frequencies and corresponding mode shapes,
2. verify the presence of a rigid body mode,
3. check the orthonormality of the mode shapes.

MATLAB example - longitudinal vibrations of free-free bar

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Eigenproblem solution methods

Transformation to standard form by inversion

Generalized eigenvalue problem

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{p} = \mathbf{0}$$

$\det \mathbf{M} \neq 0$ →

Standard eigenvalue problem

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{p} = \mathbf{0}$$

$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ dynamic stiffness matrix

$\det \mathbf{K} \neq 0$ →

Standard eigenvalue problem

$$(\mathbf{A} - \theta \mathbf{I})\mathbf{p} = \mathbf{0}$$

$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M}$ dynamic flexibility matrix

$$\theta = 1/\lambda$$

The matrices $\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{K}^{-1}\mathbf{M}$ are not symmetric.

Transformation to standard form by decomposition

To reduce to a standard eigenvalue problem:

- Let $\mathbf{L}$ be the *Cholesky factor* of $\mathbf{M}$: $\mathbf{M} = \mathbf{L}\mathbf{L}^T$.

Since $\mathbf{M}$ is symmetric positive-definite, $\mathbf{L}$ is a real lower triangular matrix with positive diagonal entries.

- Define the change of variables $\mathbf{v} = \mathbf{L}^T\mathbf{p}$. Substituting into the original problem and multiplying by $\mathbf{L}^{-1}$ yields:

$$(\mathbf{K} - \omega^2\mathbf{L}\mathbf{L}^T)\mathbf{p} = 0 \quad \Rightarrow \quad (\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T} - \omega^2\mathbf{I})\mathbf{v} = 0.$$

- **Result:** standard eigenvalue problem for symmetric and positive definite matrix $\mathbf{A} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

Transformation to standard form by decomposition

- Cholesky decomposition can be applied to $\mathbf{K}$ as well, since $\mathbf{K}$ is symmetric positive-definite:
$$\mathbf{K} = \mathbf{L}\mathbf{L}^T.$$
- **Result:** standard eigenvalue problem for symmetric and positive definite matrix $\mathbf{A} = \mathbf{L}^{-1}\mathbf{M}\mathbf{L}^{-T}$, eigenvalues $\theta = 1/\lambda$ and eigenvectors $\mathbf{v} = \mathbf{L}^T\mathbf{p}$:
$$(\mathbf{A} - \theta\mathbf{I})\mathbf{v} = \mathbf{0}.$$

Small and large scale problems

Small-scale problems

- system matrices are of modest size $n \leq 250$ or $250 \leq n \leq 2500$ and small bandwidth matrices.
- An explicit reduction to standard eigenvalue form is typically employed.

Large-scale problems

- $n \geq 2500$.
- The most effective algorithms are subspace iteration, Lanczos' method, Guyan-Irons method, Arnoldi method and Davidson method.

Numerical modal extraction algorithms

1. Rayleigh's quotient

Definition and first properties

Let $\mathbf{w}$ ba a vector

$$\mathcal{R}(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{K} \mathbf{w}}{\mathbf{w}^T \mathbf{M} \mathbf{w}}$$

Properties

1. Homogeneity. Let α a non-zero constant, then

$$\mathcal{R}(\alpha \mathbf{w}) = \frac{(\alpha \mathbf{w})^T \mathbf{K} (\alpha \mathbf{w})}{(\alpha \mathbf{w})^T \mathbf{M} (\alpha \mathbf{w})} = \frac{\alpha^2 \, \mathbf{w}^T \mathbf{K} \mathbf{w}}{\alpha^2 \, \mathbf{w}^T \mathbf{M} \mathbf{w}} = \mathcal{R}(\mathbf{w})$$

2. Rayleigh quotient of an eigenvector:

$$\mathcal{R}(\mathbf{w} = \mathbf{p}_i) = \frac{\mathbf{p}_i^T \mathbf{K} \mathbf{p}_i}{\mathbf{p}_i^T \mathbf{M} \mathbf{p}_i} = \lambda_i = \omega_i^2$$

Stationarity of Rayleigh's quotient

First variation of Rayleigh's quotient:

$$\delta\mathcal{R}(\mathbf{w}) = \frac{2(\boldsymbol{\delta}\mathbf{w}^T\mathbf{K}\mathbf{w})(\mathbf{w}^T\mathbf{M}\mathbf{w}) - 2(\mathbf{w}^T\mathbf{K}\mathbf{w})(\boldsymbol{\delta}\mathbf{w}^T\mathbf{M}\mathbf{w})}{(\mathbf{w}^T\mathbf{M}\mathbf{w})^2}$$

$$= \frac{2\,\boldsymbol{\delta}\mathbf{w}^T}{\mathbf{w}^T\mathbf{M}\mathbf{w}}\left(\mathbf{K}\mathbf{w} - \frac{\mathbf{w}^T\mathbf{K}\mathbf{w}}{\mathbf{w}^T\mathbf{M}\mathbf{w}}\mathbf{M}\mathbf{w}\right)$$

$$= \frac{2\,\boldsymbol{\delta}\mathbf{w}^T}{\mathbf{w}^T\mathbf{M}\mathbf{w}}(\mathbf{K} - \mathcal{R}(\mathbf{w})\mathbf{M})\mathbf{w}$$

If $\mathbf{w} = \mathbf{p}_i$ then

$$(\mathbf{K} - \mathcal{R}(\mathbf{w})\mathbf{M})\mathbf{w} = (\mathbf{K} - \lambda_i\mathbf{M})\mathbf{p}_i = 0$$

Stationary Rayleigh's quotient in the vicinity of an modal shape !

Rayleigh principle

Modal expansion:

$$\mathbf{w} = z_1\mathbf{p}_1 + z_2\mathbf{p}_2 + \cdots + z_i\mathbf{p}_i + \cdots + z_n\mathbf{p}_n = \mathbf{P}\mathbf{z}$$

where

$$\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_i, \ldots, \mathbf{p}_n] \quad \text{modal matrix}$$

$$\mathbf{z} = [z_1, z_2, \ldots, z_i, \ldots, z_n]^T \quad \text{coefficient vector}$$

Then

$$\mathcal{R}(\mathbf{w} = \mathbf{P}\mathbf{z}) = \frac{\mathbf{z}^T\mathbf{P}^T\mathbf{K}\mathbf{P}\mathbf{z}}{\mathbf{z}^T\mathbf{P}^T\mathbf{M}\mathbf{P}\mathbf{z}} = \frac{\mathbf{z}^T\boldsymbol{\Lambda}\mathbf{z}}{\mathbf{z}^T\mathbf{z}} = \frac{\sum_{i=1}^n \lambda_i z_i^2}{\sum_{i=1}^n z_i^2}$$

If, without loss of generality, we impose $\mathbf{w}^T\mathbf{M}\mathbf{w} = 1$, then $\mathbf{z}^T\mathbf{z} = \sum_{i=1}^n z_i^2 = 1$ and

$$\mathcal{R}(\mathbf{w} = \mathbf{P}\mathbf{z}) = \sum_{i=1}^n \lambda_i z_i^2$$

Rayleigh principle (continued)

$$\mathcal{R}(\mathbf{w}) = \sum_{i=1}^{n} \lambda_i z_i^2 = \lambda_j - \lambda_j + \sum_{i=1}^{n} \lambda_i z_i^2 = \lambda_j - \lambda_j \left(\sum_{i=1}^{n} z_i^2 \right) + \sum_{i=1}^{n} \lambda_i z_i^2$$
$$= \lambda_j + \sum_{i=1}^{n} (\lambda_i - \lambda_j) z_i^2$$

With eigenvalues ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \leq \lambda_n$, we have:

- If $j = 1$, then $\lambda_i - \lambda_1 \geq 0$ for all i, hence

$$\mathcal{R}(\mathbf{w}) = \lambda_1 + \sum_{i=1}^{n} (\lambda_i - \lambda_1) z_i^2 \geq \lambda_1.$$

- If $j = n$, then $\lambda_i - \lambda_n \leq 0$ for all i, hence

$$\mathcal{R}(\mathbf{w}) = \lambda_n + \sum_{i=1}^{n} (\lambda_i - \lambda_n) z_i^2 \leq \lambda_n.$$

Rayleigh quotient bounding theorem

- Courant-Fischer formulas:

$$\mathcal{R}(\mathbf{w}) \geq \lambda_1 = \min_{\mathbf{w}} \mathcal{R}(\mathbf{w}) \qquad \mathcal{R}(\mathbf{w}) \leq \lambda_n = \max_{\mathbf{w}} \mathcal{R}(\mathbf{w})$$

- Bounding theorem:

$$\lambda_1 \leq \mathcal{R}(\mathbf{w}) \leq \lambda_n$$

Modal frequencies and shapes search via Rayleigh quotient minimization

(1) Find λ_1 by minimization:

$$\lambda_1 = \min_{\mathbf{w}} \mathcal{R}(\mathbf{w})$$

(2) Find $\mathbf{p}_1$ by solving $(\mathbf{K} - \lambda_1 \mathbf{M})\mathbf{p}_1 = 0$.

(3) Find λ_2 by minimization:

$$\lambda_2 = \min_{\mathbf{w}} [\mathcal{R}(\mathbf{w});\ \mathbf{w}^T \mathbf{M} \mathbf{p}_1 = 0]$$

(4) Find $\mathbf{p}_2$ by solving $(\mathbf{K} - \lambda_2 \mathbf{M})\mathbf{p}_2 = 0$.

$$\vdots$$

(j) Find λ_j by minimization:

$$\lambda_j = \min_{\mathbf{w}} [\mathcal{R}(\mathbf{w});\ \mathbf{w}^T \mathbf{M} \mathbf{p}_i = 0,\ i = 1, 2, \ldots, j-1]$$

(j+1) Find $\mathbf{p}_j$ by solving $(\mathbf{K} - \lambda_j \mathbf{M})\mathbf{p}_j = 0$.

Example: Rayleigh's quotient for longitudinal vibrations of a free-free bar

MATLAB example - longitudinal vibrations of free-free bar

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Numerical modal extraction algorithms

2. Subspace iteration

Inverse iteration algorithm

Goal: compute the eigenvector associated with the smallest eigenvalue of the generalized eigenproblem $\mathbf{K}\mathbf{p} = \lambda \mathbf{M}\mathbf{p}$

Inputs:

- $\mathbf{K}$, $\mathbf{M}$: stiffness and mass matrices
- $p^{(0)}$: initial guess vector (non-zero)
- σ: spectral shift (optional)
- ε: convergence tolerance

Algorithm:

1. If $\mathbf{K}$ is singular, use shift: set $\mathbf{K}_\sigma = \mathbf{K} + \sigma\mathbf{M}$
2. For $k = 1, 2, \ldots$ until convergence:
   - Solve $\mathbf{K}_\sigma \, p^{(k)} = \mathbf{M} \, p^{(k-1)}$
   - Normalize: $p^{(k)} \leftarrow p^{(k)} / \|p^{(k)}\|$
   - Check convergence: $\|p^{(k)} - p^{(k-1)}\| < \varepsilon$

Output: approximated eigenvector $p^{(k)}$

Subspace iteration method

Goal: compute the first $m \ll n$ eigenpairs $(\mathbf{p}_i, \lambda_i)$ of the generalized eigenproblem.

Inputs:

- $\mathbf{K}$, $\mathbf{M}$: stiffness and mass matrices
- $\mathbf{P}^{(0)} \in \mathbb{R}^{n \times q}$: initial guess (matrix with $q > m$ linearly independent vectors)
- σ: spectral shift (optional)
- ε: convergence tolerance

Output:

- Approximated eigenvectors: $\mathbf{P}^{(k)} = [\mathbf{p}_1^{(k)}, \ldots, \mathbf{p}_q^{(k)}]$
- Approximated eigenvalues: $\mathbf{\Lambda}^{(k)} = \mathrm{diag}(\lambda_1^{(k)}, \ldots, \lambda_q^{(k)})$

Algorithm:

1. If $\mathbf{K}$ is singular, use shift: set $\mathbf{K}_\sigma = \mathbf{K} + \sigma \mathbf{M}$
2. For $k = 1, 2, \ldots$ until convergence:
   - Do steps 1, 2a, 2b and 2c
   - Check convergence

Subspace iteration steps

1. **Step 1:** Simultaneous inverse iteration on $q > m$ vectors: find the $(n \times q)$ matrix $\overline{\mathbf{P}^{(k)}}$ such that

$$\mathbf{K}\overline{\mathbf{P}^{(k)}} = \mathbf{M}\mathbf{P}^{(k-1)}$$

2. **Step 2a:** Compute projected stiffness and mass matrices:

$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{K}\overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{M}\overline{\mathbf{P}^{(k)}}$$

3. **Step 2b:** Solve $(q \times q)$ generalized eigenvalue problem: Find the modal matrix and the spectral matrix such that

$$\mathbf{K}^{(k)}\mathbf{Z}^{(k)} = \mathbf{M}^{(k)}\mathbf{Z}^{(k)}\mathbf{\Lambda}^{(k)}$$

4. **Step 2c:** Orthogonalization:

$$\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}}\mathbf{Z}^{(k)}$$

Subspace algorithm - step 1

Suppose that the Step 1 is replace by a simultaneous inverse iteration on m eigenvectors:

$$\mathbf{P}^{(k)} = (\mathbf{K}^{-1}\mathbf{M})\mathbf{P}^{(k-1)} = \cdots = (\mathbf{K}^{-1}\mathbf{M})^k\mathbf{P}_0$$

Define the subspace $\mathcal{S}^{(k)}$ of rank q, spanned by the vectors $\{\mathbf{p}_i^{(k)}\}$.
$\mathbf{P}^{(k)} = [\mathbf{p}_1^{(k)}, \ldots, \mathbf{p}_q^{(k)}]$ forms a *non-orthogonal* basis of $\mathcal{S}^{(k)}$.

✗ All columns of $\mathbf{P}^{(k)}$ tend toward $\mathbf{p}_1$

✗ Collinearity if no orthogonalization is applied !

- **Orthogonalization** of vectors $\mathbf{p}_i^{(k)}$ at each iteration
- Use, for instance, Gram-Schmidt method (*Note: this step is computationally expensive*)

Subspace algorithm - step 2a

- Orthogonalization by minimization of the Rayleigh quotient:

$$\mathcal{R}(\mathbf{w}^{(k)}) = \frac{(\mathbf{w}^{(k)})^T \mathbf{K} \mathbf{w}^{(k)}}{(\mathbf{w}^{(k)})^T \mathbf{M} \mathbf{w}^{(k)}}$$

- Let $\mathbf{w}^{(k)} = \overline{\mathbf{P}^{(k)}} \mathbf{z}^{(k)}$
- Projected Rayleigh's quotient:

$$\mathcal{R}(\mathbf{w}^{(k)}) = \frac{(\mathbf{z}^{(k)})^T \mathbf{K}^{(k)} \mathbf{z}^{(k)}}{(\mathbf{z}^{(k)})^T \mathbf{M}^{(k)} \mathbf{z}^{(k)}}$$

where

$$\mathbf{K}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{K} \overline{\mathbf{P}^{(k)}}, \quad \mathbf{M}^{(k)} = (\overline{\mathbf{P}^{(k)}})^T \mathbf{M} \overline{\mathbf{P}^{(k)}}$$

Subspace algorithm - step 2b

- Minimization of the Projected Rayleigh's quotient (generalized eigenvalue problem of dimension $q \times q$)
- Stationary condition:

$$\delta \mathcal{R}(\mathbf{w}^{(k)}) = 0 \qquad \Rightarrow \qquad \mathbf{K}^{(k)}\mathbf{z}^{(k)} = \lambda^{(k)}\mathbf{M}^{(k)}\mathbf{z}^{(k)}$$

- Solve via transformation method (e.g., Jacobi method):

$$\mathbf{K}^{(k)}\mathbf{Z}^{(k)} = \mathbf{M}^{(k)}\mathbf{Z}^{(k)}\mathbf{\Lambda}^{(k)}$$

- Ritz vectors and values:

$$\mathbf{Z}^{(k)} = [\mathbf{z}_1^{(k)}, \dots, \mathbf{z}_q^{(k)}], \quad \text{and} \quad \mathbf{\Lambda}^{(k)} = \text{diag}(\lambda_1^{(k)}, \dots, \lambda_q^{(k)})$$

Subspace algorithm - step 2c

- Update the modal matrix:

$$\mathbf{P}^{(k)} = \overline{\mathbf{P}^{(k)}}\mathbf{Z}^{(k)}$$

- Orthogonality check:

$$(\mathbf{P}^{(k)})^T\mathbf{M}\mathbf{P}^{(k)} = \mathbf{I}$$