Dynamic analysis of shells

Special structural elements

ME473 Dynamic finite element analysis of structures

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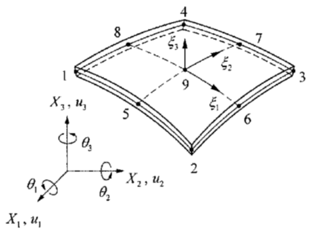
2025



Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	
7		Planar beams	Project 1 submission
8		Frames and grids	Project 2 statement
9		Kirchhoff-Love plates	
10		Reissner-Mindlin plates	
11		Shells	Project 2 submission

Summary

- Recap week 10
- Thin shell elements
- Example: modal analysis of simply supported thin shell

Recommended readings

(N) Neto et al., Engineering Computation of Structures (chap. 6.3)

(P) Petyt, Introduction to finite element vibration analysis (chap. 8)

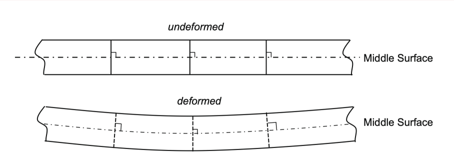
(O) Ochsner, PDE for classical structural members (chap. 7)

(G) Gmür, Méthode des éléments finis (chap. 3)

Recap week 10
Vibrations of Reissner-Mindlin plates

Plates theories

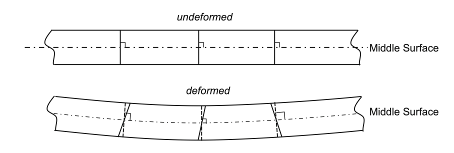
Thin plates



- Planarity and orthogonality of the cross-sectional planes.
- Generalized displacements:

$$\mathbf{u} = \begin{bmatrix} -z\partial_x u_3 & -z\partial_y u_3 & u_3 \end{bmatrix}^T$$

Thick plates



- Planarity of the cross-sectional planes.
- Generalized displacements:

$$\mathbf{u} = \begin{bmatrix} z\phi_2 & -z\phi_1 & u_3 \end{bmatrix}^T$$

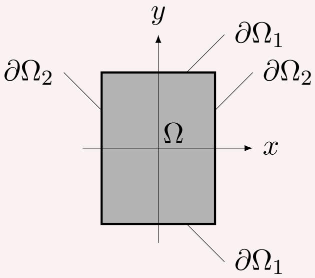
Strong form for Reissner-Mindlin plate bending

Let $\Omega = [-a, a] \times [-b, b]$ be a rectangular plate. Find the transverse displacement $u_3 \in C^2(\Omega \times [0, T])$ and the rotations $\phi_1,\ \phi_2 \in C^2(\Omega \times [0, T])$ such that

$$\nabla_m^T \overline{\boldsymbol{C}} \nabla_r \mathbf{u} + \mathbf{f} = \mathbf{I}\ddot{\mathbf{u}} \qquad \text{on } \Omega \times]0, T[$$

- boundary conditions (*simply supported*):

$$\begin{aligned} u_3 &= 0 && \text{in } \partial\Omega \times]0, T[\\ \phi_2 &= 0 && \text{in } \partial\Omega_1 \times]0, T[\\ \phi_1 &= 0 && \text{in } \partial\Omega_2 \times]0, T[\end{aligned}$$



- initial conditions:

$$\begin{aligned} \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 && \text{in } \Omega \\ \dot{\mathbf{u}}(\cdot, 0) &= \mathbf{v}_0 && \text{in } \Omega \end{aligned}$$

Definitions

- Reissner-Mindlin differential operators:

$$\nabla_r = \begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \\ \partial_x & 0 & 1 \\ \partial_y & -1 & 0 \end{bmatrix}, \qquad \text{and} \qquad \nabla_m = \begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \\ \partial_x & 0 & -1 \\ \partial_y & 1 & 0 \end{bmatrix}.$$

- Constitutive matrix for isotropic materials: $\overline{\boldsymbol{C}} = \begin{bmatrix} \overline{\boldsymbol{C}_b} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{C}_s} \end{bmatrix}$

$$\overline{\boldsymbol{C}_b} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \text{and} \qquad \overline{\boldsymbol{C}_s} = \frac{Ekh}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Mass moment of inertia matrix: $\mathbf{I} = \begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho h^3/12 & 0 \\ 0 & 0 & \rho h^3/12 \end{bmatrix}.$

Generalized displacements approximation

Since the rotations ϕ_1 and ϕ_2 are defined independently of the transversal displacement u_3, the discretization uses **bilinear Lagrangian finite elements**:

$$ {}^e\mathbf{u}^h(\mathbf{x},t) = {}^e\mathbf{H}(\mathbf{x}){}^e\mathbf{q}(t) = \sum_{i=1}^{4} {}^e h_i(\mathbf{x}){}^e\mathbf{q}^i(t) $$

- ${}^e\mathbf{H}(\mathbf{x})$ is a 3×12 matrix of **shape functions** ($\mathbf{I}$ is the 3×3 identity matrix.):

$$ {}^e\mathbf{H} = \left[\; h_1\mathbf{I} \;\vdots\; h_2\mathbf{I} \;\vdots\; h_3\mathbf{I} \;\vdots\; h_4\mathbf{I} \;\right] = \begin{bmatrix} {}^e h_1 & 0 & 0 & & {}^e h_4 & 0 & 0 \\ 0 & {}^e h_1 & 0 & \dots & 0 & {}^e h_4 & 0 \\ 0 & 0 & {}^e h_1 & & 0 & 0 & {}^e h_4 \end{bmatrix} $$

- ${}^e\mathbf{q}^i(t) = \begin{bmatrix} {}^e d^i(t) \\ {}^e\theta_1^i(t) \\ {}^e\theta_2^i(t) \end{bmatrix}$ and ${}^e\mathbf{u}^h = \begin{bmatrix} {}^e u_3^h \\ {}^e\phi_1^h \\ {}^e\phi_2^h \end{bmatrix}$ are the generalized displacements of node i and inside the element e respectively.

Elementary matrices and loads vector

- **Elementary stiffness matrix** (12×12):

$$ {}^e\mathbf{K} = \underbrace{\int_{{}^e\Omega} {}^e\mathbf{B}_b^T \overline{\boldsymbol{C}_b}\, {}^e\mathbf{B}_b \, d\Omega}_{{}^e\mathbf{K}_b} + \underbrace{\int_{{}^e\Omega} {}^e\mathbf{B}_s^T \overline{\boldsymbol{C}_s}\, {}^e\mathbf{B}_s \, d\Omega}_{{}^e\mathbf{K}_s} $$

  - Bending strain-displacement matrix: ${}^e\mathbf{B}_b = \nabla_b {}^e\mathbf{H}$.
  - Shear strain-displacement matrix: ${}^e\mathbf{B}_s = \nabla_s {}^e\mathbf{H}$.

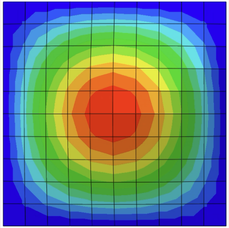
- **Elementary mass matrix** (12×12):

$$ {}^e\mathbf{M} = \int_{{}^e\Omega} {}^e\mathbf{H}^T \, \mathbf{I} \, {}^e\mathbf{H} \, d\Omega. $$

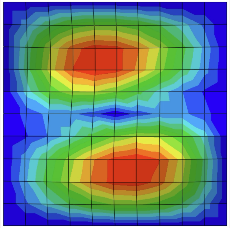
- **Elementary applied forces vector** (12×1):

$$ {}^e\mathbf{r}(t) = \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{f} \, d\Omega. $$

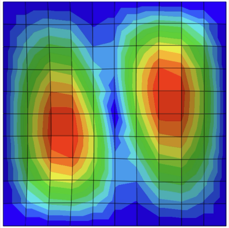
Modal shapes for SSSS thick plates (span/thickness = 0.1)



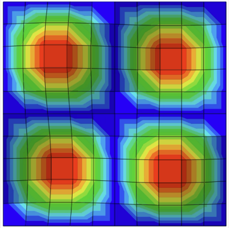
(a) Mode shape 1: $\omega_{11}^{h} = 9.672$ Hz



(b) Mode shape 2: $\omega_{12}^{h} = 23.643$ Hz

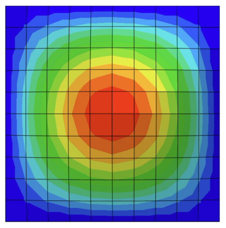


(c) Mode shape 3: $\omega_{21}^{h} = 23.643$ Hz

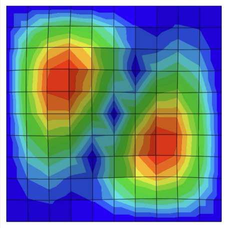


(d) Mode shape 4: $\omega_{22}^{h} = 36.114$ Hz

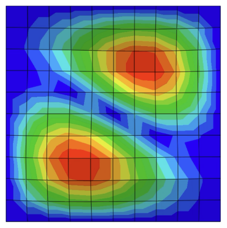
Modal shapes for SSSS thin plates (span/thickness = 0.01)



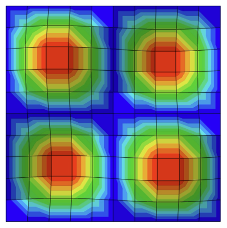
(a) Mode shape 1: $\omega_{11}^{h} = 1.001$ Hz



(b) Mode shape 2: $\omega_{12}^{h} = 2.573$ Hz



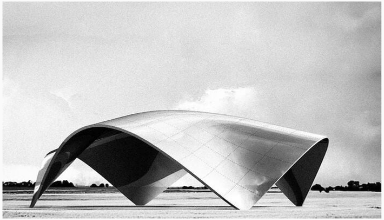
(c) Mode shape 3: $\omega_{21}^{h} = 2.573$ Hz

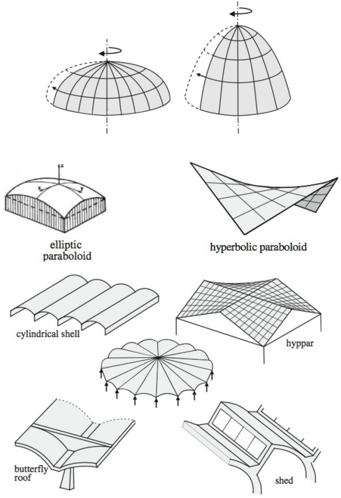


(d) Mode shape 4: $\omega_{22}^{h} = 4.103$ Hz

Thin shell elements

Examples of shell structures



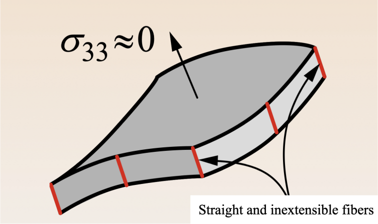


Geometry assumptions

A shell is a three-dimensional curved structural element whose thickness is very small compared to its other two dimensions.

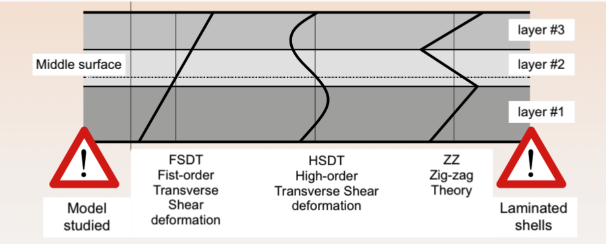
- **Rectilinearity of transverse fibers** : normals to the middle surface before deformation remain straight during deformation.
- **Inextensibility of transverse fibers**: stress in the normal direction is negligible:

$$\sigma_{33} \approx 0.$$



Others C^0 shell theories

- **First-Order Shear Deformation Theory (FSDT)**: Accounts for transverse shear deformation by introducing independent rotations, requiring a shear correction factor.
- **Higher-Order Shear Deformation Theory (HSDT)**: Includes higher-order terms in the displacement field to capture transverse shear deformation effects without requiring a shear correction factor.
- **Zig-Zag Theory**: Accounts for the piecewise linear variation of in-plane displacements through the thickness, improving accuracy for layered composites.

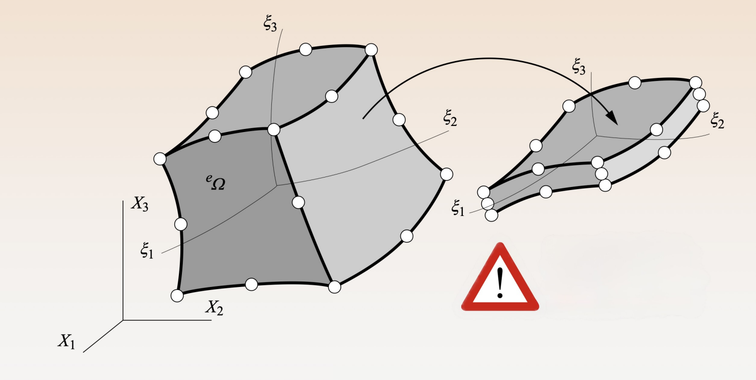


Shell element formulations

There are at least three methods of formulating C^0 shell elements:

1. Combining a membrane element with a plate bending element to form a **flat shell element**.
2. Deriving a curved element using classical **thin shell theory**.
3. Deriving a curved element which is a degenerate solid element to form a **thick shell element**.

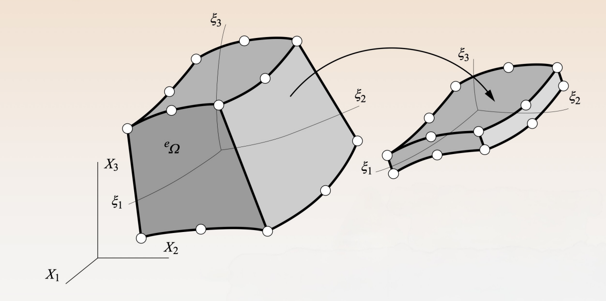
Degenerate solid elements



✗ **Rectilinearity of the transverse fibers hypothesis is not respected**

(Credit:(G))

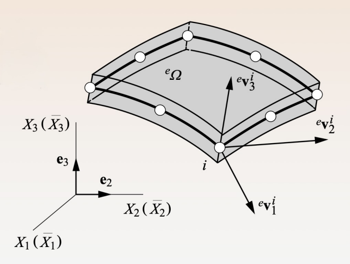
Thick shell element



- Degenerate solid-type shell elements are constructed by modifying three-dimensional solid elements.
- Nodes are situated in both the top and the bottom surfaces and have three degrees of freedom each, namely, three components of displacement.
- **Poor conditioning for thin shells due to the excessive deformation.**

(Credit:(G))

Thin shell element



- **Middle surface shell elements** have nodes only in the middle surface.
- The nodal degrees of freedom consist of three components of displacement and two components of rotation.
- These elements are similar to the ones used for thick plate elements.
- Superparametric elements : more nodes are required to define the geometry than the deformation.
- They can be used to model thin shells provided selective integration is used.

Dyanmic equilibrium equation for thin shells

The governing equation for thin shell theory is a *reduced* elastodynamic equilibrium equation:

$$\nabla^T \overline{\mathbf{C}} \nabla \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}} \qquad \text{on } \Omega \times]0, T[$$

where Ω is characterized by the degenerate hypothesis that one dimension is significantly smaller than the other two.

Elasticity, linearity and isotropic hypothesis

- Linear strain-displacement relationship: $\boldsymbol{\varepsilon} = \nabla \mathbf{u}$ with

$$\nabla = \begin{bmatrix} \partial_{x_1} & 0 & 0 \\ 0 & \partial_{x_2} & 0 \\ 0 & 0 & \partial_{x_3} \\ 0 & \partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & \partial_{x_1} \\ \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

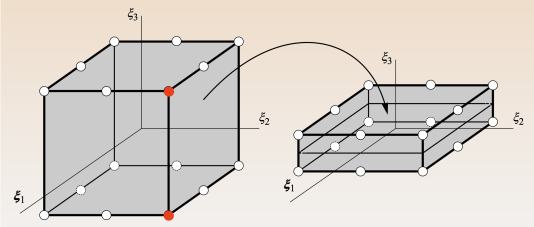
- **Reduced constitutive equation** for isotropic materials: $\boldsymbol{\sigma} = \overline{\mathbf{C}}\boldsymbol{\varepsilon}$:

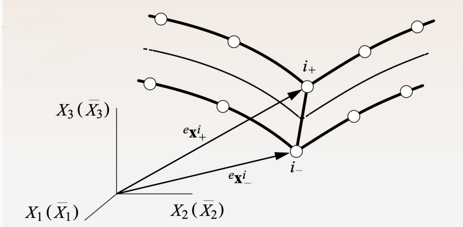
$$\overline{\mathbf{C}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k(1-\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k(1-\nu)/2 \end{bmatrix}$$

E is the Young's modulus, ν is the Poisson's ratio, and k is the shear correction factor.

Coordinate transformation

Finite solid element with linear edges along the ξ_3 direction.



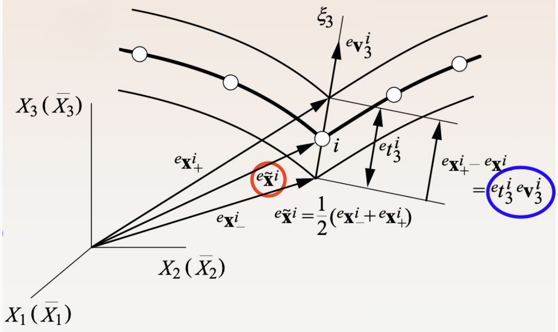


Nodes on top and bottom faces in pairs. Let ${}^{e}q$ be the total number of nodes.

$$ {}^{e}T: \quad \mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^{{}^{e}q/2} {}^{a}h_i(\xi_1, \xi_2) \left[\frac{1-\xi_3}{2} {}^{e}\mathbf{x}_{-}^{i} + \frac{1+\xi_3}{2} {}^{e}\mathbf{x}_{+}^{i} \right] $$

${}^{e}\mathbf{x}_{-}^{i}$ and ${}^{e}\mathbf{x}_{+}^{i}$ are the coordinates of node i on the bottom and top faces respectively.

Coordinate transformation



Let ${}^ep = {}^eq/2$ be the total number of nodes on the midsurface.

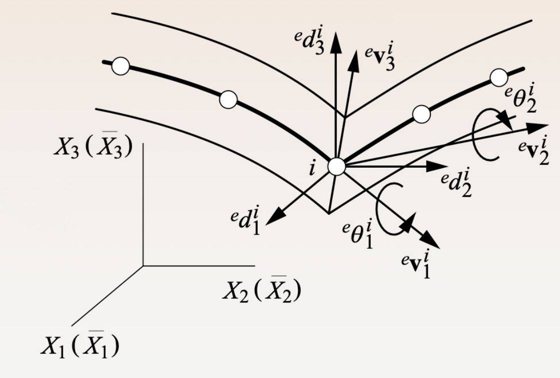
- ${}^e\tilde{\mathbf{x}}^i$ coordinate of node i on the midsurface of ${}^e\Omega$
- ${}^et_3^i$ thickness at node i
- ${}^e\mathbf{v}_3^i$ normal vector to the midsurface at node i

$$
\begin{aligned}
{}^eT : \mathbf{x}(\boldsymbol{\xi}) &= \sum_{i=1}^{{}^ep} {}^ah_i(\xi_1, \xi_2)\left[\frac{1}{2}({}^e\mathbf{x}_-^i + {}^e\mathbf{x}_+^i) + \frac{1}{2}\xi_3({}^e\mathbf{x}_-^i - {}^e\mathbf{x}_+^i)\right] \\
&= \sum_{i=1}^{{}^ep} {}^ah_i(\xi_1, \xi_2)\left[{}^e\tilde{\mathbf{x}}^i + \frac{1}{2}\xi_3\,{}^et_3^i\,{}^e\mathbf{v}_3^i\right]
\end{aligned}
$$

Approximate displacements

Analogy with transformation of coordinates:

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{^e p} {}^a h_i(\xi_1, \xi_2) \left[{}^e\mathbf{d}^i + \frac{1}{2}\xi_3 \, {}^e t_3^i \left(-{}^e\theta_1^i \, {}^e\mathbf{v}_2^i + {}^e\theta_2^i \, {}^e\mathbf{v}_1^i \right) \right]$$



$${}^e\mathbf{d}^i = [{}^e d_1^i, {}^e d_2^i, {}^e d_3^i]^T$$

- ${}^e d_1^i, {}^e d_2^i, {}^e d_3^i$ displacements of node i, oriented along the *global axis*.
- ${}^e\theta_1^i, {}^e\theta_2^i$ rotations of node i, oriented along the *local axis*:
  - ${}^e\theta_1^i$ rotation about ${}^e\mathbf{v}_1^i$.
  - ${}^e\theta_2^i$ rotation about ${}^e\mathbf{v}_2^i$.

Construction of local vectors

- We have:

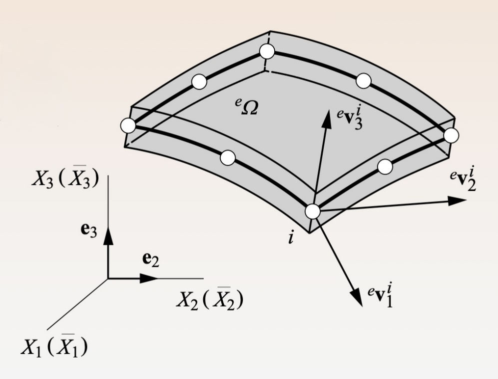
$$ {}^e\mathbf{v}_3^i = {}^e\mathbf{x}_-^i - {}^e\mathbf{x}_+^i $$

the normal vector to the midsurface at node i.

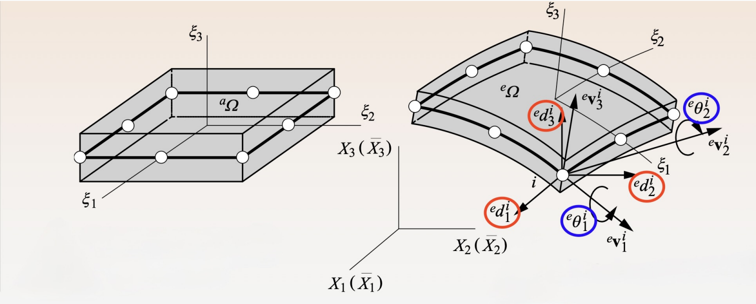
- We define the local vectors:

$$ {}^e\mathbf{v}_1^i = \begin{cases} \mathbf{e}_2 \wedge {}^e\mathbf{v}_3^i & \text{if } {}^e\mathbf{v}_3^i \neq \pm\mathbf{e}_2 \\ \pm\mathbf{e}_3 & \text{if } {}^e\mathbf{v}_3^i = \pm\mathbf{e}_2 \end{cases} $$

$$ {}^e\mathbf{v}_2^i = {}^e\mathbf{v}_3^i \wedge {}^e\mathbf{v}_1^i $$



Thin shell elements



- ✓ 5 DOFs per node, no rotation $^e\theta_3^i$ w.r.t the normal local vector $^e\mathbf{v}_3^i$.
- ✓ Lead to huge computational time savings since allow modeling with fewer nodes.
- ✓ Less prone to negative Jacobian errors which might occur when using extremely thin 3d solid elements.

Shape functions matrix

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left[{}^e\mathbf{d}^i + \frac{1}{2}\xi_3 \, {}^e t_3^i \left(-{}^e\theta_1^i \, {}^e\mathbf{v}_2^i + {}^e\theta_2^i \, {}^e\mathbf{v}_1^i \right) \right]$$

$$= \sum_{i=1}^{{}^e p} {}^a\mathbf{H}_i(\boldsymbol{\xi}) \, {}^e\mathbf{q}^i(t)$$

$$= \sum_{i=1}^{{}^e p} \underbrace{\left[\begin{array}{ccc|cc} {}^a h_i & 0 & 0 & -\frac{1}{2}\xi_3 \, {}^e t_3^i \, {}^a h_i \, {}^e v_{21}^i & \frac{1}{2}\xi_3 \, {}^e t_3^i \, {}^a h_i \, {}^e v_{11}^i \\ 0 & {}^a h_i & 0 & -\frac{1}{2}\xi_3 \, {}^e t_3^i \, {}^a h_i \, v_{22}^i & \frac{1}{2}\xi_3 \, {}^e t_3^i \, {}^a h_i \, {}^e v_{12}^i \\ 0 & 0 & {}^a h_i & -\frac{1}{2}\xi_3 \, {}^e t_3^i \, {}^a h_i \, {}^e v_{23}^i & \frac{1}{2}\xi_3 \, {}^e t_3^i \, {}^a h_i \, {}^e v_{13}^i \end{array} \right]}_{{}^a\mathbf{H}_i} \underbrace{\begin{bmatrix} {}^e d_1^i \\ {}^e d_2^i \\ {}^e d_3^i \\ {}^e \theta_1^i \\ {}^e \theta_2^i \end{bmatrix}}_{{}^e\mathbf{q}^i}$$

Deformation matrix

$$
\begin{aligned}
{}^{a}\mathbf{B}_i =& \nabla {}^{a}\mathbf{H}_i \\
=& \begin{bmatrix}
\frac{\partial {}^{e}h_i}{\partial x_1} & 0 & 0 & {}^{e}g^{i}_{11} & {}^{e}f^{i}_{11} \\
0 & \frac{\partial {}^{e}h_i}{\partial x_2} & 0 & {}^{e}g^{i}_{22} & {}^{e}f^{i}_{22} \\
0 & 0 & \frac{\partial {}^{e}h_i}{\partial x_3} & {}^{e}g^{i}_{33} & {}^{e}f^{i}_{33} \\
0 & \frac{\partial {}^{e}h_i}{\partial x_3} & \frac{\partial {}^{e}h_i}{\partial x_2} & {}^{e}g^{i}_{23} + {}^{e}g^{i}_{32} & {}^{e}f^{i}_{23} + {}^{e}f^{i}_{32} \\
\frac{\partial {}^{e}h_i}{\partial x_3} & 0 & \frac{\partial {}^{e}h_i}{\partial x_1} & {}^{e}g^{i}_{31} + {}^{e}g^{i}_{13} & {}^{e}f^{i}_{31} + {}^{e}f^{i}_{13} \\
\frac{\partial {}^{e}h_i}{\partial x_2} & \frac{\partial {}^{e}h_i}{\partial x_1} & 0 & {}^{e}g^{i}_{12} + {}^{e}g^{i}_{21} & {}^{e}f^{i}_{12} + {}^{e}f^{i}_{21}
\end{bmatrix} \qquad (i = 1, 2, \ldots, {}^{e}p)
\end{aligned}
$$

Deformation matrix

Derivative with respect to global variables:

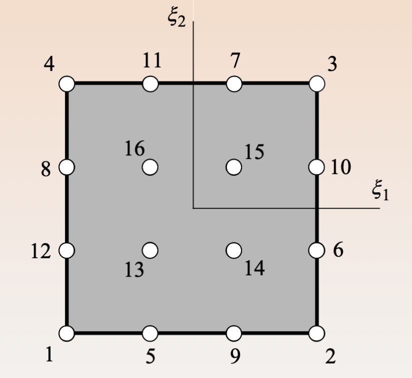
$$\frac{\partial^e h_i}{\partial x_k} = \frac{\partial^a h_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_k} + \frac{\partial^a h_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_k} = {}^e J^{-1}_{k1}\frac{\partial^a h_i}{\partial \xi_1} + {}^e J^{-1}_{k2}\frac{\partial^a h_i}{\partial \xi_2}$$

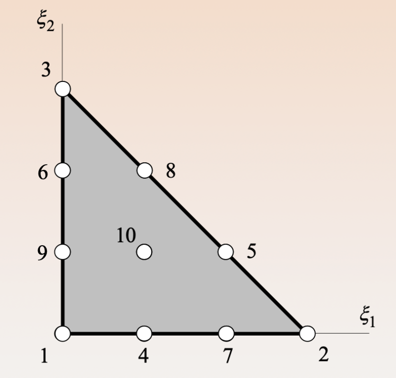
and

$$\begin{aligned}
{}^e f^i_{jk} &= \frac{1}{2}\,{}^e t^i_3 v^i_{1j}\frac{\partial(\xi_3\,{}^e h_i)}{\partial x_k}, \\
{}^e g^i_{jk} &= -\frac{1}{2}\,{}^e t^i_3 v^i_{2j}\,\frac{\partial(\xi_3\,{}^e h_i)}{\partial x_k} \\
\frac{\partial(\xi_3\,{}^e h_i)}{\partial x_k} &= \xi_3\frac{\partial^e h_i}{\partial x_k} + {}^a h_i\frac{\partial \xi_3}{\partial x_k} = \xi_3\left({}^e J^{-1}_{k1}\frac{\partial^a h_i}{\partial \xi_1} + {}^e J^{-1}_{k2}\frac{\partial^a h_i}{\partial \xi_2}\right) + {}^e J^{-1}_{k3}{}^a h_i
\end{aligned}$$

Examples of thin shell elements

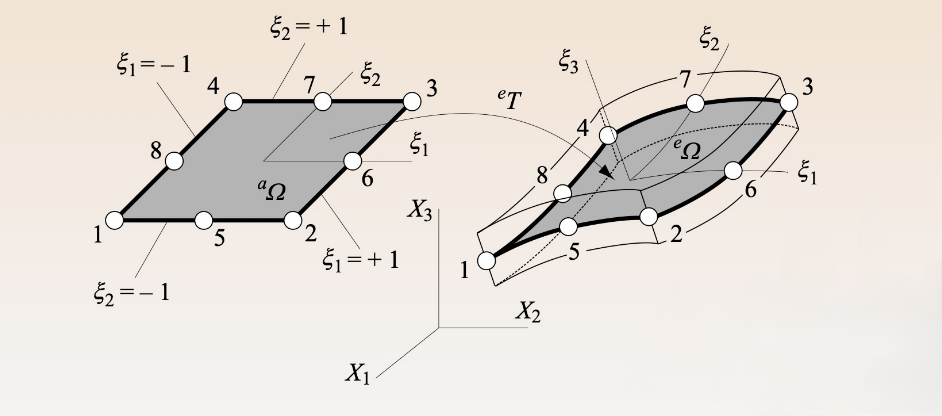
Along the midsurface of the shell we can use Lagrangian 2D finite elements such as, for example, bicubic quadrilateral or bicubic triangular elements.



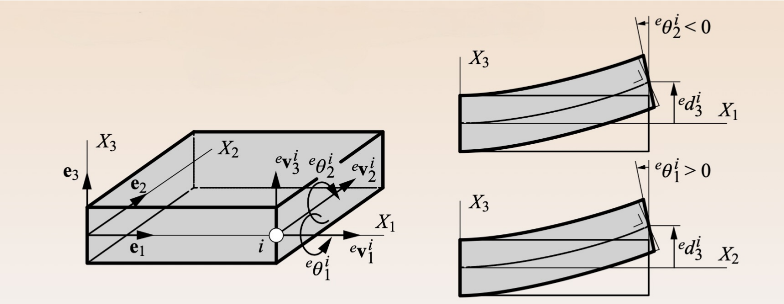


Example of thin shell elements

Biquadratic quadrangular shell element (8 nodes, 5 DOF per node)



Example: flat shell element

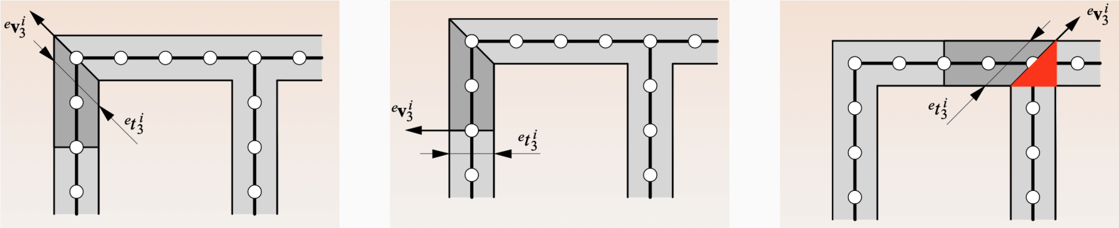


$$\mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left(\begin{bmatrix} {}^e x^i \\ {}^e y^i \\ 0 \end{bmatrix} + \frac{1}{2}\xi_3 {}^e t_3^i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\mathbf{u}^h(\boldsymbol{\xi}) = \sum_{i=1}^{{}^e p} {}^a h_i(\xi_1, \xi_2) \left(\begin{bmatrix} 0 \\ 0 \\ {}^e d_3^i \end{bmatrix} + \frac{1}{2}\xi_3 {}^e t_3^i \left(-{}^e\theta_1^i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + {}^e\theta_2^i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \right)$$

Limitation of shells with 5 degrees of freedom per node

Cross-section of a structure discretized with shell elements with 5 DOFs per node.

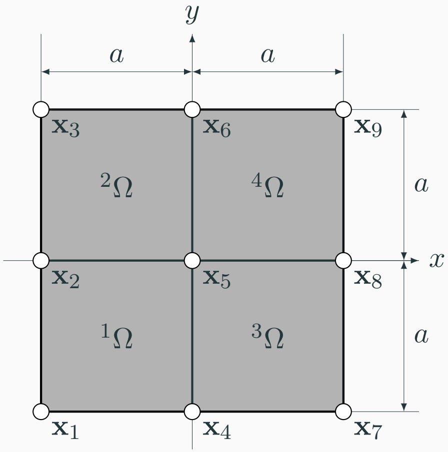


- To overcome this limitation, we need shell elements with 6 DOFs per node: 3 displacements and 3 rotations, all defined with respect to the global axis.
- This eliminates the need to define a local normal vector allowing for more complex geometries.

Example: modal analysis of simply supported thin shell

Simply supported isotropic thin shell

Discretization with 4 bilinear quadrilateral shell elements (4 nodes each).



- $2a$ length
- $2a$ height
- t thickness
- E Young's modulus
- ν Poisson's ratio
- k shear correction factor
- ρ material density

Objective: determine the first natural frequency of the flat shell and compare it with the exact one.

Approximate displacements for flat shell elements

- Nodal displacements in the plane of the elements are suppressed:

$$^{e}\mathbf{d}^{i} = [0, 0, {}^{e}d_3^i]^T$$

Thus only 3 DOFs per node: $^{e}\mathbf{q}^{i} = [{}^{e}d_3^i,\ {}^{e}\theta_1^i,\ {}^{e}\theta_2^i]$.

- Local vectors are oriented along the principal axes of the shell:

$$^{e}\mathbf{v}_1^i = [1, 0, 0]^T, \qquad {}^{e}\mathbf{v}_2^i = [0, 1, 0]^T, \qquad {}^{e}\mathbf{v}_3^i = [0, 0, 1]^T$$

- Local shape functions matrices:

$$^{a}\mathbf{H}_i = \begin{bmatrix} 0 & 0 & \frac{1}{2}\xi_3 t\, {}^{a}h_i \\ 0 & -\frac{1}{2}\xi_3 t\, {}^{a}h_i & 0 \\ {}^{a}h_i & 0 & 0 \end{bmatrix} \qquad (i = 1, 2, 3, 4)$$

Coordinate transformation for ${}^1\Omega$

- Bilinear base functions for quadrilateral shell element:

$$
\begin{aligned}
{}^a h_1(\xi_1, \xi_2) &= (1-\xi_1)(1-\xi_2)/4 \\
{}^a h_2(\xi_1, \xi_2) &= (1+\xi_1)(1-\xi_2)/4 \\
{}^a h_3(\xi_1, \xi_2) &= (1+\xi_1)(1+\xi_2)/4 \\
{}^a h_4(\xi_1, \xi_2) &= (1-\xi_1)(1+\xi_2)/4
\end{aligned}
$$

- Coordinates ${}^1\tilde{\mathbf{x}} = [[0,0,0],[a,0,0],[a,a,0],[0,a,0]]$

$$
{}^1T : \mathbf{x}(\boldsymbol{\xi}) = \sum_{i=1}^{4} {}^a h_i(\xi_1, \xi_2) \left({}^e\tilde{\mathbf{x}}^i + \frac{1}{2}\xi_3 {}^e t_3^i \, {}^e\mathbf{v}_3^i \right) = \left[a\frac{1+\xi_1}{2},\ a\frac{1+\xi_2}{2}, t\frac{\xi_3}{2} \right]^T
$$

- Jacobian matrix ${}^1J = \mathrm{diag}(a/2,\ a/2,\ t/2)$ and determinant ${}^1j = a^2 t/8$.

Local mass matrices

$$
\begin{aligned}
{}^1\mathbf{M}_{ij} &= \frac{\rho a^2 t}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} {}^a\mathbf{H}_i^T\,{}^a\mathbf{H}_j\, d\xi_1 d\xi_2 d\xi_3 \\
&= \frac{\rho a^2 t}{4}\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} {}^a h_i\,{}^a h_j & 0 & 0 \\ 0 & t^2\,{}^a h_i\,{}^a h_j/12 & 0 \\ 0 & 0 & t^2\,{}^a h_i\,{}^a h_j/12 \end{bmatrix} d\xi_1 d\xi_2
\end{aligned}
$$

Local mass matrix for ${}^1\Omega$ via exact integration:

$$
{}^1\mathbf{M} = \begin{bmatrix} {}^1\mathbf{M}_{11} & {}^1\mathbf{M}_{12} & {}^1\mathbf{M}_{13} & {}^1\mathbf{M}_{14} \\ & {}^1\mathbf{M}_{22} & {}^1\mathbf{M}_{23} & {}^1\mathbf{M}_{24} \\ & & {}^1\mathbf{M}_{33} & {}^1\mathbf{M}_{34} \\ sym. & & & {}^1\mathbf{M}_{44} \end{bmatrix}
$$

Local deformation matrices

$$
{}^{1}\mathbf{B}_i = \nabla^{a}\mathbf{H}_i = \begin{bmatrix} \partial_{x_1} & 0 & 0 \\ 0 & \partial_{x_2} & 0 \\ 0 & 0 & \partial_{x_3} \\ 0 & \partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & \partial_{x_1} \\ \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2}\xi_3 t\,{}^{a}h_i \\ 0 & -\frac{1}{2}\xi_3 t\,{}^{a}h_i & 0 \\ {}^{a}h_i & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & \frac{t}{a}\xi_3\,\partial_{\xi_1}{}^{a}h_i \\ 0 & -\frac{t}{a}\xi_3\,\partial_{\xi_2}{}^{a}h_i & 0 \\ 0 & 0 & 0 \\ \frac{2}{a}\,\partial_{\xi_2}{}^{a}h_i & -{}^{a}h_i & 0 \\ \frac{2}{a}\,\partial_{\xi_1}{}^{a}h_i & 0 & {}^{a}h_i \\ 0 & -\frac{t}{a}\xi_3\,\partial_{\xi_1}{}^{a}h_i & \frac{t}{a}\xi_3\,\partial_{\xi_2}{}^{a}h_i \end{bmatrix}
$$

Local stiffness matrices

$$^1\mathbf{K}_{ij} = \frac{a^2 t}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} {}^1\mathbf{B}_i^T \overline{\mathbf{C}}\, {}^1\mathbf{B}_j d\xi_1 d\xi_2 d\xi_3$$

Local stiffness matrix for $^1\Omega$ via exact integration

$$^1\mathbf{K} = \begin{bmatrix} {}^1\mathbf{K}_{11} & {}^1\mathbf{K}_{12} & {}^1\mathbf{K}_{13} & {}^1\mathbf{K}_{14} \\ & {}^1\mathbf{K}_{22} & {}^1\mathbf{K}_{23} & {}^1\mathbf{M}_{24} \\ & & {}^1\mathbf{K}_{33} & {}^1\mathbf{K}_{34} \\ sym. & & & {}^1\mathbf{K}_{44} \end{bmatrix}$$

Assembly and boundary conditions

- Since ${}^eJ = {}^1J$ and thus ${}^ej = {}^1j$ for every $e = 2, 3, 4$, we have

$$ {}^e\mathbf{K} = {}^1\mathbf{K} \qquad \text{and} \qquad {}^e\mathbf{M} = {}^1\mathbf{M} $$

- The assembly of the global stiffness $\mathbf{K}$ (27×27) and mass matrices $\mathbf{M}$ (27×27) can be performed using the connectivity table:

${}^e\Omega$	${}^1\Omega$	${}^2\Omega$	${}^3\Omega$	${}^4\Omega$
1	1	2	4	5
2	4	5	7	8
3	5	6	8	9
4	2	3	5	6

- The following 20 DOFs are constrained due the the fact that the plate is simply supported along its perimeter:

$d_3^1,\ \theta_1^1,\ \theta_2^1$ $\quad$ $d_3^3,\ \theta_1^3,\ \theta_2^3$ $\quad$ $d_3^7,\ \theta_1^7,\ \theta_2^7$ $\quad$ $d_3^9,\ \theta_1^9,\ \theta_2^9$

$d_3^2,\ \theta_1^2$ $\quad$ $d_3^4,\ \theta_2^4$ $\quad$ $d_3^6,\ \theta_2^6$ $\quad$ $d_3^8,\ \theta_1^8$

Modal analysis

- The semi-discrete weak form is a system of $27 - 20 = 7$ differential equations for the 7 free DOFs: $d_3^5,\ \theta_1^5,\ \theta_2^5$, and $\theta_2^2,\ \theta_1^4,\ \theta_1^6,\ \theta_2^8$

$$\mathbf{K}\mathbf{q}(t) + \mathbf{M}\ddot{\mathbf{q}}(t) = \mathbf{0}$$

- The first fundamental frequency $\omega_1 = \sqrt{\lambda_1}$ (in rad/s) can be computed solving the generalized eigenvalue problem: $(\mathbf{K} + \lambda\mathbf{M})\,\mathbf{p} = \mathbf{0}$.
- Assuming a thin plate $t/2a = 0.01$ and a shear coefficient $k = 5/6$ we obtain

$$\omega_1^{approx} = 0.6620\sqrt{E/(1-\nu^2)\rho a}$$

- From the analytical solution we obtain

$$\omega_1^{exact} = 0.0285\sqrt{E/(1-\nu^2)\rho a}$$

- Exact integration of transverse shear terms in the stiffness matrix $\mathbf{K}$ leads to *element locking*.

Selective integration

- Separate the transverse shear contributions terms ${}^e\mathbf{B}_i^\tau$ in the deformation matrix:

$${}^e\mathbf{B}_i = {}^e\mathbf{B}_i^\sigma + {}^e\mathbf{B}_i^\tau$$

- Split the stiffness matrix into flexural stiffness ${}^e\mathbf{K}_{ij}^\sigma$ and transverse shear stiffness ${}^e\mathbf{K}_{ij}^\tau$:

$${}^e\mathbf{K}_{ij} = {}^e\mathbf{K}_{ij}^\sigma + {}^e\mathbf{K}_{ij}^\tau = \int_\Omega {}^e\mathbf{B}_i^\sigma \overline{\mathbf{C}}{}^e\mathbf{B}_j^\sigma \, d\Omega + \int_\Omega {}^e\mathbf{B}_i^\tau \overline{\mathbf{C}}{}^e\mathbf{B}_j^\tau \, d\Omega$$

- Perform a **selective integration**: exact integration of bending contributions ${}^e\mathbf{K}_{ij}^\sigma$ and reduced integration (a single Gauss point located at the center of the element) for shear contributions ${}^e\mathbf{K}_{ij}^\tau$.
- Assuming a thin squared plate $t/2a = 0.01$ and a shear coefficient $k = 5/6$ with selective integration we obtain

$$\omega_1 = 0.0383\sqrt{E/(1-\nu^2)\rho a} \qquad \text{(error} \approx 34\%)$$

Error estimates

Assuming a thin squared plate $t/2a = 0.01$ and a shear coefficient $k = 5/6$ with selective integration we obtain:

$$\omega_1 = 0.0383\sqrt{E/(1-\nu^2)\rho a} \qquad \text{(Rel. error} \approx 34\%)$$

Meshing	Integration	Elements	Rel. error
2×2	Exact	bilinear	$> 20'000\%$
2×2	Selective	bilinear	34%
4×4	Selective	bilinear	7.2%
1×1	Selective	biquadratic	6.2%
2×2	Selective	biquadratic	1.0%