

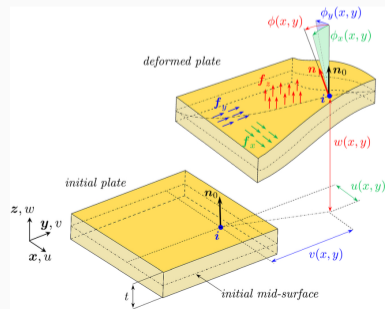
Dynamic analysis of Reissner-Mindlin plates

Special structural elements

ME473 Dynamic finite element analysis of structures

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Where do we stand?

Week	Module	Lecture topic	Mini-projects
1	Linear elastodynamics	Strong and weak forms	
2		Galerkin method	
3		Finite element method	Groups formation
4		Systematization of the procedure	Project 1 statement
5		3d elements, numerical integration	
6	Special structural elements	Bars and trusses	
7		Planar beams	Project 1 submission
8		Frames and grids	Project 2 statement
9		Kirchhoff-Love plates	
10		Reissner-Mindlin plates	

Summary

- Recap week 9
- Reissner-Mindlin plate theory
- Thick plate bending elements

Recommended readings

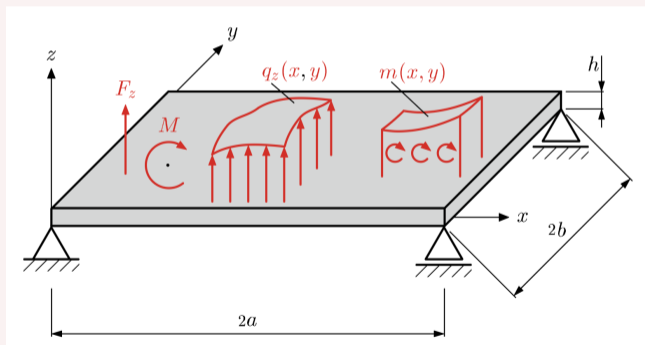
- (N) Neto et al., Engineering Computation of Structures (chap. 6.2)
- (P) Petyt, Introduction to finite element vibration analysis (chap. 6)
- (O) Ochsner, PDE for classical structural members (chap. 7)

Recap week 9

Vibrations of Kirchhoff-Love plates

Plate structure

- Plate structures are geometrically similar to structures of the 2D plane stress problem, but it usually carries only **transversal loads** that lead to bending deformation of the plate.
- For example: floors of a building, aerospace and ships structures, etc...

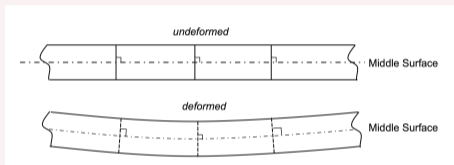


(Credit: (O))

Kirchhoff assumption

Planarity and orthogonality of the cross-sectional planes

Bernoulli's hypothesis is valid, i.e. a cross-sectional plane stays plane and perpendicular to the middle surface in the deformed state.



- Transverse shear strains are neglected $\varepsilon_{13} = \varepsilon_{23} = 0$ due to the Bernoulli's hypothesis.
- Transverse normal strain is negligible $\varepsilon_{33} = 0$ due to the inextensibility of transverse fibers assumption.

(Credit: (N))

Finite element approximation of Kirchhoff-Love plate

- Strong form

$$\frac{h^3}{12} \nabla_k^T \mathbf{C} \nabla_k u_3 + \rho h \ddot{u}_3 = f_3 \quad \text{on } \Omega \times]0, T[$$

- Weak form equation

$$\frac{h^3}{12} \int_{\Omega} (\nabla_k \delta u_3)^T \mathbf{C} \nabla_k u_3 d\Omega + \int_{\Omega} \rho h \ddot{u}_3 \delta u_3 d\Omega = \int_{\Omega} f_3 \delta u_3 d\Omega$$

- Semi-discrete weak form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{r}(t)$$

- **Adini-Melosh-Clough element (AMC):**
12 DOFs quadrangular, not conforming, thin plate bending element.
- **Crouzeix–Raviart (CR):**
16 DOFs quadrangular, conforming, thin plate bending element.

Comparison: AMC vs CR plate bending element

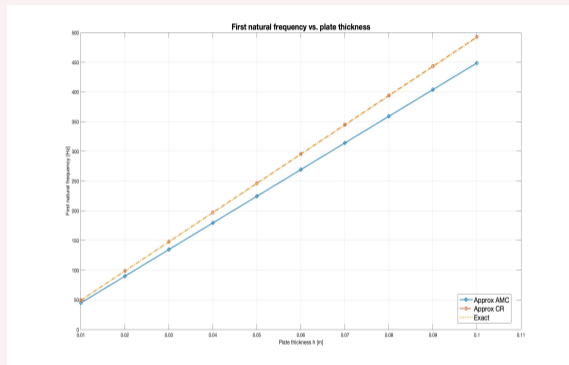


Figure 1: Evolution of the approximated and exact fundamental frequencies with thickness for a SSSS square plate.

AMC element (*Non conforming*)

- 3 degrees of freedom per node: u_3 , θ_1 , θ_2 .
- Only displacement u_3 is continuous across elements; rotations may have jumps.

CR element (*Conforming*)

- 4 degrees of freedom per node: u_3 , θ_1 , θ_2 , and θ_{12} .
- Displacement u_3 and rotations θ_1 , θ_2 are continuous across element boundaries.

Selection of the displacement function

Displacement approximation for AMC element

$${}^e u_3^h(x_1, x_2, t) = a_1 + a_2 x_1 + a_3 x_2 + a_4 x_1^2 + a_5 x_1 x_2 + a_6 x_2^2 + \\ + a_7 x_1^3 + a_8 x_1^2 x_2 + a_9 x_1 x_2^2 + a_{10} x_2^3 + a_{11} x_1^3 x_2 + a_{12} x_1 x_2^3$$

Displacement approximation for CR element

$${}^e u_3^h(x_1, x_2, t) = a_1 + a_2 x_1 + a_3 x_2 + a_4 x_1^2 + a_5 x_1 x_2 + a_6 x_2^2 + \\ + a_7 x_1^3 + a_8 x_1^2 x_2 + a_9 x_1 x_2^2 + a_{10} x_2^3 + \\ + a_{11} x_1^3 x_2 + a_{12} x_1 x_2^3 + a_{13} x^2 y^2 + a_{14} x^3 y^2 + a_{15} x^2 y^3 + a_{16} x^3 y^3$$

Approximate displacement and shape functions for the AMC element

The approximate displacement in local coordinate is defined as:

$${}^e u_3^h(\boldsymbol{\xi}, t) = \sum_{i=1}^4 {}^a \mathbf{h}_i(\boldsymbol{\xi}) {}^e \mathbf{q}^i(t) = {}^a \mathbf{H}(\boldsymbol{\xi}) {}^e \mathbf{q}(t)$$

■ 3 dofs per node: ${}^e \mathbf{q}^i(t) = \begin{bmatrix} {}^e d^i(t) \\ {}^e \theta_1^i(t) \\ {}^e \theta_2^i(t) \end{bmatrix} = \begin{bmatrix} {}^e u_3^h(\boldsymbol{\xi}^i, t) \\ \partial_{\xi_2} {}^e u_3^h(\boldsymbol{\xi}^i, t)/b \\ -\partial_{\xi_1} {}^e u_3^h(\boldsymbol{\xi}^i, t)/a \end{bmatrix}$

■ The shape function matrix is ${}^a \mathbf{H} = [{}^a \mathbf{h}_1 \quad {}^a \mathbf{h}_2 \quad {}^a \mathbf{h}_3 \quad {}^a \mathbf{h}_4]$ where

$${}^a \mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} (1 + \xi_1^i \xi_1)(1 + \xi_2^i \xi_2)(2 + \xi_1^i \xi_1 + \xi_2^i \xi_2 - \xi_1^2 - \xi_2^2)/8 \\ b(1 + \xi_1^i \xi_1)(\xi_2^i + \xi_2)(\xi_2^2 - 1)/8 \\ -a(\xi_1^i + \xi_1)(\xi_1^2 - 1)(1 + \xi_2^i \xi_2)/8 \end{bmatrix}^T$$

Approximate displacement and shape functions for the CR element

The approximate displacement in local coordinate is defined as:

$${}^e u_3^h(\boldsymbol{\xi}, t) = \sum_{i=1}^4 {}^a \mathbf{h}_i(\boldsymbol{\xi}) {}^e \mathbf{q}^i(t) = {}^a \mathbf{H}(\boldsymbol{\xi}) {}^e \mathbf{q}(t)$$

■ 4 dofs per node: ${}^e \mathbf{q}^i(t) = \begin{bmatrix} {}^e d^i(t) \\ {}^e \theta_1^i(t) \\ {}^e \theta_2^i(t) \\ {}^e \theta_{12}^i(t) \end{bmatrix} = \begin{bmatrix} {}^e u_3^h(\boldsymbol{\xi}^i, t) \\ \partial_{\xi_2} {}^e u_3^h(\boldsymbol{\xi}^i, t)/b \\ -\partial_{\xi_1} {}^e u_3^h(\boldsymbol{\xi}^i, t)/a \\ \partial_{\xi_1 \xi_2}^2 {}^e u_3^h(\boldsymbol{\xi}^i, t)/(ab) \end{bmatrix}$

■ The shape function matrix is ${}^a \mathbf{H} = [{}^a \mathbf{h}_1 \quad {}^a \mathbf{h}_2 \quad {}^a \mathbf{h}_3 \quad {}^a \mathbf{h}_4]$ where

$${}^a \mathbf{h}_i(\boldsymbol{\xi}) = \begin{bmatrix} f_i(\xi_1) f_i(\xi_2) \\ b f_i(\xi_1) g_i(\xi_2) \\ -a g_i(\xi_1) f_i(\xi_2) \\ a b g_i(\xi_1) g_i(\xi_2) \end{bmatrix}^T$$

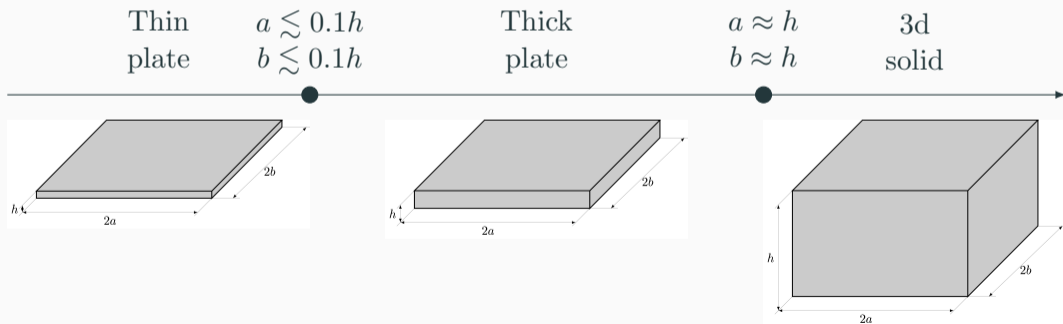
Hermite functions: $f_i(\xi) = (-\xi^i \xi^3 + 3\xi^i \xi + 2)/4$, $g_i(\xi) = (\xi^3 + \xi^i \xi^2 - \xi - \xi^i)/4$.

Reissner-Mindlin plate theory

Limitations of classical plate theory

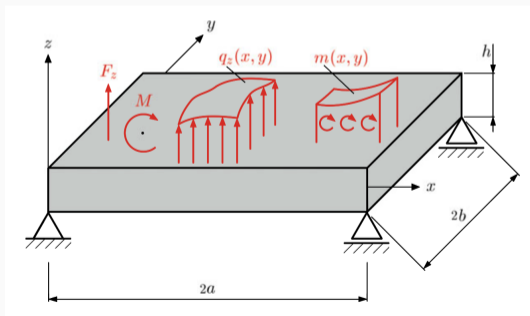
The validity of the classical (Kirchhoff-Love) plate theory depends on a number of factors:

- 1 the curvatures are small,
- 2 the in-plane plate dimensions are large compared to the thickness,
- 3 membrane strains are neglected.



Shear deformable plates

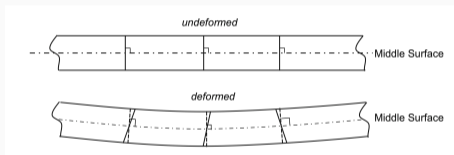
- Thick plates are structures in which shear deformation and rotary inertia effects are important: *transverse shear* becomes an integral part of the formulation.
- The material is isotropic, homogenous and linear-elastic according to Hooke's law for a plane stress state where $\sigma_{33} = 0$,
- Plates carry only transversal loads and in-plane moments that lead to bending deformation of the plate.



Reissner-Mindlin assumption

Planarity of the cross-sectional planes

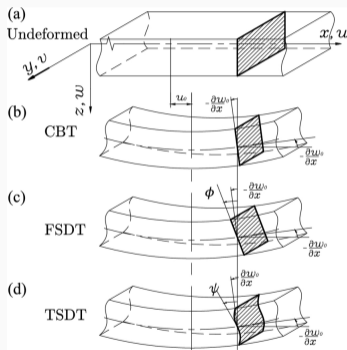
Timoshenko's hypothesis is valid, i.e. a cross-sectional plane stays planar and *but not necessarily perpendicular* to the middle surface in the deformed state.



Shear strains ε_{13} and ε_{23} , due to the distributed shear forces q_x and q_y , are **constant** through the thickness of the plate.

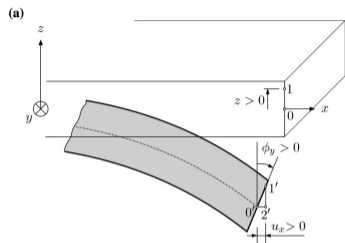
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Higher-order deformation theories for plates



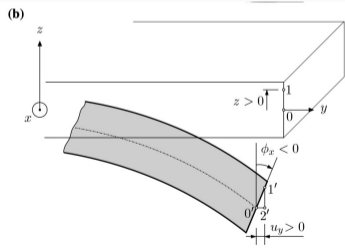
- **Classical plate theory** (Kirchhoff-Love) is a zero-order shear deformation theory.
- **Reissner-Mindlin theory** is a first-order shear deformation theory. Cross-sectional planes remain straight but not necessarily perpendicular to the mid-surface.
 - Displacements are linear functions through thickness.
 - In-plane strains are linear in z .
 - Shear strains are constant in z .
- **Third-order shear deformation theory**: normal cross-sectional planes to the mid-surface can rotate and deform.
 - Displacements are cubic functions through thickness.
 - In-plane strains are cubic in z .
 - Shear strains are quadratic in z .

Kinematics assumptions



$$\phi_2 \approx \sin(\phi_2) = \frac{u_1}{z}$$

$$\implies u_1 = z\phi_2$$



$$\phi_1 \approx \sin(\phi_1) = -\frac{u_2}{z}$$

$$\implies u_2 = -z\phi_1$$

Independent variables:

- Transverse displacement u_3
- Rotation w.r.t. Ox axis ϕ_1
- Rotation w.r.t. Oy axis ϕ_2

Deformation is exaggerated in the figures for better illustration.

Strain-displacement relations

Using engineering definitions of strain: $\varepsilon_{ii} = \partial_i u_i$ and $\gamma_{ij} = \partial_i u_j + \partial_j u_i$ we obtain:

- in-plane or bending strains:

$$\underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}}_{\varepsilon_b} = z \underbrace{\begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \end{bmatrix}}_{\nabla_b} \underbrace{\begin{bmatrix} u_3 \\ \phi_1 \\ \phi_2 \end{bmatrix}}_{\mathbf{u}},$$

- transverse shear strains:

$$\underbrace{\begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}}_{\varepsilon_s} = \underbrace{\begin{bmatrix} \partial_x & 0 & 1 \\ \partial_y & -1 & 0 \end{bmatrix}}_{\nabla_s} \underbrace{\begin{bmatrix} u_3 \\ \phi_1 \\ \phi_2 \end{bmatrix}}_{\mathbf{u}}.$$

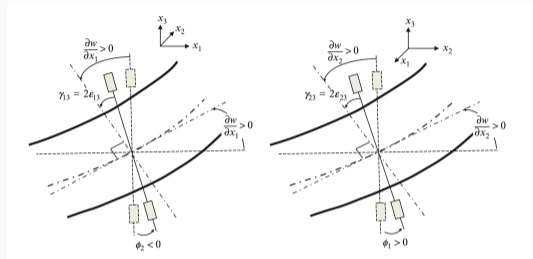
Note that $\varepsilon_{33} = 0$ due to the inextensibility of transverse fibers assumption.

Reduction to Kirchhoff-Love plate theory

The transverse shear are exactly equal to the additional rotations of the normal to the reference surface after deformation:

$$\gamma_{13} = \frac{\partial u_3}{\partial x} + \phi_2$$

$$\gamma_{23} = \frac{\partial u_3}{\partial y} - \phi_1$$



If the transverse shear strains are negligible, $\gamma_{13} = 0$ and $\gamma_{23} = 0$, then, as in the Kirchhoff-Love theory:

$$\frac{\partial u_3}{\partial x} = -\phi_2 \quad \text{and} \quad \frac{\partial u_3}{\partial y} = \phi_1$$

Constitutive equation for isotropic material

The constitutive equation for isotropic material is $\bar{\sigma} = \bar{\mathbf{C}}\bar{\varepsilon}$ where

- bending stresses:

$$\underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}}_{\sigma_b} = \frac{E}{1-\nu^2} \underbrace{\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\mathbf{C}_b} \underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}}_{\varepsilon_b},$$

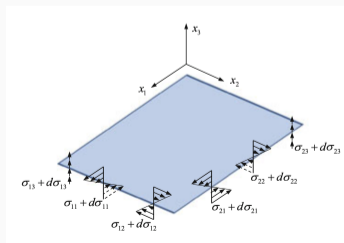
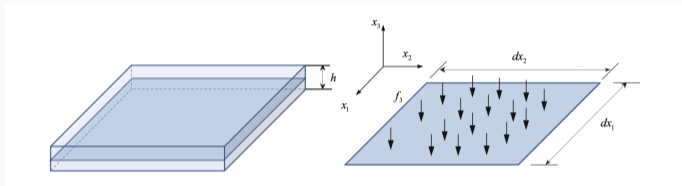
- transverse shear stresses:

$$\underbrace{\begin{bmatrix} \sigma_{13} \\ \sigma_{23} \end{bmatrix}}_{\sigma_s} = G \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{C}_s} \underbrace{\begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}}_{\varepsilon_s}.$$

Recall that, for isotropic materials, the shear modulus G is related to Young's modulus E and Poisson's ratio ν by: $G = \frac{E}{2(1+\nu)}$.

External forces and moments

Consider a plate cell of dimensions $dx_1 \times dx_2 \times h$ that is submitted to external forces, here denoted by f_3 , and area distributed moments m_1 and m_2 (not shown).



Normal and shear stresses distributions through the thickness of the plate element:

- linear distributed normal stresses σ_{11} and σ_{22} ,
- linear distributed shear stresses σ_{12} and σ_{21} ,
- parabolic distributed transverse shear stresses σ_{13} and σ_{23} .

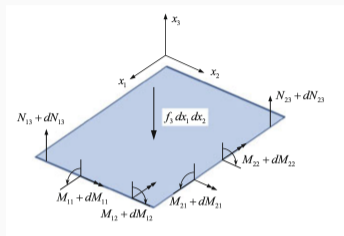
Normal and shear stresses distributions

- Transverse shear strains ε_{13} and ε_{23} are constant through the plate thickness (independent of z).
- Transverse shear stresses σ_{13} and σ_{23} also have a constant distribution through the plate thickness.
- *Transverse shear correction coefficient k* : accounts for the discrepancy in transverse shear stress between plate theory and 3D elasticity. It ensures the strain energy from shear stresses in plate theory matches that from 3D elasticity:

$$\underbrace{\begin{bmatrix} \sigma_{13} \\ \sigma_{23} \end{bmatrix}}_{\sigma_s} = kG \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C_s} \underbrace{\begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}}_{\varepsilon_s}.$$

For homogeneous isotropic rectangular plates : $k = 5/6$.

Moments and shear forces



Moments and shear forces acting along the edge of the plate:

- bending moments M_{11} and M_{22} ,
- twisting moment M_{12} ,
- shear forces N_{13} and N_{23} .

$$\mathbf{M} = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 \boldsymbol{\sigma}_b dx_3 = \mathbf{C}_b \nabla_b \mathbf{u} \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3^2 dx_3 = \underbrace{\frac{h^3}{12} \mathbf{C}_b}_{\overline{\mathbf{C}}_b} \nabla_b \mathbf{u}$$

$$\mathbf{N} = \begin{bmatrix} N_{13} \\ N_{23} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\sigma}_s dx_3 = \underbrace{h \mathbf{C}_s}_{\overline{\mathbf{C}}_s} \nabla_s \mathbf{u}$$

Moments and shear forces

In matrix form:

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{\mathbf{C}}_b & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{C}}_s \end{bmatrix}}_{\overline{\mathbf{C}}} \underbrace{\begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix}}_{\nabla_r} \mathbf{u},$$

Here

$$\nabla_r = \begin{bmatrix} \nabla_b \\ \nabla_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & -\partial_y & 0 \\ 0 & -\partial_x & \partial_y \\ \partial_x & 0 & 1 \\ \partial_y & -1 & 0 \end{bmatrix},$$

$$\overline{\mathbf{C}}_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{C}}_s = \frac{Ekh}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Dynamic equilibrium equation

- Equilibrium condition for the vertical forces:

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + f_3 = \rho h \ddot{u}_3$$

- Equilibrium of moments:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} + m_2 - N_{13} = \rho \frac{h^3}{12} \ddot{\phi}_2$$

$$\frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_1} - m_1 - N_{23} = -\rho \frac{h^3}{12} \ddot{\phi}_1$$

Dynamic equilibrium equation

In matrix form:

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & \partial_x & \partial_y \\ 0 & -\partial_y & -\partial_x & 0 & 1 \\ \partial_x & 0 & \partial_y & -1 & 0 \end{bmatrix}}_{\nabla_m^T} \begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} + \underbrace{\begin{bmatrix} f_3 \\ m_1 \\ m_2 \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho h^3/12 & 0 \\ 0 & 0 & \rho h^3/12 \end{bmatrix}}_{\mathbf{I}} \ddot{\mathbf{u}}$$

- \mathbf{I} mass moment of inertia matrix:
 - ρh translational inertia (in the transverse x_3 -direction),
 - $\rho h^3/12$: rotational inertia (about the in-plane x_1 - and x_2 -axes).
- \mathbf{f} applied forces and moments.
- Linear elastic stress-strain relation and the constitutive relation:

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \end{bmatrix} = \bar{\mathbf{C}} \nabla_r \mathbf{u}$$

Strong form for Reissner-Mindlin plate bending

Let $\Omega = [-a, a] \times [-b, b]$ be a rectangular plate. Find the transverse displacement $u_3 \in C^2(\Omega \times [0, T])$ and the rotations $\phi_1, \phi_2 \in C^2(\Omega \times [0, T])$ such that

$$\nabla_m^T \bar{\mathbf{C}} \nabla_r \mathbf{u} + \mathbf{f} = \mathbf{I} \ddot{\mathbf{u}} \quad \text{on } \Omega \times]0, T[$$

- boundary conditions (simply supported):

$$u_3 = 0 \quad \text{in } \partial\Omega \times]0, T[$$

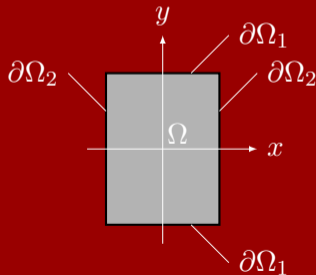
$$\phi_2 = 0 \quad \text{in } \partial\Omega_1 \times]0, T[$$

$$\phi_1 = 0 \quad \text{in } \partial\Omega_2 \times]0, T[$$

- initial conditions:

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega$$

$$\dot{\mathbf{u}}(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \Omega$$



Strong form for Reissner-Mindlin plate bending

Expanding the strong form, for isotropic material, the dynamic equation in matrix form results in the following system of partial differential equations:

$$\begin{bmatrix} D_s (\partial_{xx}^2 + \partial_{yy}^2) & -D_s \partial_y & D_s \partial_x \\ D_s \partial_y & D_b \left(\frac{1-\nu}{2} \partial_{xx}^2 + \partial_{yy}^2 \right) - D_s & -\frac{1+\nu}{2} D_b \partial_{xy}^2 \\ -D_s \partial_x & -\frac{1+\nu}{2} D_b \partial_{xy}^2 & D_b \left(\partial_{xx}^2 + \frac{1-\nu}{2} \partial_{yy}^2 \right) - D_s \end{bmatrix} \begin{bmatrix} u_3 \\ \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} f_3 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \rho h \ddot{u}_3 \\ \rho h^3 / 12 \ddot{\phi}_1 \\ \rho h^3 / 12 \ddot{\phi}_2 \end{bmatrix}$$

where $D_s = khG$, and $D_b = \frac{Eh^3}{12(1-\nu^2)}$.

Weak form for Reissner-Mindlin plate bending

The weak form consists of finding the transverse displacement $u_3 \in \mathcal{U}$ and the rotations $\phi_1, \phi_2 \in \mathcal{U}$ such that the following equation is satisfied for every $\delta \mathbf{u} \in \mathcal{V}$:

$$\int_{\Omega} (\nabla_r \delta \mathbf{u})^T \bar{\mathbf{C}} \nabla_r \mathbf{u} d\Omega + \int_{\Omega} \delta \mathbf{u}^T \mathbf{I} \ddot{\mathbf{u}} d\Omega = \int_{\Omega} \delta \mathbf{u}^T \mathbf{f} d\Omega$$

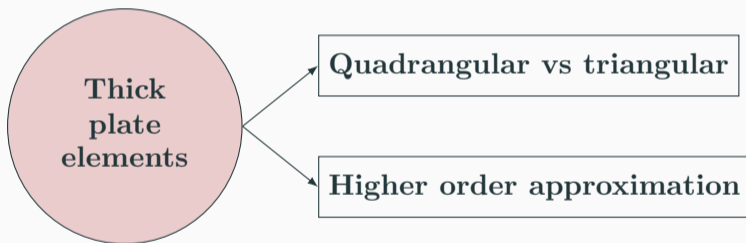
$$\mathcal{U} = \{ \mathbf{u}(\cdot, t) \in H^1(\Omega) \mid u_3(\cdot, t) = 0 \text{ in } \partial\Omega, \phi_1(\cdot, t) = 0 \text{ in } \partial\Omega_2, \phi_2(\cdot, t) = 0 \text{ in } \partial\Omega_1 \}$$

$$\mathcal{V} = \{ \delta \mathbf{u} \in H^1(\Omega) \mid \delta u_3 = 0 \text{ in } \partial\Omega, \delta \phi_1 = 0 \text{ in } \partial\Omega_2, \delta \phi_2 = 0 \text{ in } \partial\Omega_1 \}$$

Thick plate bending elements

Overview of thick plate bending elements

Numerous finite elements for plate bending have been developed: **more than 88 distinct types** (thick and thin combined) can be identified.



- 1977 - **Hughes-Taylor-Kanoknukulcha element (HTK)**: 12 dofs quadrangular, thick plate bending element.

Generalized displacements approximation

Since the rotations ϕ_1 and ϕ_2 are defined independently of the transversal displacement u_3 , the discretization procedure uses 2D bilinear finite elements.

$${}^e u_3(x, y, t) = \sum_{i=1}^4 {}^e h_i(x, y) {}^e d^i(t)$$

$${}^e \phi_1(x, y, t) = \sum_{i=1}^4 {}^e h_i(x, y) {}^e \theta_1^i(t)$$

$${}^e \phi_2(x, y, t) = \sum_{i=1}^4 {}^e h_i(x, y) {}^e \theta_2^i(t)$$

The shape functions ${}^e h_i$ are bilinear Lagrangian functions defined over the element.

Generalized displacements approximation

$${}^e\mathbf{u}^h(\mathbf{x}, t) = {}^e\mathbf{H}(\mathbf{x}){}^e\mathbf{q}(t) = \sum_{i=1}^4 {}^e h_i(\mathbf{x}) {}^e\mathbf{q}^i(t)$$

- ${}^e\mathbf{H}(\mathbf{x})$ is a 3×12 matrix of **shape functions**:

$${}^e\mathbf{H} = \left[\begin{array}{c|c|c|c} h_1\mathbf{I} & h_2\mathbf{I} & h_3\mathbf{I} & h_4\mathbf{I} \end{array} \right] = \begin{bmatrix} {}^e h_1 & 0 & 0 & \dots & {}^e h_4 & 0 & 0 \\ 0 & {}^e h_1 & 0 & \dots & 0 & {}^e h_4 & 0 \\ 0 & 0 & {}^e h_1 & \dots & 0 & 0 & {}^e h_4 \end{bmatrix}$$

\mathbf{I} is the 3×3 identity matrix.

- ${}^e\mathbf{q}^i(t) = \begin{bmatrix} {}^e d^i(t) \\ {}^e \theta_1^i(t) \\ {}^e \theta_2^i(t) \end{bmatrix}$ is the vector of generalized displacements of node i .

Elementary stiffness matrix

$${}^e\mathbf{K} = \int_{{}^e\Omega} {}^e\mathbf{B}^T \bar{\mathbf{C}} {}^e\mathbf{B} d\Omega$$

- Elementary deformation matrix (5×12):

$${}^e\mathbf{B} = \nabla_r {}^e\mathbf{H} = \left[\nabla_r {}^e h_1 \quad \vdots \quad \dots \quad \vdots \quad \nabla_r {}^e h_4 \right] = \left[\begin{array}{c|c|c} \nabla_b {}^e h_1 & \dots & \nabla_b {}^e h_4 \\ \nabla_s {}^e h_1 & \dots & \nabla_s {}^e h_4 \end{array} \right] = \left[\begin{array}{c} \nabla_b {}^e \mathbf{H} \\ \nabla_s {}^e \mathbf{H} \end{array} \right]$$

- Bending strain-displacement matrix: ${}^e\mathbf{B}_b = \nabla_b {}^e\mathbf{H}$.
- Shear strain-displacement matrix: ${}^e\mathbf{B}_s = \nabla_s {}^e\mathbf{H}$.

- Constitutive matrix (5×5):

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}}_b & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{C}}_s \end{bmatrix}.$$

Elementary deformation matrix

- Bending strain-displacement matrix:

$${}^e\mathbf{B}_b = \nabla_b {}^e\mathbf{H} = \left[\begin{array}{ccc|ccc} 0 & 0 & \partial_x {}^e h_1 & \dots & 0 & 0 & \partial_x {}^e h_4 \\ 0 & -\partial_y {}^e h_1 & 0 & \dots & 0 & -\partial_y {}^e h_4 & 0 \\ 0 & -\partial_x {}^e h_1 & \partial_y {}^e h_1 & \dots & 0 & -\partial_x {}^e h_4 & \partial_y {}^a h_4 \end{array} \right].$$

- Shear strain-displacement matrix:

$${}^e\mathbf{B}_s = \nabla_s {}^e\mathbf{H} = \left[\begin{array}{ccc|ccc} \partial_x {}^e h_1 & 0 & {}^e h_1 & \dots & \partial_x {}^e h_4 & 0 & {}^a h_4 \\ \partial_y {}^e h_1 & -{}^e h_1 & 0 & \dots & \partial_y {}^e h_4 & -{}^e h_4 & 0 \end{array} \right].$$

Elementary stiffness matrix

The elementary stiffness matrix is split in two:

$$\begin{aligned} {}^e\mathbf{K} &= \int_{e\Omega} \begin{bmatrix} {}^e\mathbf{B}_b \\ {}^e\mathbf{B}_s \end{bmatrix}^T \begin{bmatrix} \overline{\mathbf{C}}_b & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{C}}_s \end{bmatrix} \begin{bmatrix} {}^e\mathbf{B}_b \\ {}^e\mathbf{B}_s \end{bmatrix} d\Omega \\ &= \underbrace{\int_{e\Omega} {}^e\mathbf{B}_b^T \overline{\mathbf{C}}_b {}^e\mathbf{B}_b d\Omega}_{{}^e\mathbf{K}_b} + \underbrace{\int_{e\Omega} {}^e\mathbf{B}_s^T \overline{\mathbf{C}}_s {}^e\mathbf{B}_s d\Omega}_{{}^e\mathbf{K}_s} \end{aligned}$$

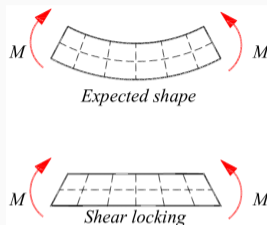
- bending stiffness matrix:

$${}^e\mathbf{K}_b = \int_{e\Omega} {}^e\mathbf{B}_b^T \overline{\mathbf{C}}_b {}^e\mathbf{B}_b d\Omega,$$

- shear stiffness matrix:

$${}^e\mathbf{K}_s = \int_{e\Omega} {}^e\mathbf{B}_s^T \overline{\mathbf{C}}_s {}^e\mathbf{B}_s d\Omega.$$

Selective integration to avoid shear locking



- Reissner-Mindlin theory has demonstrated to suffer from *shear locking*: as the thickness of the plate is reduced, the element becomes over-stiff and the computed displacements are much smaller than the analytical solution.
- The simplest remedy to this numerical behavior is to perform reduced integration of the shear component (*selective integration*).
- For instance, if bilinear elements are used, then: **2×2 Gauss integration (exact) for ${}^e\mathbf{K}_b$ and single point quadrature (reduced) for ${}^e\mathbf{K}_s$.**

- Elementary mass matrix (12×12):

$${}^e\mathbf{M} = \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{I} {}^e\mathbf{H} d\Omega.$$

- Elementary applied forces vector (12×1):

$${}^e\mathbf{r}(t) = \int_{{}^e\Omega} {}^e\mathbf{H}^T \mathbf{f} d\Omega.$$

Once the nodal generalized displacements ${}^e\mathbf{q}^i$ is computed out stresses can be recovered from constitutive equations as:

$$\boldsymbol{\sigma}_b^h = \mathbf{C}_b \boldsymbol{\varepsilon}_b^h = z \mathbf{C}_b \nabla_b {}^e\mathbf{H}^e \mathbf{q} = z \mathbf{C}_b {}^e\mathbf{B}_b {}^e\mathbf{q},$$

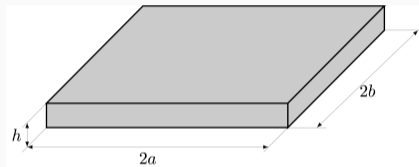
$$\boldsymbol{\sigma}_s^h = \mathbf{C}_s \boldsymbol{\varepsilon}_s^h = z \mathbf{C}_s \nabla_s {}^e\mathbf{H}^e \mathbf{q} = z \mathbf{C}_s {}^e\mathbf{B}_s {}^e\mathbf{q}.$$

- Bending stresses are linear through the plate thickness.
- On the contrary, shear stresses are constant through the thickness, thus they are independent on z .

Example: modal analysis of a simply supported thick plate

Example: Simply supported square isotropic plate

- 1 Approximate the first natural frequency of a $(1 \times 1 \times 0.1)$ plate, which is simply supported on all four edges.

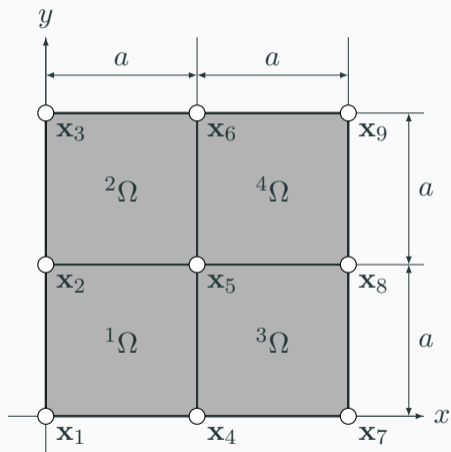


- 2 Compare the results with the analytical solution (as a function of h):

$$\omega_{1,1}^{\text{exact}}(h) = 10\pi h \sqrt{\frac{70}{4\pi^2 h^2 + 7}} \text{ Hz.}$$

Notice that this formula is only valid for the following choice of the plate geometry and materials.

Discretization with 4 HTK elements



- $2a = 1$ length
- $2a = 1$ height
- $h = 0.1$ thickness
- $E = 10920$ Young's modulus
- $\nu = 0.3$ Poisson's ratio
- $\rho = 1$ material density
- $k = 5/6$ shear correction coefficient

The values for ρ and E is only a practical convenience to obtain non-dimensional flexural rigidity of the plate:

$$D = \frac{Eh^3}{12(1-\nu^2)} = 1.$$

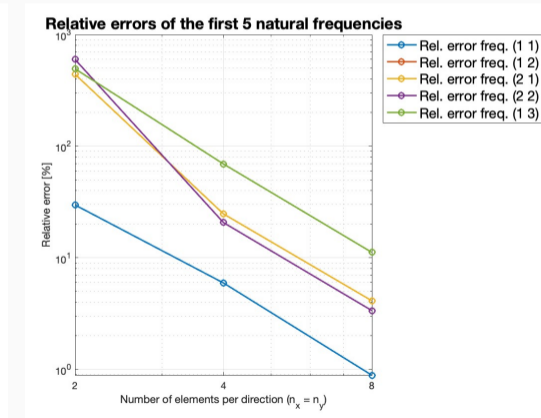
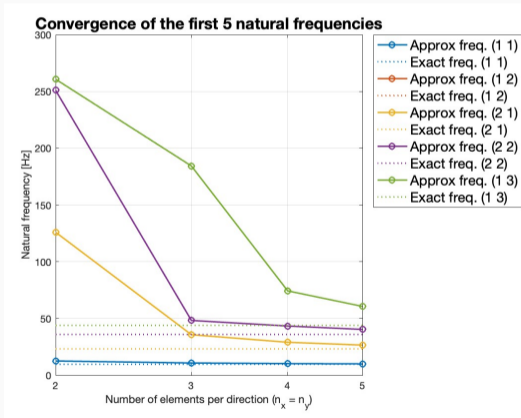
Frequencies approximation for square plate with HTK elements

Comparison of exact and approximate natural frequencies with a mesh of 2×2 HTK elements for a thick SSSS square plate.

m	n	Exact Freq. [Hz]	Approx. Freq. [Hz]	Rel. Error (%)
1	1	9.6658	12.543	29.768
1	2	23.251	125.96	441.75
2	1	23.251	125.96	441.75
2	2	35.895	250.98	599.19
1	3	43.871	260.74	494.33
3	1	43.871	284.72	549
2	3	55.239	284.72	415.44

Frequencies approximation for square plate with HTK elements

In these figures, we present the convergence and relative errors of the first five natural frequencies of a square plate, obtained using HTK elements for different mesh refinements.



Modal patterns

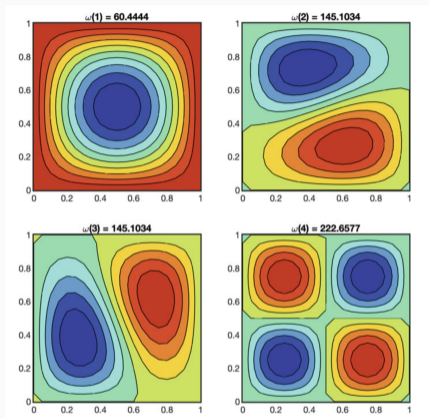


Figure 2: Modes of vibration for a SSSS plate with $h/a = 0.1$, using 20×20 bilinear elements.

(Credit: Ferreira, Fantuzzi - MATLAB Codes for Finite Element Analysis)