

# Strong and weak forms

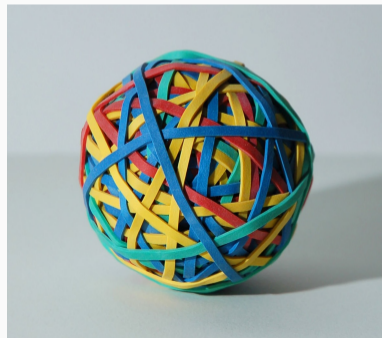
## Linear elastodynamics

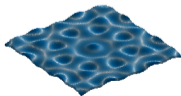
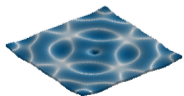
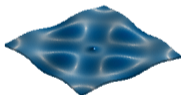
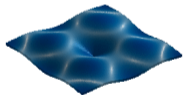
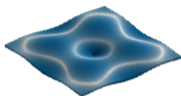
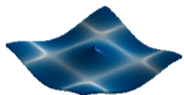
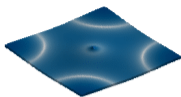
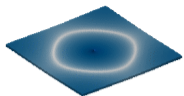
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ME473 Dynamic finite element analysis of structures

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2025





**Linear:** infinitesimally small deformations relative to the solid's size.

**Elasticity:** deformable material that returns to its original shape and size when the forces causing the deformation are removed.

**Dynamics:** study bodies in motion under the influence of time-dependent loads applied to them.

## Summary

- Underlying hypothesis and notations
- Equilibrium equations
- Strong formulation of equilibrium equations
- Weak formulation of equilibrium equations
- Example 1: longitudinal vibrations of bars
- Example 2: transversal vibrations of beams

## Recommended readings

- ① Gmür, Dynamique des structures (§2.1 and §2.2)
- ② Gmür, Méthode des éléments finis (§5.1 and §5.2)
- ③ Neto et al., Engineering Computation of Structures (§1.1, §1.3 and §2.1)

## Underlying hypothesis and notations

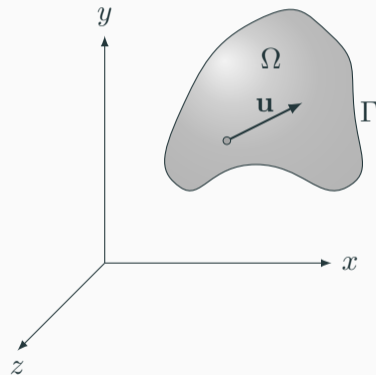
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## Hypothesis and notations

- Inertial orthonormal reference frame  $O(x, y, z)$ . Sometimes will use  $O(x_1, x_2, x_3)$ .
- Continuous three-dimensional deformable finite (compact) **body** with volume  $\Omega$  and surface  $\Gamma$ .
- **Material**: continuous and homogenous.
- **Deformations**: small (proportional to stress).
- Continuous and derivable time-dependent (*unknown*) **displacement field**:

$$\mathbf{u}(\mathbf{x}, t) = \begin{bmatrix} u_1(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) \\ u_3(\mathbf{x}, t) \end{bmatrix}$$

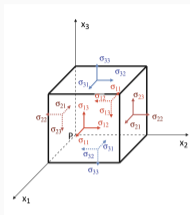
where  $\mathbf{x} \in \bar{\Omega}$  and  $t \in [0, T]$ .



# Stress and strain tensors

## Stress tensor

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \begin{bmatrix} \sigma_{11}(\mathbf{x}, t) \\ \sigma_{22}(\mathbf{x}, t) \\ \sigma_{33}(\mathbf{x}, t) \\ \sigma_{23}(\mathbf{x}, t) \\ \sigma_{31}(\mathbf{x}, t) \\ \sigma_{12}(\mathbf{x}, t) \end{bmatrix}$$



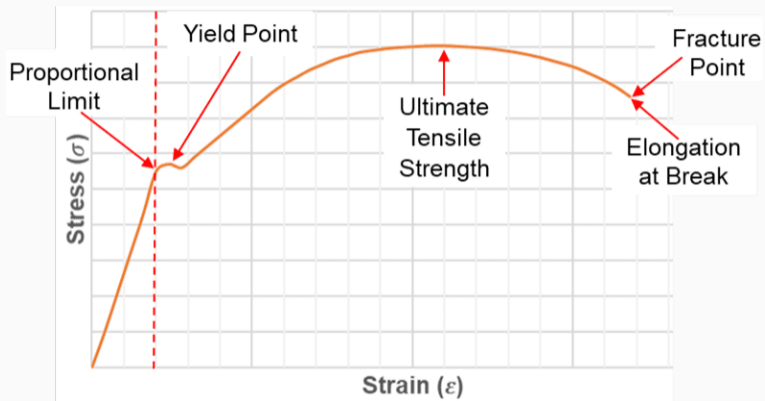
*Credit: Neto et al.*

## Strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{x}, t) = \begin{bmatrix} \varepsilon_{11}(\mathbf{x}, t) \\ \varepsilon_{22}(\mathbf{x}, t) \\ \varepsilon_{33}(\mathbf{x}, t) \\ 2\varepsilon_{23}(\mathbf{x}, t) \\ 2\varepsilon_{31}(\mathbf{x}, t) \\ 2\varepsilon_{12}(\mathbf{x}, t) \end{bmatrix}$$

- **Stress** expresses the internal forces that neighboring particles of a material exert on each other (SI units of Pascal.)
- **Strain** is defined as relative deformation, compared to a reference position configuration (SI units of meter per meter.)
- First index specifies the normal of the surface on which the stress/strain is acting, the second index specifies the direction of the stress/strain.
- if  $i \neq j$  then  $\sigma_{ij} = \tau_{ij}$  (shear stress) and  $2\varepsilon_{ij} = \gamma_{ij}$  (engineering shear strain).
- Symmetric tensors:  $\tau_{ij} = \tau_{ji}$  (Cauchy's stress theorem) and  $\gamma_{ij} = \gamma_{ji}$ .

# The stress-strain curve



Credit: *Fidelis*

## Generalised Hooke's law

The constitutive stress-strain relationship of linear elasticity is given by the Hooke's law:

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{C} \boldsymbol{\varepsilon}(\mathbf{x}, t)$$

or equivalently

$$\boldsymbol{\varepsilon}(\mathbf{x}, t) = \mathbf{S} \boldsymbol{\sigma}(\mathbf{x}, t)$$

where  $\mathbf{C}$  is called the *stiffness material matrix* and  $\mathbf{S} = \mathbf{C}^{-1}$ .

# 1 - Anisotropic materials

**No symmetry:** these materials have properties that vary with direction and do not have symmetry. Their stiffness differ depending on the axis of measurement.

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix}$$

- Most of the elements are non-zero, the matrix  $\mathbf{S}$  is considered dense.
- 21 independent constants (symmetry).

## 2 - Orthotropic materials

**Three planes of symmetry:** a special case of anisotropic materials where properties vary along three mutually perpendicular directions. These materials have three different principal material properties along three axes.

$$\mathbf{S} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}$$

- $E_i$  is the Young's modulus along axis  $Ox_i$ .
- $G_{ij}$  is the shear modulus in direction  $Ox_i$  on the plane whose normal is  $Ox_j$ .
- $\nu_{ij}$  is the Poisson's ratio that corresponds to a contraction in direction  $Ox_j$  when an extension is applied in direction  $Ox_i$ .

### 3 - Isotropic materials

**Infinite planes of symmetry:** these materials have identical properties in all directions. Their mechanical and physical properties do not change regardless of the direction in which they are measured.

$$\mathbf{S} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix}$$

- $E$  is the Young's modulus,  $G$  is the shear modulus, and  $\nu$  is the Poisson's ratio.
- Only two independent constants since  $E = 2G(1 + \nu)$ .

# Strain-displacement relationship

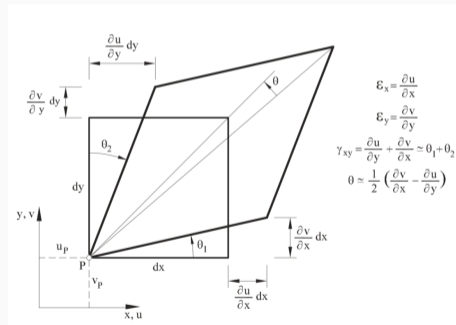
- Both normal strain and shear strain can be interpreted as measures of displacement change per unit length.
- Strain is obtained by derivatives of displacements for small deformations.
- The strain-displacement relation can be written in the three equivalent forms as follows:

## Kinematics

a)  $\boldsymbol{\varepsilon}(\mathbf{x}, t) = \nabla \mathbf{u}(\mathbf{x}, t)$

b) 
$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ 0 & \partial_z & \partial_y \\ \partial_z & 0 & \partial_x \\ \partial_y & \partial_x & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

c)  $\varepsilon_{ii} = \partial_{x_i} u_i$  and  $2\varepsilon_{ij} = \partial_{x_i} u_j + \partial_{x_j} u_i$



Credit: Neto et al.

## Equilibrium equations

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# Differential equations of motion

- Consider an infinitely small cube subjected to a force represented by the vector:

$$\mathbf{f}(\mathbf{x}, t) = \begin{bmatrix} f_1(\mathbf{x}, t) \\ f_2(\mathbf{x}, t) \\ f_3(\mathbf{x}, t) \end{bmatrix}.$$

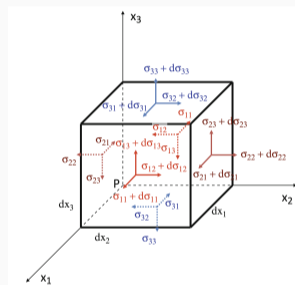
- Stress is not uniform. The variation of the stress between two opposite sides is linear and

$$d\sigma_{ij} = \partial_{x_k} \sigma_{ij} dx_k$$

- Newton 2nd law of motion taking into account internal forces  $\mathbf{f}^{int}$  is

$$\mathbf{f}^{int}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho \ddot{\mathbf{u}}(\mathbf{x}, t) dx_1 dx_2 dx_3$$

$\rho$  is the material density,  $\ddot{\mathbf{u}} = \partial_{tt} \mathbf{u}$  is the acceleration,  $dx_1 dx_2 dx_3$  is the cube's volume.



*Credit: Neto et al.*

- The equilibrium equation of elastodynamics in matrix form is:

$$\nabla^T \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho \ddot{\mathbf{u}}(\mathbf{x}, t)$$

- Using the strain-displacement relation we can write it in terms of the displacement field:

$$\nabla^T \mathbf{C} \nabla \mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho \ddot{\mathbf{u}}(\mathbf{x}, t)$$

## Boundary conditions

Boundary conditions are appointed for each point on the solid surface  $\Gamma = \Gamma_u \cup \Gamma_\sigma$ .

### ■ Prescribed displacement:

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_u \times ]0, T[$$

where  $\hat{\mathbf{u}} = [\hat{u}_1, \hat{u}_2, \hat{u}_3]^T$  is the given displacement prescribed on  $\Gamma_u$ .

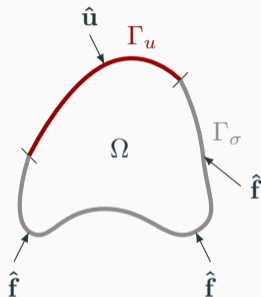
### ■ Prescribed surface force:

$$\mathbf{N}^T \boldsymbol{\sigma} = \hat{\mathbf{f}} \quad \text{on } \Gamma_\sigma \times ]0, T[$$

$\hat{\mathbf{f}} = [\hat{f}_1, \hat{f}_2, \hat{f}_3]^T$  is the given surface load prescribed on  $\Gamma_\sigma$  and

$$\mathbf{N}^T = \begin{bmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{bmatrix}$$

$n_i$  being the *direction cosines* for the outward-pointing normal  $\mathbf{n} = [n_1, n_2, n_3]^T$  to the boundary  $\Gamma_\sigma$ .



# Initial conditions

The initial conditions are set at  $t = 0$ :

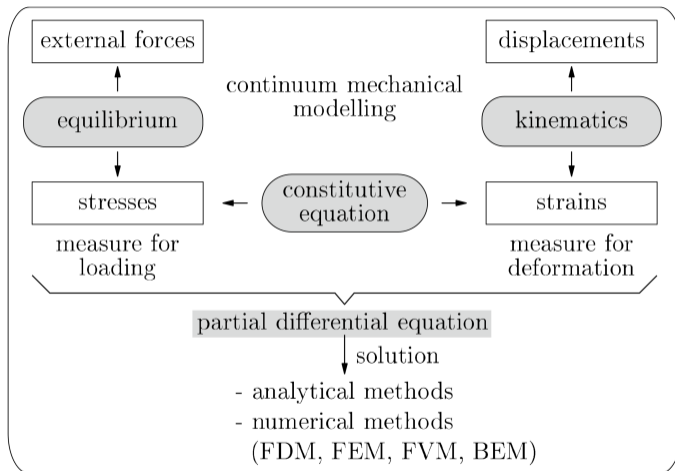
- **Imposed initial displacement field:**

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega$$

- **Imposed initial velocity field:**

$$\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega$$

# Continuum mechanical modelling



Credit: A. Öchsner - PDE for classical structural members

## Strong formulation of equilibrium equations

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## Strong form of elastodynamics

Given a deformable solid  $\Omega$  with boundary  $\Gamma$ , and

- stiffness material matrix  $\mathbf{C}$  and material density  $\rho$ ,
- vector of body load  $\mathbf{f}$  applied on  $\Omega$ ,
- prescribed boundary displacement  $\hat{\mathbf{u}}$  on  $\Gamma_u$  and surface load  $\hat{\mathbf{f}}$  on  $\Gamma_\sigma$ ,
- prescribed initial (at  $t = 0$ ) displacement  $\mathbf{u}_0$  and initial velocity  $\mathbf{v}_0$ .

find the displacement  $\mathbf{u} \in C^2(\bar{\Omega} \times [0, T], \mathbb{R}^3)$  such that

$$\left\{ \begin{array}{ll} \nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}} & \text{on } \Omega \times ]0, T[ \\ \mathbf{u} = \hat{\mathbf{u}} & \text{on } \Gamma_u \times ]0, T[ \\ \mathbf{N}^T \mathbf{C} \nabla \mathbf{u} = \hat{\mathbf{f}} & \text{on } \Gamma_\sigma \times ]0, T[ \\ \mathbf{u} = \mathbf{u}_0 & \text{for } t = 0 \\ \dot{\mathbf{u}} = \mathbf{v}_0 & \text{for } t = 0 \end{array} \right.$$

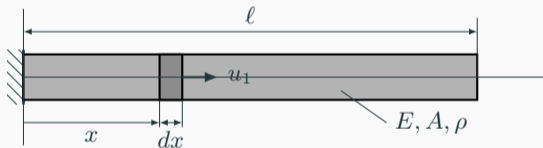
## Example 1: longitudinal vibrations of bars

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# Strong form for longitudinal vibrations of bars

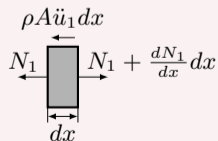
## Kinematic assumptions:

- The bar cross-section is infinitely rigid in its own plane, remaining plane after deformation.
- Loads, which are uniform in every cross-section, can only be applied axially.
- The influence on the axial movement of the bar of lateral displacements due to the Poisson effect is negligible ( $\sigma_{22} = \sigma_{33} = 0$ ). Thus the bar can undergo only axial stress  $\sigma_{11}$ , which is uniform in every cross-section.



- $A$  cross-sectional area
- $E$  Young's modulus
- $\rho$  material density
- $\ell$  length

- $x$  axial coordinate
- $u_1(x, t)$  axial displacement
- $N_1(x, t)$  normal internal force
- $\sigma_{11}(x, t)$  normal stress
- $\varepsilon_{11}(x, t)$  normal strain



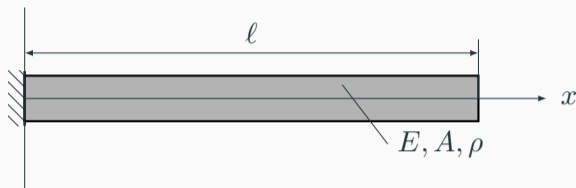
Equilibrium equation:

$$N_1 + \partial_x N_1 dx - N_1 = \rho A \ddot{u}_1 dx$$

Stress-strain-displacement relation:

$$N_1 = A \sigma_{11} = EA \varepsilon_{11} = EA \partial_x u_1$$

## Strong form for longitudinal vibrations of bars



Find  $u_1 \in C^2([0, \ell] \times [0, T])$  such that

$$\partial_x(EA\partial_x u_1) = \rho A \ddot{u}_1$$

boundary conditions:

$$u_1(0, t) = 0$$

$$EA\partial_x u_1(\ell, t) = 0$$

initial conditions:

$$u_1(x, 0) = u_0(x)$$

$$\dot{u}_1(x, 0) = v_0(x)$$

## Disclaimer - exact or closed-form solution

The strong form for longitudinal vibrations of a bar admits an exact solution in the following decoupled form

$$u_1(x, t) = \sum_{k=1}^{\infty} v_k(x) \phi_k(t)$$

where

$$v_k(x) = \sin\left(\frac{\pi(2k+1)}{2\ell}x\right)$$
$$\phi_k(t) = a_k \sin(\omega_k t) + b_k \cos(\omega_k t)$$

$$\omega_k = \frac{\pi(2k+1)}{2\ell} \sqrt{\frac{E}{\rho}}$$

- $a_k$  and  $b_k$  depends respectively on the initial conditions  $u_0$  and  $v_0$ .
- $k$  represents a mode of vibration ( $k = 1$  first or *fundamental* mode).

# Weak formulation of equilibrium equations

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## Road to weak formulation

- Strong formulations lead to strong solutions in the sense that they require strong continuity in the field variables.
  - The weak form is often expressed as an integral equation that requires weaker continuity on the variables.
  - The weak form can be obtained using the *Virtual Work Principle*.
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The weak form is obtained through the following steps:

- ① Multiply each differential equation by an appropriate arbitrary function.
- ② Integrate over the space domain of the problem.
- ③ Reduce the order of the involved derivatives using the divergence theorem.
- ④ Apply the boundary conditions reasonably.

# Derivation of weak formulation

- ① Introduce an admissible **virtual displacement**:

$$\delta \mathbf{u}(\mathbf{x}) = \begin{bmatrix} \delta u_1(\mathbf{x}) \\ \delta u_2(\mathbf{x}) \\ \delta u_3(\mathbf{x}) \end{bmatrix}$$

- ② Multiply the differential equation in the strong form by  $\delta \mathbf{u}^T$  and integrate it over  $\Omega$ :

$$\int_{\Omega} \delta \mathbf{u}^T (\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f}) d\Omega = \int_{\Omega} \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} d\Omega$$

- ③ Apply the divergence theorem to the first term:

$$- \int_{\Omega} (\nabla \delta \mathbf{u})^T \mathbf{C} \nabla \mathbf{u} d\Omega + \int_{\Gamma} \delta \mathbf{u}^T \mathbf{N}^T \mathbf{C} \nabla \mathbf{u} d\Gamma + \int_{\Omega} \delta \mathbf{u}^T \mathbf{f} d\Omega = \int_{\Omega} \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} d\Omega$$

- ④ Use the boundary conditions:  $\mathbf{N}^T \mathbf{C} \nabla \mathbf{u} = \hat{\mathbf{f}}$  on  $\Gamma_{\sigma}$  and *impose*  $\delta \mathbf{u} = \mathbf{0}$  on  $\Gamma_u$ :

$$- \int_{\Omega} (\nabla \delta \mathbf{u})^T \mathbf{C} \nabla \mathbf{u} d\Omega + \int_{\Gamma_{\sigma}} \delta \mathbf{u}^T \hat{\mathbf{f}} d\Gamma + \int_{\Omega} \delta \mathbf{u}^T \mathbf{f} d\Omega = \int_{\Omega} \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} d\Omega$$

To ensure the integrals remain finite, we impose that:

$$\mathbf{u} \in \mathcal{U} \quad \text{and} \quad \delta \mathbf{u} \in \mathcal{V}$$

$$\mathcal{U} = \{ \mathbf{u}(\cdot, t) \in H^1(\Omega, \mathbb{R}^3) \mid \mathbf{u}(\cdot, t) = \hat{\mathbf{u}} \text{ on } \Gamma_u \forall t \in ]0, T[ \}$$

$$\mathcal{V} = \{ \mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u \}$$

- Recall that  $H^1(\Omega)$  is the Sobolev space defined as

$$H^1(\Omega) = \{ \mathbf{w} \in L^2(\Omega, \mathbb{R}^3) \mid \int_{\Omega} (\nabla \mathbf{w})^T \nabla \mathbf{w} \, d\Omega < \infty \}.$$

- Note that the difference between spaces  $\mathcal{U}$  and  $\mathcal{V}$  is that only the functions in  $\mathcal{U}$  are time-dependent.
- The spaces are designed to incorporate *only* the displacement boundary condition on  $\Gamma_u$ .

## Weak form of elastodynamics

Given  $\Omega$ ,  $\Gamma$ ,  $\mathbf{C}$ ,  $\rho$ ,  $\mathbf{f}$ ,  $\hat{\mathbf{u}}$ ,  $\hat{\mathbf{f}}$ ,  $\mathbf{u}_0$ ,  $\mathbf{v}_0$  as in the previous slide, find the displacement  $\mathbf{u} \in \mathcal{U}$  such that for any virtual displacement  $\delta\mathbf{u} \in \mathcal{V}$  we have

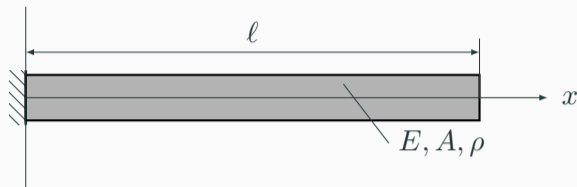
$$\int_{\Omega} (\nabla \delta\mathbf{u})^T \mathbf{C} \nabla \mathbf{u} \, d\Omega + \int_{\Omega} \rho \delta\mathbf{u}^T \ddot{\mathbf{u}} \, d\Omega = \int_{\Gamma_{\sigma}} \delta\mathbf{u}^T \hat{\mathbf{f}} \, d\Gamma + \int_{\Omega} \delta\mathbf{u}^T \mathbf{f} \, d\Omega,$$

$$\left. \begin{array}{l} \mathbf{u} = \mathbf{u}_0 \quad \text{for } t = 0 \\ \dot{\mathbf{u}} = \mathbf{v}_0 \quad \text{for } t = 0 \end{array} \right\} \text{Initial conditions}$$

## Example 1: longitudinal vibrations of bars

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## Strong form for longitudinal vibrations of bars (reminder)



- $A$  cross-sectional area
- $E$  Young's modulus (isotropic bar)
- $\rho$  material density
- $l$  length
- $x$  axial coordinate
- $u_1(x, t)$  axial displacement

Find  $u_1 \in C^2([0, \ell] \times [0, T])$  such that

$$\partial_x(EA\partial_x u_1) = \rho A \ddot{u}_1$$

boundary conditions:

$$\begin{aligned} u_1(0, t) &= 0 \\ EA\partial_x u_1(\ell, t) &= 0 \end{aligned}$$

initial conditions:

$$\begin{aligned} u_1(x, 0) &= u_0(x) \\ \dot{u}_1(x, 0) &= v_0(x) \end{aligned}$$

## Derivation of weak form for longitudinal vibrations of bars

- 1 Define the virtual displacement  $\delta u_1$  so that  $\delta u_1 \in H^1(]0, \ell[)$  and  $\delta u_1(0) = 0$ .
- 2 Multiply the differential equation by the virtual displacement and integrate it over the interval  $]0, \ell[$ .

$$\int_0^\ell \partial_x (EA \partial_x u_1) \delta u_1 \, dx = \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \, dx.$$

- 3 Use the integration by parts formula on the left hand side:

$$- \int_0^\ell EA \partial_x u_1 \partial_x (\delta u_1) \, dx + [EA \partial_x u_1 \delta u_1]_0^\ell = \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \, dx.$$

- 4 Make use of the boundary condition  $EA \partial_x u_1(\ell, t) = 0$  and  $\delta u_1(0) = 0$  to simplify

$$- \int_0^\ell EA \partial_x u_1 \partial_x (\delta u_1) \, dx = \int_0^\ell \rho A \ddot{u}_1 \delta u_1 \, dx.$$

## Weak form for longitudinal vibrations of bars

Find  $u_1 \in \mathcal{U}$  such that for any  $\delta u_1 \in \mathcal{V}$  we have

$$\int_0^\ell EA \partial_x u_1 \partial_x (\delta u_1) dx + \int_0^\ell \rho A \ddot{u}_1 \delta u_1 dx = 0,$$

$$\left. \begin{array}{l} u_1 = u_0 \quad \text{for } t = 0 \\ \dot{u}_1 = v_0 \quad \text{for } t = 0 \end{array} \right\} \text{Initial conditions}$$

Recall:

$$\mathcal{U} = \{u_1(\cdot, t) \in H^1(]0, \ell[) \mid u_1(0, t) = 0 \forall t \in ]0, T[\}$$

$$\mathcal{V} = \{\delta u_1 \in H^1(]0, \ell[) \mid \delta u_1(0) = 0\}$$

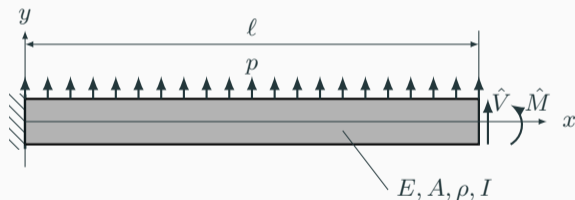
## Example 2: transversal vibrations of beams

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# Transversal vibrations of beams

## Kinematic assumptions:

- The analysis will be restricted to the dynamic behavior of the beam in the  $O(x, y)$  plane.
- A normal cross-section to the neutral fiber remains planar after deformation, but not necessarily orthogonal to it
- Shear strain  $\varepsilon_{12}$  are taken into account (Timoshenko or thick beam).



### Model parameters:

- $A$  cross-sectional area
- $E$  Young's modulus
- $\rho$  material density
- $I$  moment of inertia
- $\ell$  length

### Loads:

- $\hat{M}$  bending moment at free end
- $\hat{V}$  shear force at free end
- $p$  distributed transversal load

## Variables

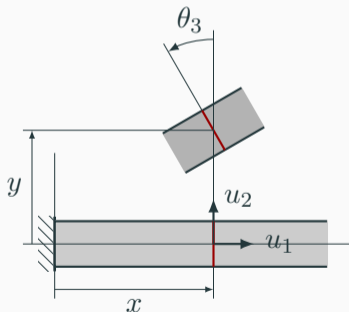
- $u_1(x, y, t)$  axial displacement
- $u_2(x, y, t)$  transversal displacement

## Strain-displacement relationships

- Introduce an auxiliary variable  $\theta_3(x, t)$  representing the total rotation of the section around the  $Oz$  axis:

$$u_1(x, y, t) = -y\theta_3(x, t).$$

- Assume  $u_2(x, y, t) = u_2(x, t)$  i.e. independent of  $y$ .



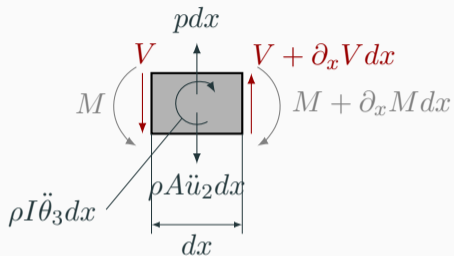
**Normal strain :**

$$\varepsilon_{11} = \partial_x u_1 = -y\partial_x \theta_3$$

**Shear strain :**

$$\varepsilon_{12} = \partial_x u_2 + \partial_y u_1 = \partial_x u_2 - \theta_3$$

# Equilibrium equations



Shear force  $V$  and bending moment  $M$  satisfy the following equations of motion:

$$\begin{aligned}\partial_x V + p &= \rho A \ddot{u}_2 \\ -\partial_x M + V &= \rho I \ddot{\theta}_3\end{aligned}$$

Since normal and shear stresses are  $\sigma_{11} = E\varepsilon_{11}$  and  $\sigma_{12} = G\varepsilon_{12}$  where  $E$  is the Young's modulus and  $G$  is the shear modulus, then

$$\begin{aligned}V &= \int_A \sigma_{12} dA = kGA\varepsilon_{12} = kGA(\partial_x u_2 - \theta_3) \\ M &= \int_A y\sigma_{11} dA = \int_A yE\varepsilon_{11} dA = -EI\partial_x \theta_3\end{aligned}$$

where the constant  $k$  is the shear correction factor (parabolic distribution of shear stresses).

## Strong form for transversal vibrations of beams

The strong form for transversal vibrations of beams consists of finding the functions  $u_2 \in C^2([0, \ell] \times [0, T])$  and  $\theta_3 \in C^2([0, \ell] \times [0, T])$  such that the following equilibrium equations, boundary and initial conditions are satisfied.

### Equilibrium equations

$$\begin{aligned}\partial_x(kGA(\partial_x u_2 - \theta_3)) + p &= \rho A \ddot{u}_2 \\ \partial_x(EI\partial_x \theta_3) + kGA(\partial_x u_2 - \theta_3) &= \rho I \ddot{\theta}_3\end{aligned}$$

In matrix form:

$$\underbrace{\begin{bmatrix} \partial_x & 0 \\ 1 & \partial_x \end{bmatrix}}_{\nabla_\sigma^T} \underbrace{\begin{bmatrix} kGA & 0 \\ 0 & EI \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \partial_x & -1 \\ 0 & \partial_x \end{bmatrix}}_{\nabla_u} \underbrace{\begin{bmatrix} u_2 \\ \theta_3 \end{bmatrix}}_{\mathbf{u}} + \underbrace{\begin{bmatrix} p \\ 0 \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{u}_2 \\ \ddot{\theta}_3 \end{bmatrix}}_{\ddot{\mathbf{u}}}$$

$$\nabla_\sigma^T \mathbf{C} \nabla_u \mathbf{u} + \mathbf{f} = \mathbf{M} \ddot{\mathbf{u}}$$

## Boundary conditions

$$u_2(0, t) = 0$$

$$\theta_3(0, t) = 0$$

$$kGA(\partial_x u_2(\ell, t) - \theta_3(\ell, t)) = \hat{V}$$

$$EI\partial_x \theta_3(\ell, t) = \hat{M}$$

In matrix form: let  $\hat{\mathbf{f}} = [\hat{V}, \hat{M}]^T$  then

$$\mathbf{u}(0, t) = \mathbf{0}$$

$$\mathbf{C}\nabla_u \mathbf{u}(\ell, t) = \hat{\mathbf{f}}$$

## Initial conditions

$$u_2(x, 0) = u_0(x)$$

$$\theta_3(x, 0) = \theta_0(x)$$

$$\dot{u}_2(x, 0) = v_0(x)$$

$$\dot{\theta}_3(x, 0) = \phi_0(x)$$

In matrix form:  $\mathbf{u}_0 = [u_0, \theta_0]^T$  and  $\mathbf{v}_0 = [v_0, \phi_0]^T$  then

$$\mathbf{u}(x, 0) = \mathbf{u}_0$$

$$\dot{\mathbf{u}}(x, 0) = \mathbf{v}_0$$