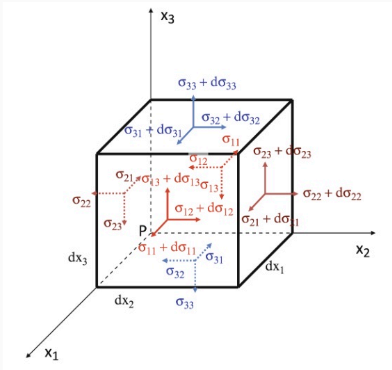
Differential equations of motion

- Consider an infinitely small cube subjected to a force represented by the vector:

$$\mathbf{f}(\mathbf{x},t) = \begin{pmatrix} f_1(\mathbf{x},t) \\ f_2(\mathbf{x},t) \\ f_3(\mathbf{x},t) \end{pmatrix}.$$



Credit: [N]

- Stress is not uniform. The variation of the stress between two opposite sides is linear and

$$d\sigma_{ij} = \partial_{x_k}\sigma_{ij}\, dx_k$$

- Newton 2nd law of motion taking into account internal forces $\mathbf{f}^i$ is

$$\mathbf{f}^i(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t)\, dxdydz$$

where $\rho(\mathbf{x})$ is the material density, $\ddot{\mathbf{u}} = \partial_{tt}\mathbf{u}$ is the acceleration, $dxdydz$ is the volume of the cube.

Equilibrium of forces in the x_1 direction:

$$(\cancel{\sigma_{11}} + d\sigma_{11} - \cancel{\sigma_{11}})\, dx_2\, dx_3 + (\cancel{\sigma_{21}} + d\sigma_{21} - \cancel{\sigma_{21}})\, dx_1 dx_3 +$$

$$+ (\cancel{\sigma_{31}} + d\sigma_{31} - \cancel{\sigma_{31}})\, dx_1\, dx_2 + f_1\, dx_1 dx_2 dx_3 = \rho\, \ddot{u}_1\, dx_1 dx_2 dx_3$$

After accounting ⊛, the latter equation is:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho\, \ddot{u}_1$$

Similarly, the equilibrium of forces in the x_2 and x_3 directions leads to:

$$\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} + f_j = \rho\, \ddot{u}_j \qquad (j = 1, 2, 3)$$

The equilibrium equations can be written in matrix form as

$$\nabla^T \sigma + f = \rho \ddot{u}$$

where
$$\nabla^T = \begin{pmatrix} \partial_x & 0 & 0 & 0 & \partial_z & \partial_y \\ 0 & \partial_y & 0 & \partial_z & 0 & \partial_x \\ 0 & 0 & \partial_z & \partial_y & \partial_x & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}.$$

Since $\sigma = C \varepsilon = C \nabla u$ we obtain

$$\boxed{\nabla^T C \nabla u + f = \rho \ddot{u}}$$

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❶ Introduce an admissible (avoid divergence of integral) **virtual displacement**:

$$\boldsymbol{\delta}\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \delta u_1(\mathbf{x}) \\ \delta u_2(\mathbf{x}) \\ \delta u_3(\mathbf{x}) \end{pmatrix}$$

❷ Multiply the differential equation by $\boldsymbol{\delta}\mathbf{u}^T$ and integrate it over the spatial domain Ω.

$$\int_\Omega \boldsymbol{\delta}\mathbf{u}^T (\nabla^T \mathbf{C} \nabla \mathbf{u} + \mathbf{f}) \, d\Omega = \int_\Omega \rho \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}} \, d\Omega$$

❸ Apply the divergence theorem to the first term:

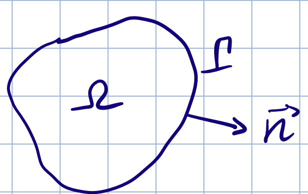
$$-\int_\Omega (\nabla \boldsymbol{\delta}\mathbf{u})^T \mathbf{C} \nabla \mathbf{u} \, d\Omega + \int_\Gamma \boldsymbol{\delta}\mathbf{u}^T \mathbf{N}^T \mathbf{C} \nabla \mathbf{u} \, d\Gamma + \int_\Omega \boldsymbol{\delta}\mathbf{u}^T \mathbf{f} \, d\Omega = \int_\Omega \rho \boldsymbol{\delta}\mathbf{u}^T \ddot{\mathbf{u}} \, d\Omega$$

Let's clarify how the term $\int_\Omega (\delta u)^T \nabla^T C \nabla u \, d\Omega$ is transformed into

$$-\int_\Omega (\nabla \delta u)^T C \nabla u \, d\Omega + \int_\Gamma (\delta u)^T N^T C \nabla u \, d\Gamma$$

by applying the divergence theorem:

<u>Div. theorem</u>: $\displaystyle\int_\Omega \partial_{x_i} u \, d\Omega = \int_\Gamma u \, n_i \, d\Gamma$



$\vec{n}$ outward pointing normal vector.

$$\int_\Omega (\delta u)^T \nabla^T C \nabla u \, d\Omega = \int_\Omega (\delta u)^T \nabla^T \sigma \, d\Omega$$

$$= \sum_{j=1}^{3} \int_\Omega \delta u_j \left(\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} \right) d\Omega$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \delta u_j \frac{\partial \sigma_{ij}}{\partial x_i} \, d\Omega$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \underbrace{\frac{\partial}{\partial x_i} (\delta u_j \, \sigma_{ij})}_{\text{apply div. thm.}} - \frac{\partial \delta u_j}{\partial x_i} \sigma_{ij} \, d\Omega$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Gamma \delta u_j \, \sigma_{ij} \, n_i \, d\Gamma - \int_\Omega \frac{\partial \delta u_j}{\partial x_i} \sigma_{ij} \, d\Omega$$

$$= \int_\Gamma (\delta u)^T N^T \sigma \, d\Gamma - \int_\Omega (\nabla \delta u)^T \sigma \, d\Omega$$

$$= \int_\Gamma (\delta u)^T N^T C \nabla u \, d\Gamma - \int_\Omega (\nabla \delta u)^T C \nabla u \, d\Omega$$