

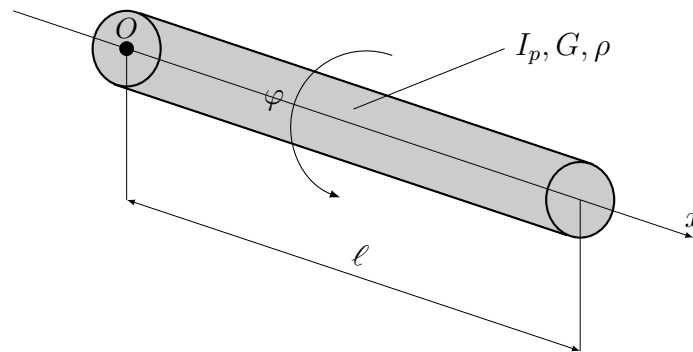
Mini-project 1

Natural frequencies and mode shapes of a uniform shaft

Project organization:

- Groups: 3 to 4 students
 - 10% of final grade
 - Pdf report: maximum 10 pages
 - Programming language: MATLAB
 - Submission: October 31, 2025
 - Total workload: 32h (approx. 8h per student)
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Consider a free-free cylindrical steel shaft of length $\ell = 2$ m, with a uniform cross-section of radius $r = 0.1$ m, having a polar moment of inertia I_p , shear modulus $G = 79$ GPa, and mass density $\rho = 7850$ kg/m³, as shown in the figure below.



The unknown $\varphi(x, t)$ represents the torsional vibration: the angular displacement (twist angle) at position $x \in [0, \ell]$ and time $t \in [0, T]$.

a) Given the strong formulation for the free vibration problem of the shaft

$$\begin{cases} \frac{\partial}{\partial x} (GI_p \frac{\partial}{\partial x} \varphi(x, t)) = \rho I_p \ddot{\varphi}(x, t) \\ GI_p \frac{\partial}{\partial x} \varphi(0, t) = 0 \\ GI_p \frac{\partial}{\partial x} \varphi(\ell, t) = 0 \end{cases}$$

Write the weak formulations, and discretize it using the local approach of the finite element method with n finite elements. Write the expressions for the element mass and stiffness matrices for:

- **Group A:** linear finite elements,
- **Group B:** quadratic finite elements,
- **Group C:** cubic finite elements.

- b) Determine the first 5 approximate natural frequencies and mode shapes of the shaft for $n = 1, 2, 4, 8,$ and 16 finite elements. Compare the results with the exact natural frequencies obtained from the analytical solution, and analyze the convergence of the numerical predictions.
- c) Discuss qualitatively the influence of the boundary conditions (free-free, clamped-free, clamped-clamped) on the approximated natural frequencies and mode shapes.