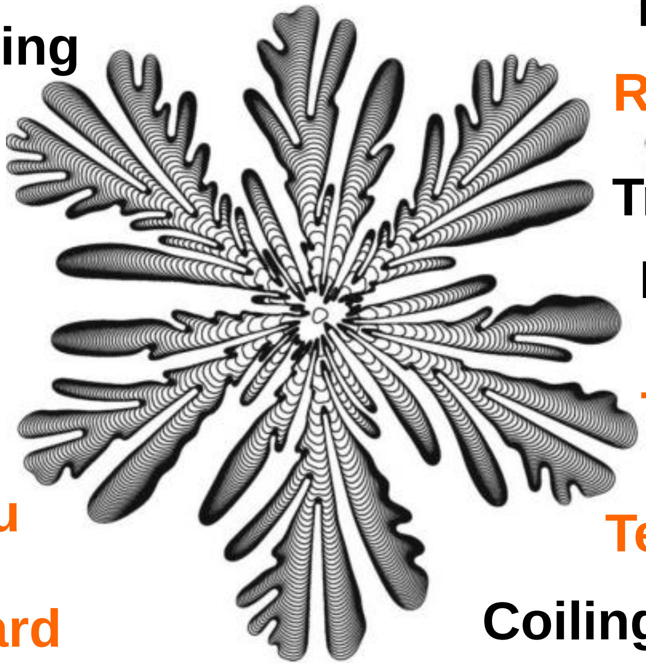
Most flows are unstable...



Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

Tearing instability

Rayleigh-Benard

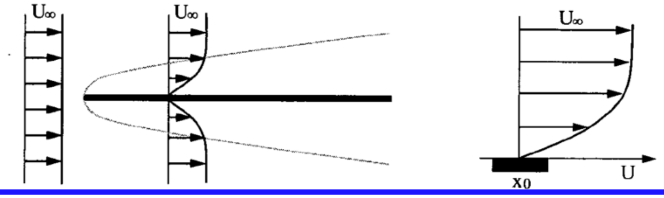
Coiling instability

Benard-Marangoni

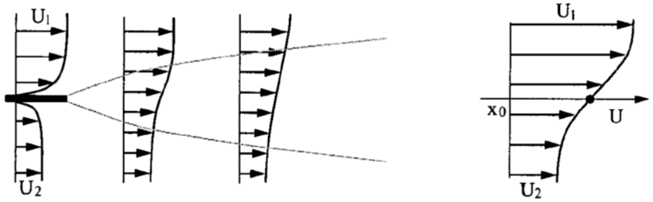
Kelvin-Helmholtz

SPATIALLY DEVELOPING SHEAR FLOWS

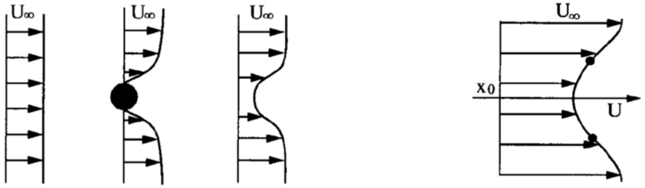
Flat plate boundary layer



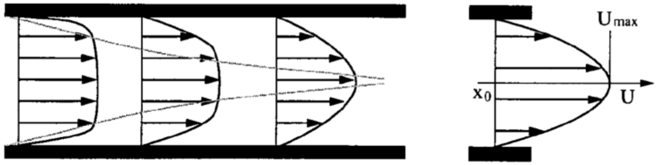
Mixing layer



Cylinder wake



Plane channel flow

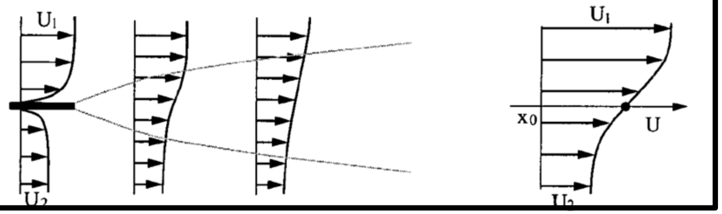


2D jet

Kelvin-Helmholtz

SPATIALLY DEVELOPING SHEAR FLOWS

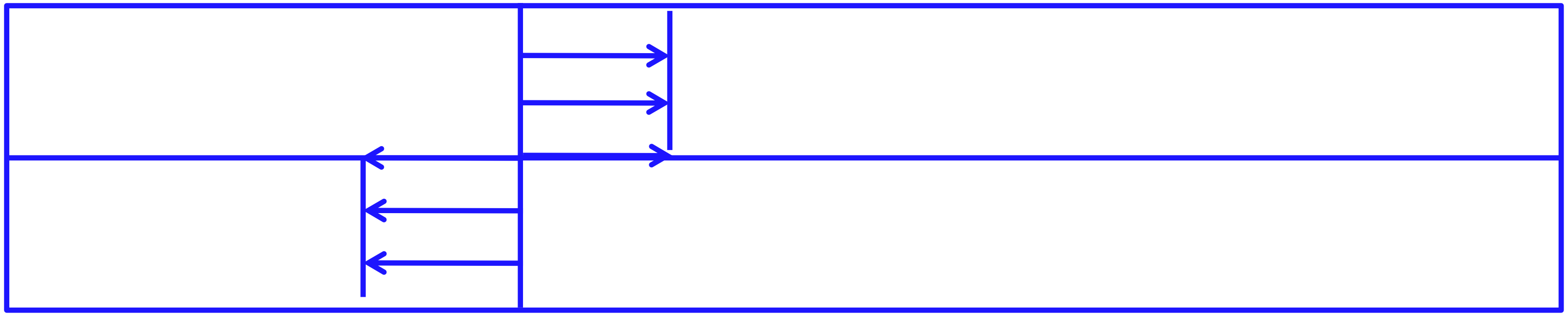
Mixing layer



Holmboe’s tube
(Film A. Garcia)

Holmboe’s tube

Inviscid assumption



1. Equations
Inviscid assumption

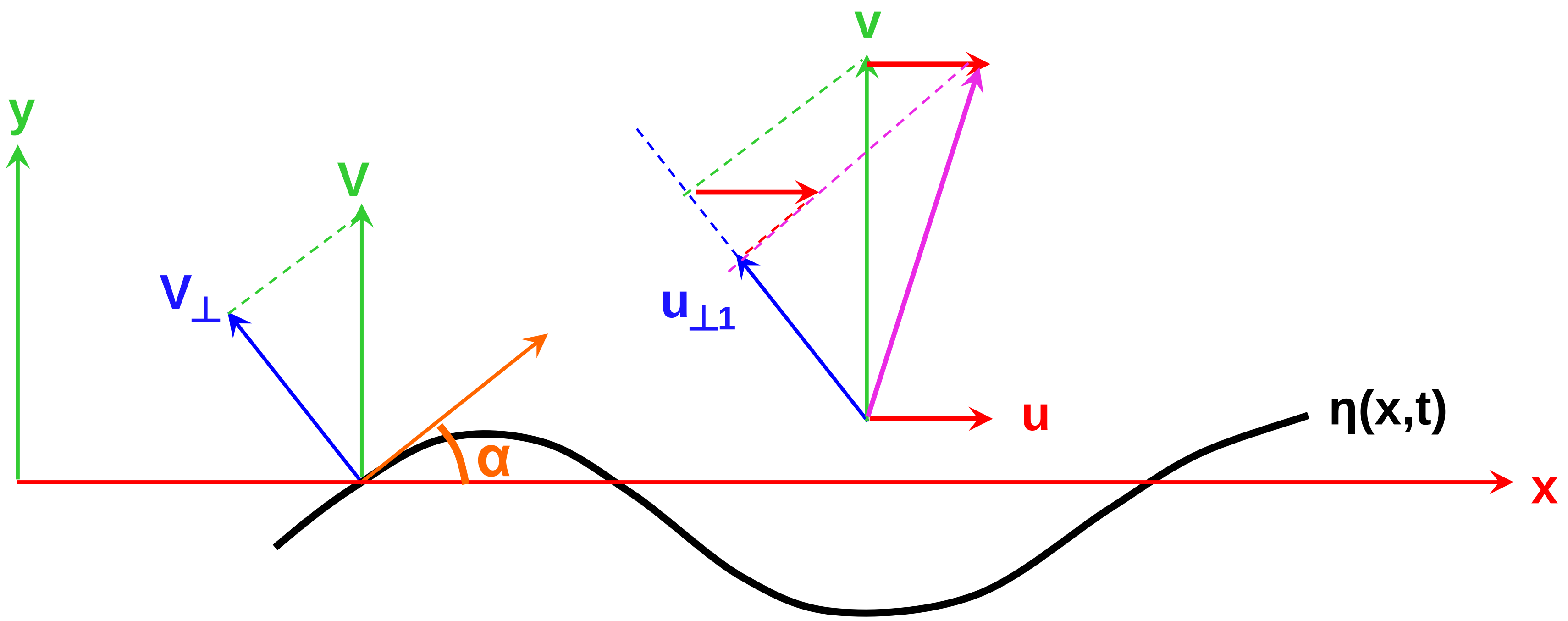
$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}+V_1\frac{\partial}{\partial y}\right)\Delta\Psi_1 &= 0,\\
\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}+V_2\frac{\partial}{\partial y}\right)\Delta\Psi_2 &= 0
\end{aligned}
$$

Streamfunction formulation

$$
\begin{aligned}
U_1 &= \frac{\partial\Psi_1}{\partial y}, \qquad V_1 = -\frac{\partial\Psi_1}{\partial x},\\
U_2 &= \frac{\partial\Psi_2}{\partial y}, \qquad V_2 = -\frac{\partial\Psi_2}{\partial x}.
\end{aligned}
$$

Velocity field

1. Kinematic boundary condition



Kinematic condition : impermeability (no penetration)

No fluid particles going across the interface through the normal direction

$$\left.\begin{aligned} V_\perp &= \partial\eta/\partial t \cos(\alpha) \\ u_{\perp 1} &= v_1 \cos(\alpha) - u_1 \sin(\alpha) \end{aligned}\right\} \partial\eta/\partial t = v_1 - u_1 \tan(\alpha) \quad \Rightarrow \quad \boxed{\partial\eta/\partial t = v_1 - u_1 \partial\eta/\partial x}$$

1. Kinematic boundary conditions

$$\boxed{\begin{aligned}\Psi_1 &= W_1 y \text{ at } y = -\infty,\\ \Psi_2 &= W_2 y \text{ at } y = +\infty.\end{aligned}}$$

far-field

$$\boxed{\begin{aligned}-U_1\frac{\partial\eta}{\partial x} + V_1 &= \frac{\partial\eta}{\partial t}\\ -U_2\frac{\partial\eta}{\partial x} + V_2 &= \frac{\partial\eta}{\partial t}\end{aligned}} \quad \text{at } z = \eta$$

1. Dynamic boundary conditions

$$P_1 - P_2 = -\gamma \frac{\frac{\partial^2 \eta}{\partial x^2}}{\left(1 + \frac{\partial \eta}{\partial x}^2\right)^{3/2}} \ \text{at}\ z = \eta$$

2. Base state

$$\Psi_1 = W_1 y$$

$$\Psi_2 = W_2 y$$

$$\eta = 0$$

$$P_1 = -\rho_1 g z$$

$$P_2 = -\rho_2 g z$$

3. Perturb and linearize perturbation expansion

$$
\begin{aligned}
\Psi_1 &= W_1 y &&+ \epsilon\psi_1 \\
\Psi_2 &= W_2 y &&+ \epsilon\psi_2 \\
U_1 &= W_1 &&+ \epsilon u_1 \\
V_1 &= 0 &&+ \epsilon v_1 \\
U_2 &= W_2 &&+ \epsilon u_2 \\
V_2 &= 0 &&+ \epsilon v_2 \\
P_1 &= -\rho_1 g y &&+ \epsilon p_1 \\
P_2 &= -\rho_2 g y &&+ \epsilon p_2 \\
\eta &= 0 &&+ \epsilon\sigma
\end{aligned}
$$

$$\epsilon \ll 1$$

Variables **Base state** **Small perturbation**

2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi = 0$$

Basic flow + perturbation

$$\Psi(x,t) = \int U(y)dy + \psi(x,y,t)$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi - U''(y)\frac{\partial \psi}{\partial x} = 0$$

3. Linearized equations

$$\boxed{\begin{aligned} \Delta\psi_1 &= 0, \\ \Delta\psi_2 &= 0, \end{aligned}}$$

linear vorticity equation

$$\boxed{\begin{aligned} u_1 &= \frac{\partial\psi_1}{\partial y}, & v_1 &= -\frac{\partial\psi_1}{\partial x}, \\ u_2 &= \frac{\partial\psi_2}{\partial y}, & v_2 &= -\frac{\partial\psi_2}{\partial x}. \end{aligned}}$$

velocity field

3. Perturbed kinematic boundary conditions

$$\psi_1 = 0 \text{ at } z = -\infty;$$
$$\psi_2 = 0 \text{ at } z = +\infty.$$

$$-\epsilon W_1 \frac{\partial \sigma}{\partial x} - \epsilon^2 u_1 \frac{\partial \sigma}{\partial x} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } y = \epsilon \sigma,$$
$$-\epsilon W_2 \frac{\partial \sigma}{\partial x} - \epsilon^2 u_2 \frac{\partial \sigma}{\partial x} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } y = \epsilon \sigma,$$

3. Perturbed kinematic boundary conditions

$$\boxed{\begin{aligned} \psi_1 &= 0 \text{ at } z = -\infty; \\ \psi_2 &= 0 \text{ at } z = +\infty. \end{aligned}}$$

$$\boxed{\begin{aligned} -\epsilon W_1 \frac{\partial \sigma}{\partial x} - \xcancel{\epsilon^2 u_1 \frac{\partial \sigma}{\partial x}} + \epsilon v_1 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } y = \epsilon \sigma, \\ -\epsilon W_2 \frac{\partial \sigma}{\partial x} - \xcancel{\epsilon^2 u_2 \frac{\partial \sigma}{\partial x}} + \epsilon v_2 = \epsilon \frac{\partial \sigma}{\partial t} \text{ at } y = \epsilon \sigma, \end{aligned}}$$

3. Flattened kinematic boundary conditions

$$\boxed{\begin{aligned} -W_1\frac{\partial\sigma}{\partial x}-\frac{\partial\psi_1}{\partial x} &= \frac{\partial\sigma}{\partial t}\ \text{at}\ y=\epsilon\sigma,\\ -W_2\frac{\partial\sigma}{\partial x}-\frac{\partial\psi_2}{\partial x} &= \frac{\partial\sigma}{\partial t}\ \text{at}\ y=\epsilon\sigma.\end{aligned}}$$

Taylor expansion around 0: $\psi(\epsilon\sigma)=\psi(0)+(\epsilon\sigma)\left.\frac{\partial\psi}{\partial y}\right|_{\mathbf{0}}$

$$\boxed{\begin{aligned} -W_1\frac{\partial\sigma}{\partial x}-\frac{\partial\psi_1}{\partial x} &= \frac{\partial\sigma}{\partial t}\ \text{at}\ y=0\\ -W_2\frac{\partial\sigma}{\partial x}-\frac{\partial\psi_2}{\partial x} &= \frac{\partial\sigma}{\partial t}\ \text{at}\ y=0\end{aligned}}$$

⇨transforms a b.c. at an unkwown interface into a fixed place!

3. Perturbed and linearized Euler

$$\frac{\partial u_1}{\partial t} + W_1 \frac{\partial u_1}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}$$

$$\frac{\partial u_2}{\partial t} + W_2 \frac{\partial u_2}{\partial x} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}$$

4. Normal mode expansion

Fourier transform in x and t

$$\boxed{\begin{aligned} \psi_1 &= f_1(y)\exp(i(kx-\omega t)), \\ \psi_2 &= f_2(y)\exp(i(kx-\omega t)), \\ \sigma &= C\exp(i(kx-\omega t)), \end{aligned}}$$

k is the wavenumber and ω the frequency (in rad/s)

$$\boxed{\lambda = 2\pi/k} \qquad \boxed{T = 2\pi/\omega}$$

$$\boxed{f = \omega/(2\pi)}$$

4. Normal mode expansion

Solution to Laplace equation:

4. Normal mode expansion

Solution to Laplace equation:

$$\boxed{\begin{aligned} \psi_1 &= A\exp(ky)\exp(i(kx-\omega t)), \\ \psi_2 &= B\exp(-ky)\exp(i(kx-\omega t)), \\ \sigma &= C\exp(i(kx-\omega t)). \end{aligned}}$$

4. Perturbed and linearized Euler

$$\begin{aligned} \frac{-ik}{\rho_1} p_1 &= (-i\omega + W_1 ik) k A \exp(ky) \exp(i(kx - \omega t)), \\ \frac{-ik}{\rho_2} p_2 &= -(-i\omega + W_2 ik) k B \exp(-ky) \exp(i(kx - \omega t)), \end{aligned}$$

$$\begin{aligned} p_1(0) &= \rho_1(\omega - W_1 k) A, \\ p_2(0) &= -\rho_2(\omega - W_2 k) B \end{aligned}$$

4. Normal mode expansion

Replace in boundary conditions

$$\boxed{\begin{aligned} g(\rho_2-\rho_1)C+\rho_1(\omega-W_1k)A+\rho_2(\omega-W_2k)B &= \gamma k^2 C \\ -ikW_1C-ikA &= -i\omega C \\ -ikW_2C-ikB &= -i\omega C. \end{aligned}}$$

This is an eigenvalue problem iωX=MX!

$$\boxed{\rho_1(\omega/k-W_1)^2+\rho_2(\omega/k-W_2)^2+g(\rho_2-\rho_1)/k-\gamma k=0}$$

5. Dispersion relation

$$\boxed{\omega/k = \frac{\rho_1 W_1 + \rho_2 W_2}{\rho_1 + \rho_2} + \frac{\sqrt{(\rho_1 W_1 + \rho_2 W_2)^2 - (\rho_1 W_1^2 + \rho_2 W_2^2 - g(\rho_1 - \rho_2)/k - \gamma k)(\rho_1 + \rho_2)}}{\rho_1 + \rho_2}}$$

- **Unstable if there exists one ω, Im(ω)>0**

- **Neutral if for all ω, Im(ω)=0:**

- **Stable (or damped) if for all ω, Im(ω)<0:**

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

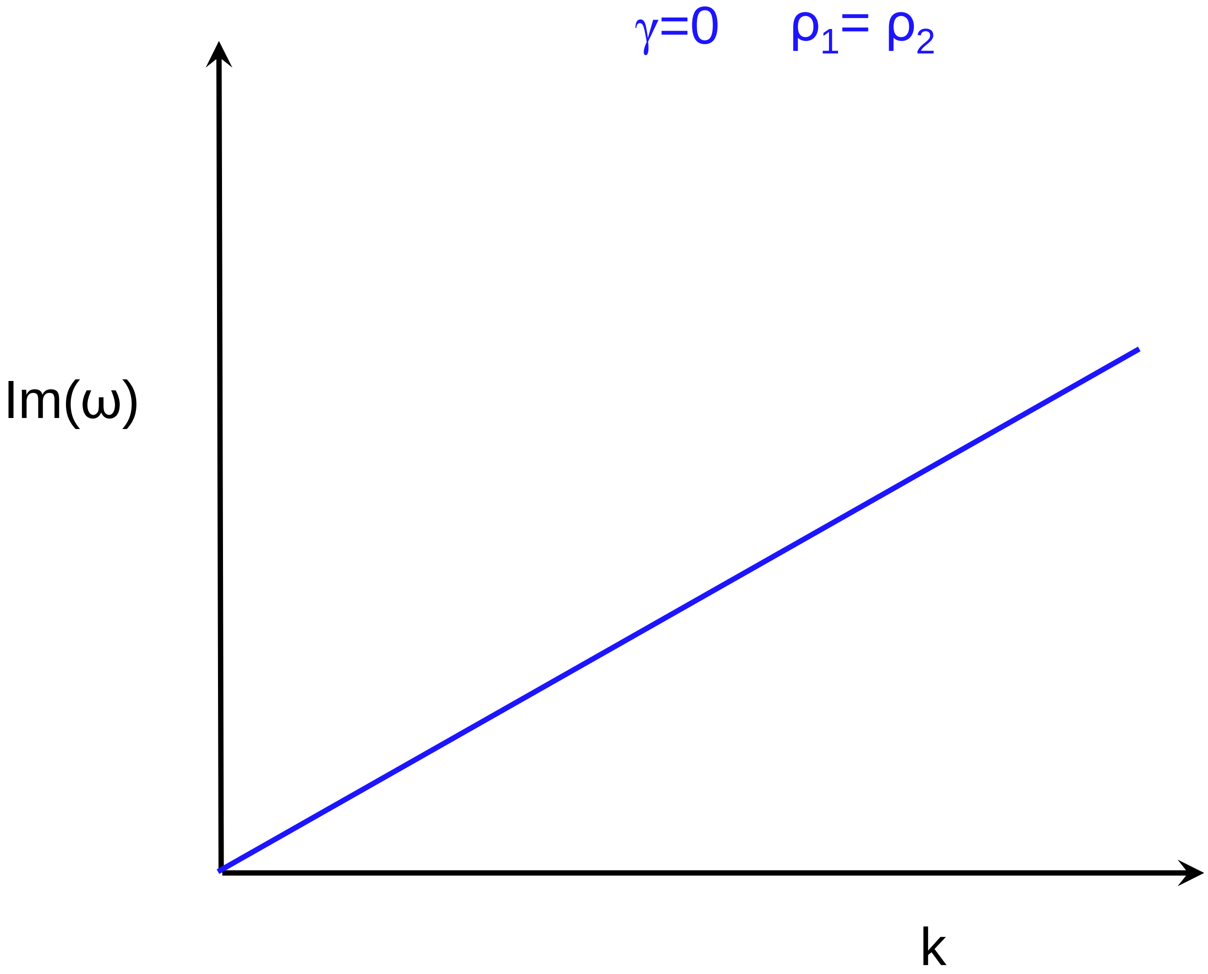
5. Dispersion relation

$$\omega/k = \frac{\rho_1 \cancelto{}{W_1} + \rho_2 \cancelto{}{W_2}}{\rho_1 + \rho_2} + \frac{\sqrt{(\rho_1 \cancelto{}{W_1} + \rho_2 \cancelto{}{W_2})^2 - (\rho_1 \cancelto{}{W_1^2} + \rho_2 \cancelto{}{W_2^2} - g(\rho_1 - \rho_2)/k - \gamma k)(\rho_1 + \rho_2)}}{\rho_1 + \rho_2}$$

If $W_1=W_2=0$, Rayleigh-Taylor instability

$$\boxed{\omega^2 = \frac{-kg(\rho_2 - \rho_1) + \gamma k^3}{\rho_1 + \rho_2}}$$

Kelvin-Helmholz instability of a vortex sheet

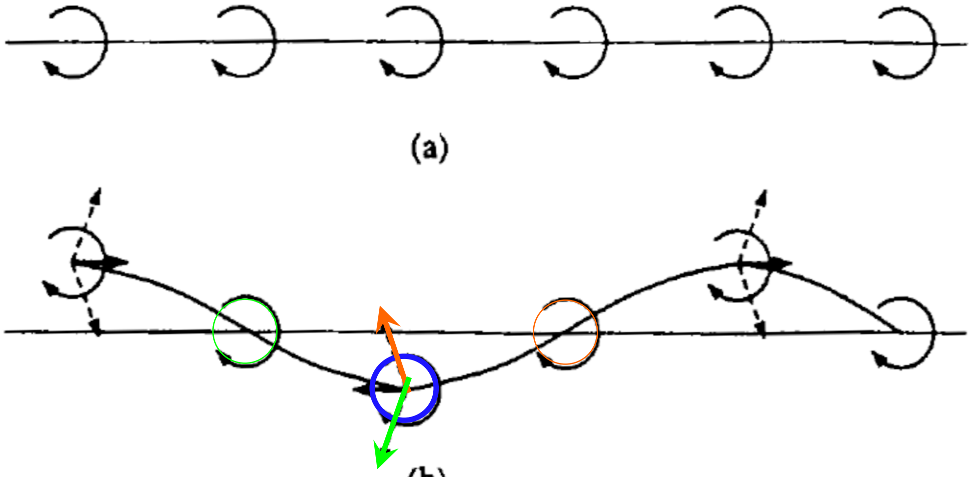


2D PARALLEL FLOW CONCEPTS

Inviscid instabilities

Vortex sheet

Instability mechanism

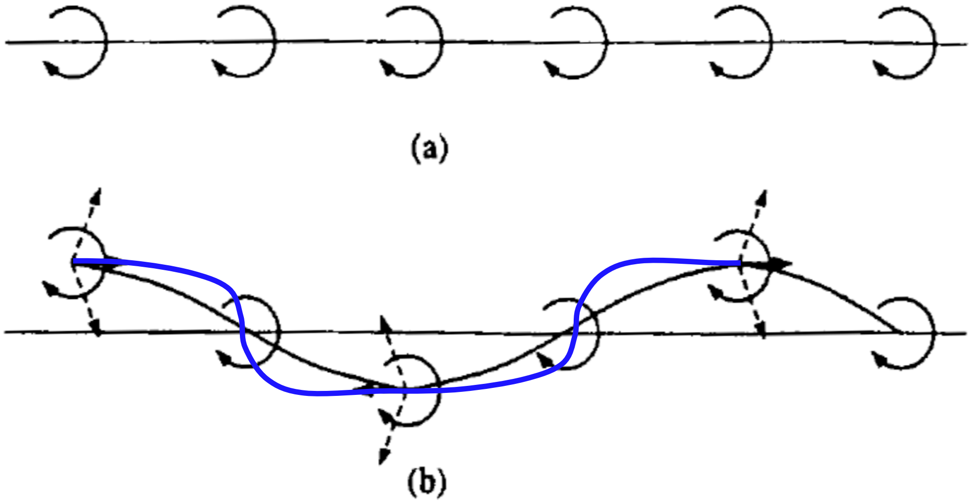


2D PARALLEL FLOW CONCEPTS

Inviscid instabilities

Vortex sheet

Instability mechanism

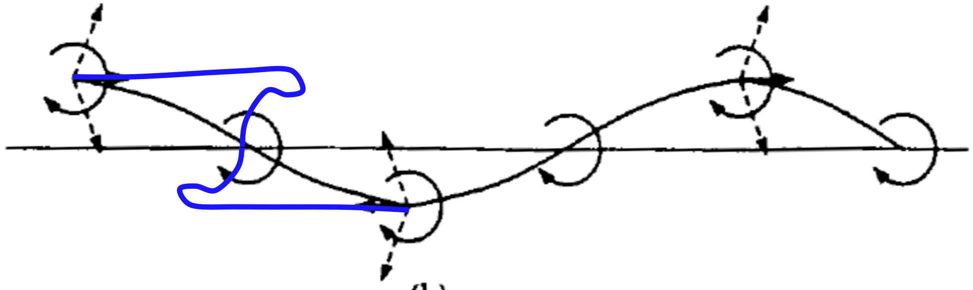


2D PARALLEL FLOW CONCEPTS

Inviscid instabilities

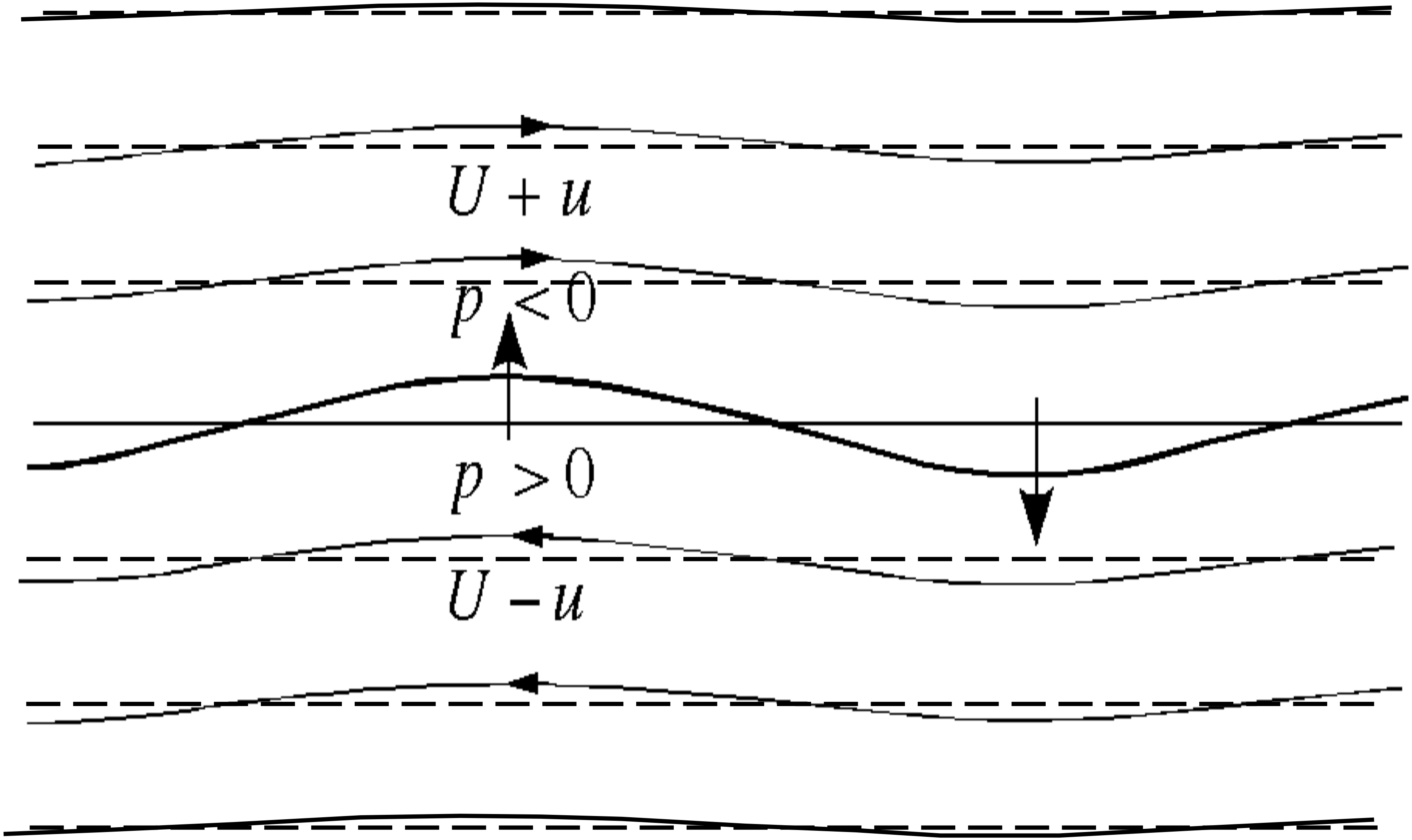
Vortex sheet

Instability mechanism



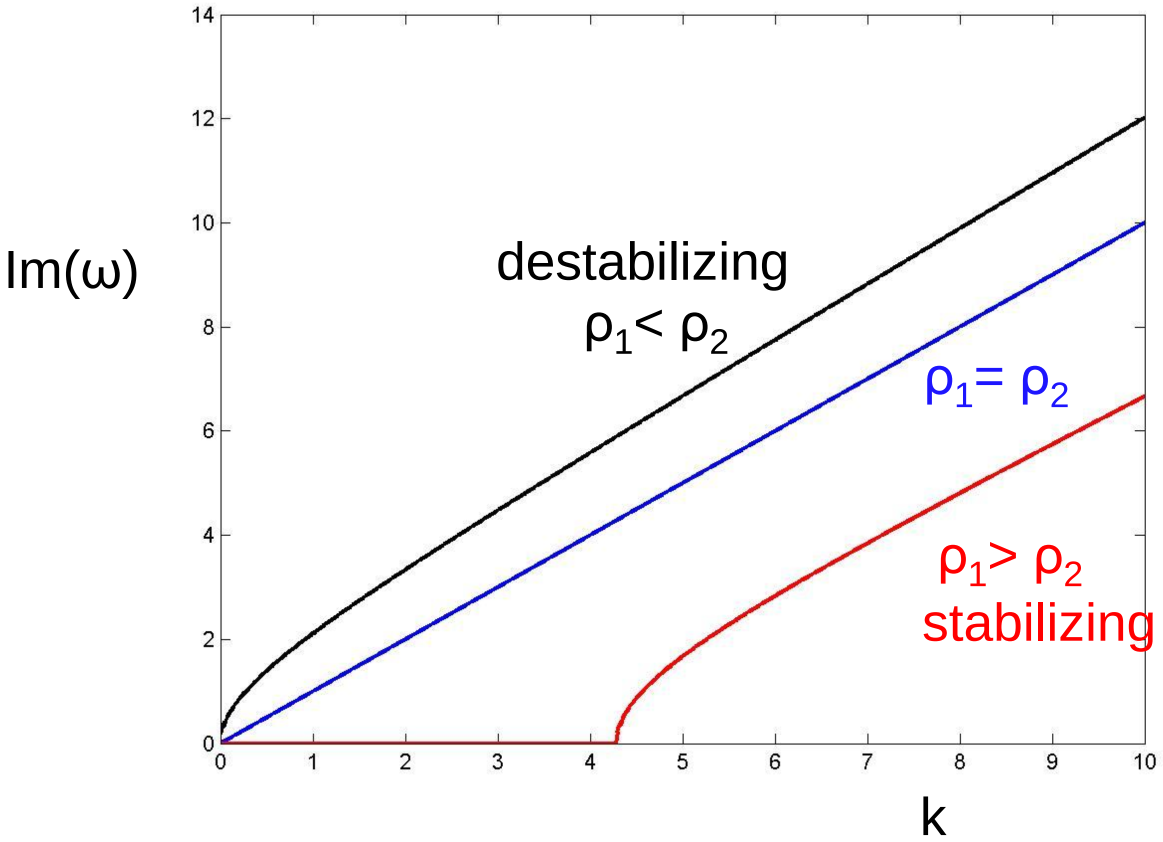
Interpretation “Bernoulli”

Interpretation “Bernoulli”



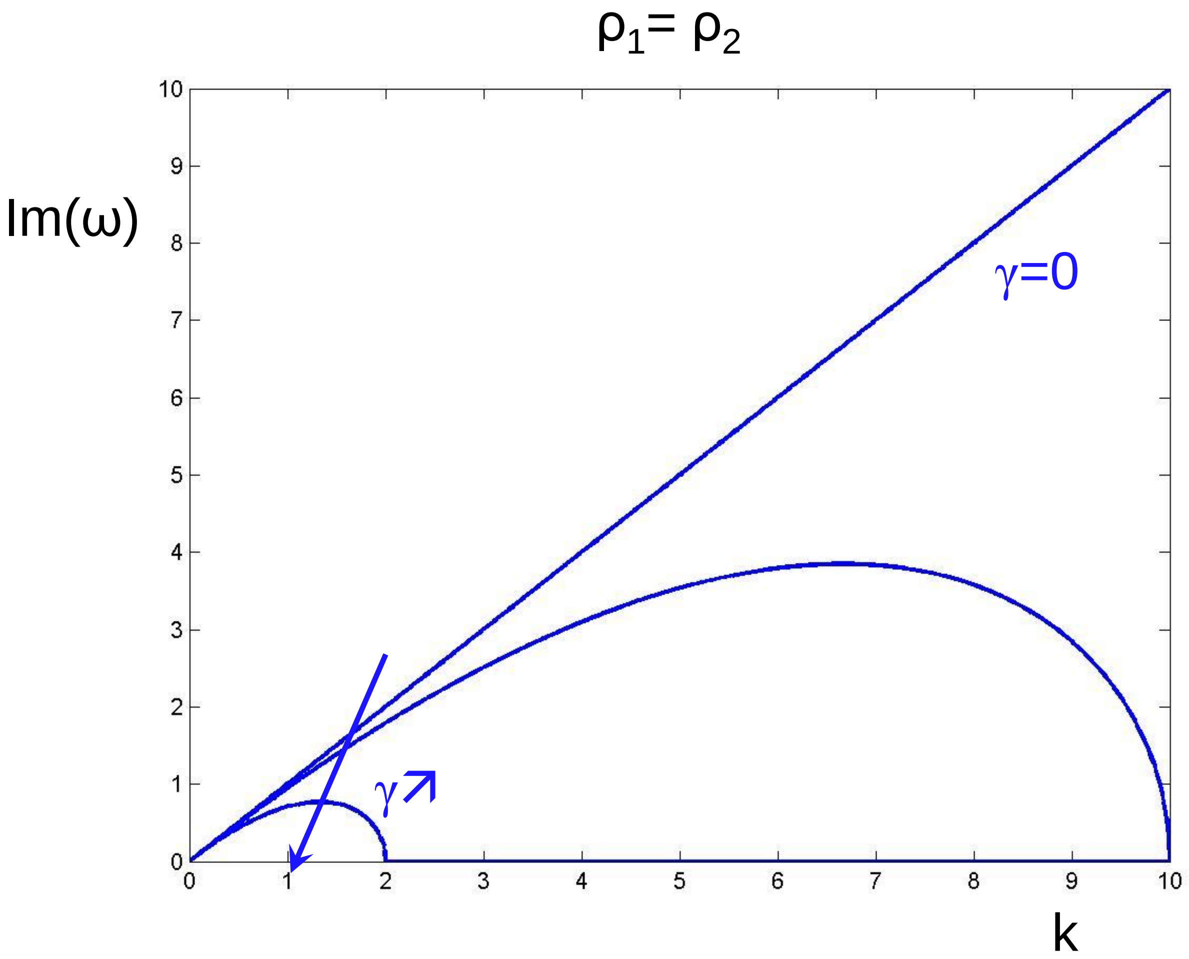
Influence of stratification

$\gamma=0$



Buoyancy stabilizes large wavelengths!

Influence of surface tension

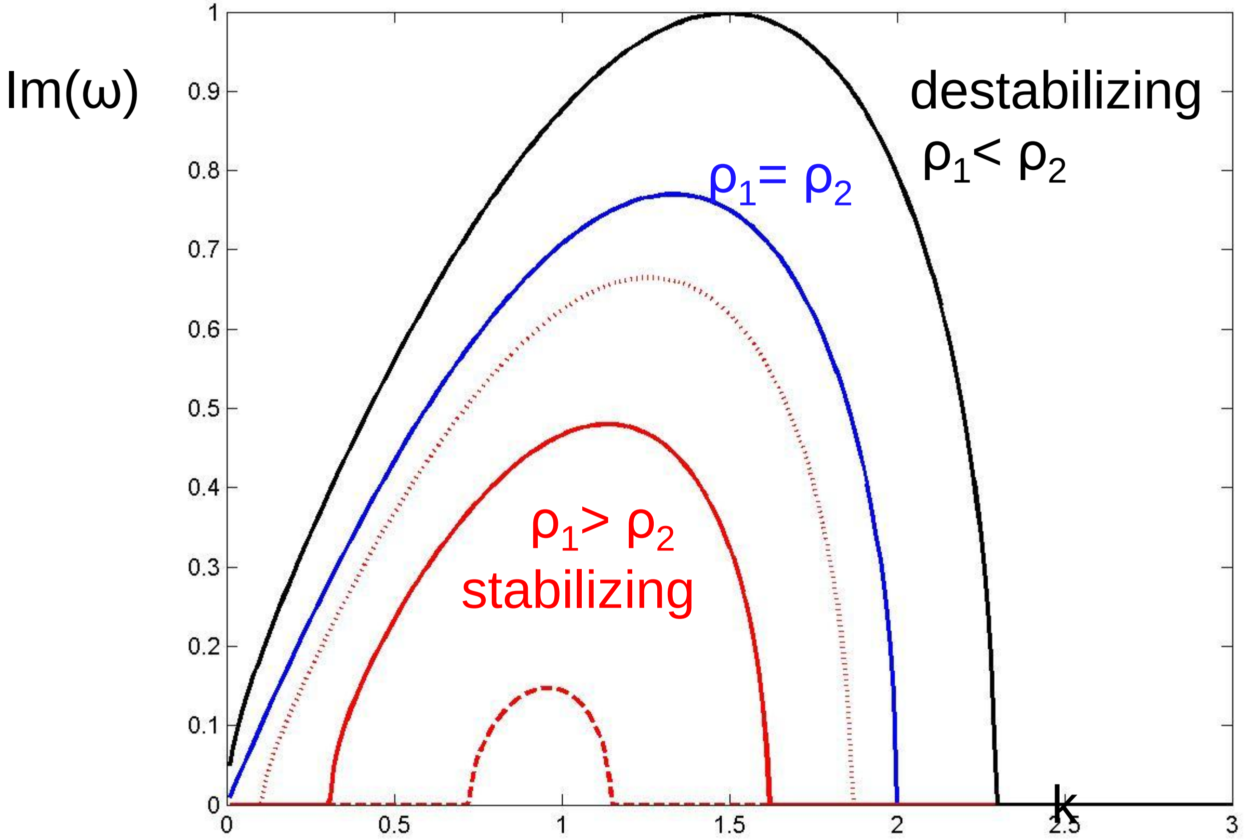


surface tension creates a cut off

high wavenumbers create large curvatures!

Joint influence of surface tension and stratification

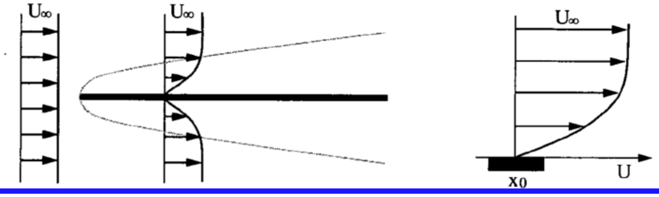
$\gamma > 0$!



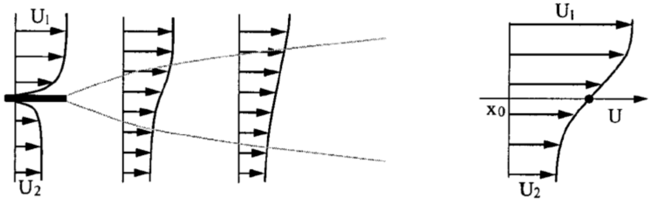
Stratification and surface tension can restabilize the flow

SPATIALLY DEVELOPING SHEAR FLOWS

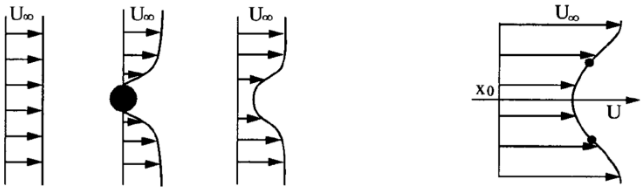
Flat plate boundary layer



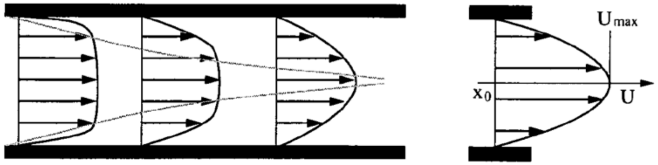
Mixing layer



Cylinder wake



Plane channel flow



2D jet

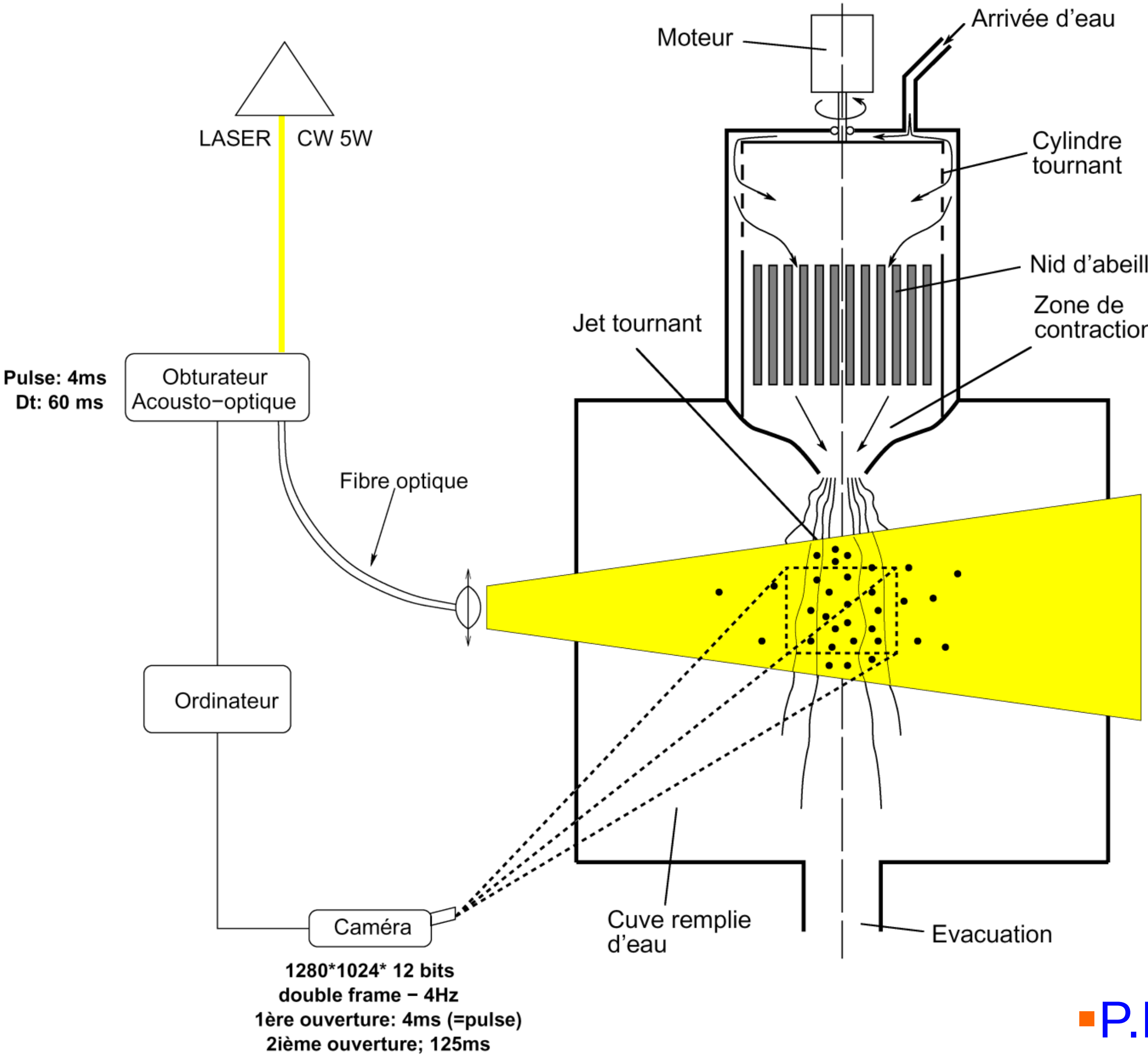
Kelvin Helmholtz in geophysics

Rio Negro
(slow and clean) meets
amazon (quick and dirty)

Kelvin Helmholtz in geophysics

La jonction/Geneve

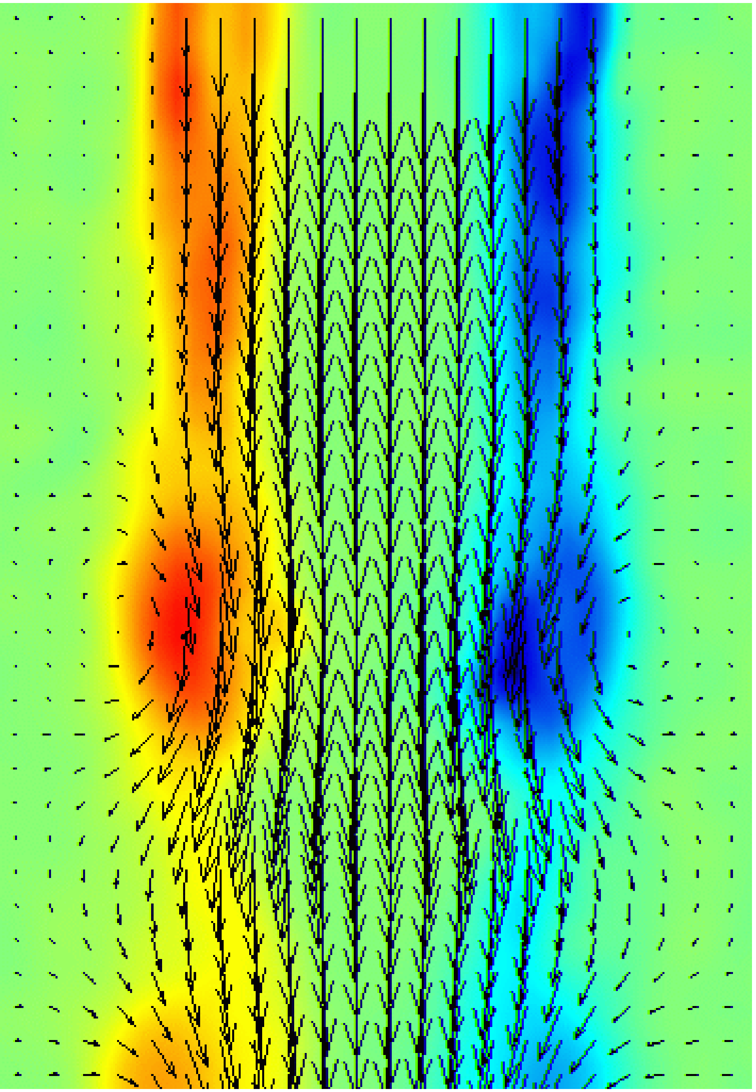
Round jet



- P.Billant
- T.Loiseleux

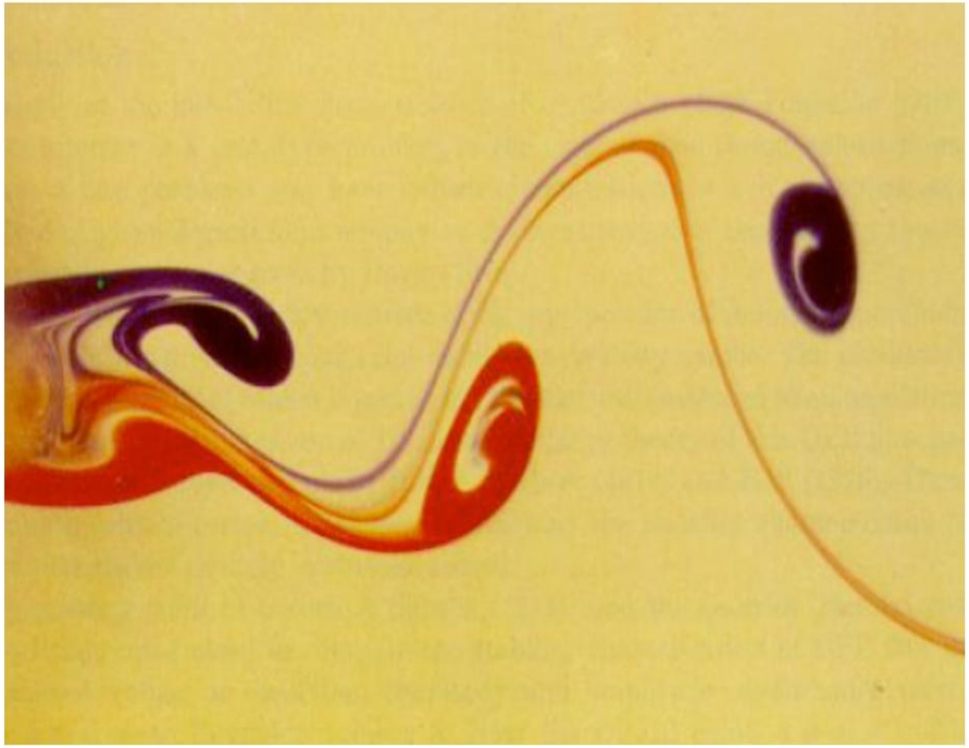
Toroidal vortices – vortex rings

Vorticity field measured by PIV



1 point de vitesse sur 4 est représenté

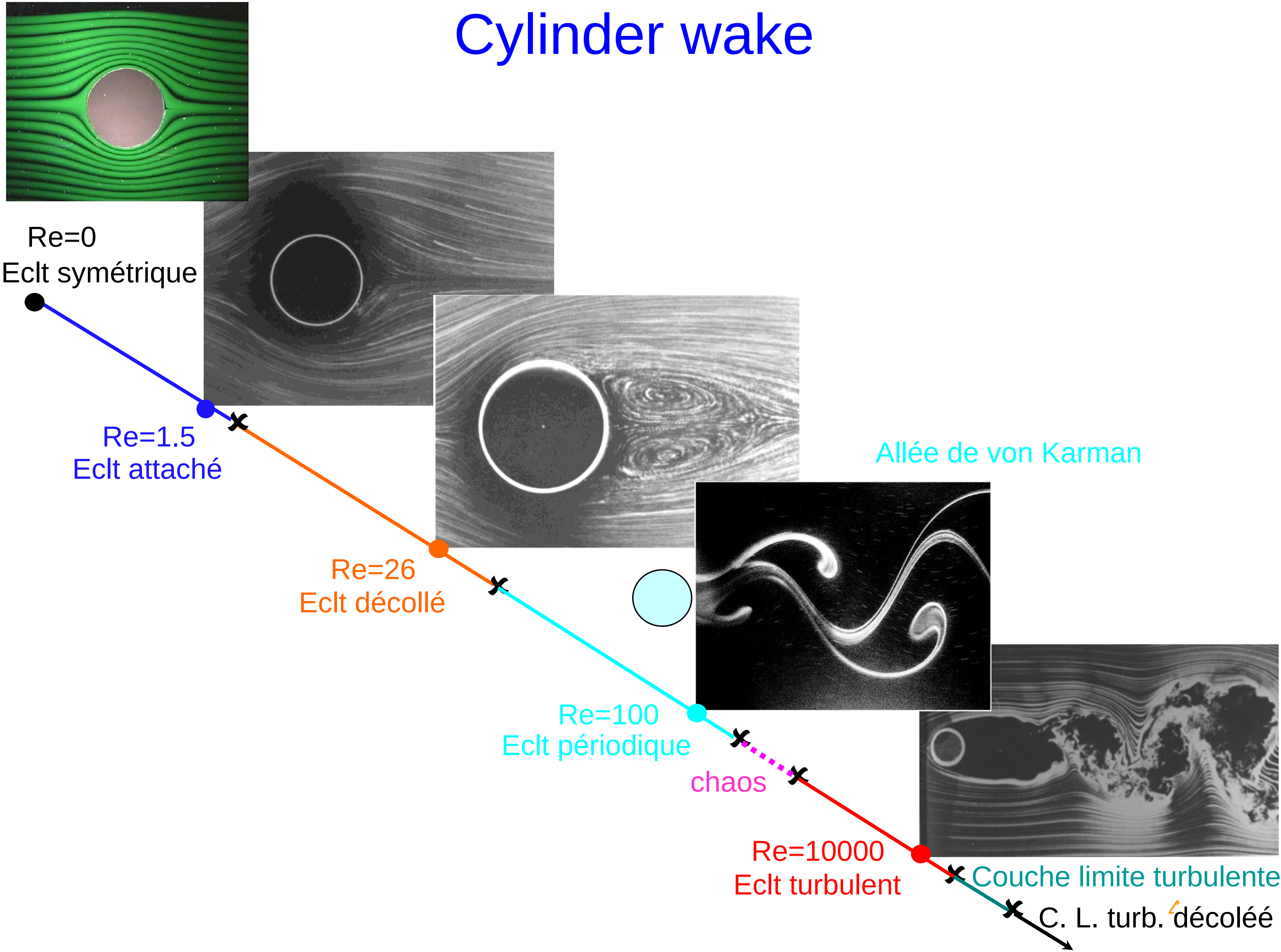
Allée de von Karman



(Perry, Chong & Lim (1982))

Sinuous mode

Cylinder wake



Re=0
Eclt symétrique

Re=1.5
Eclt attaché

Re=26
Eclt décollé

Allée de von Karman

Re=100
Eclt périodique

chaos

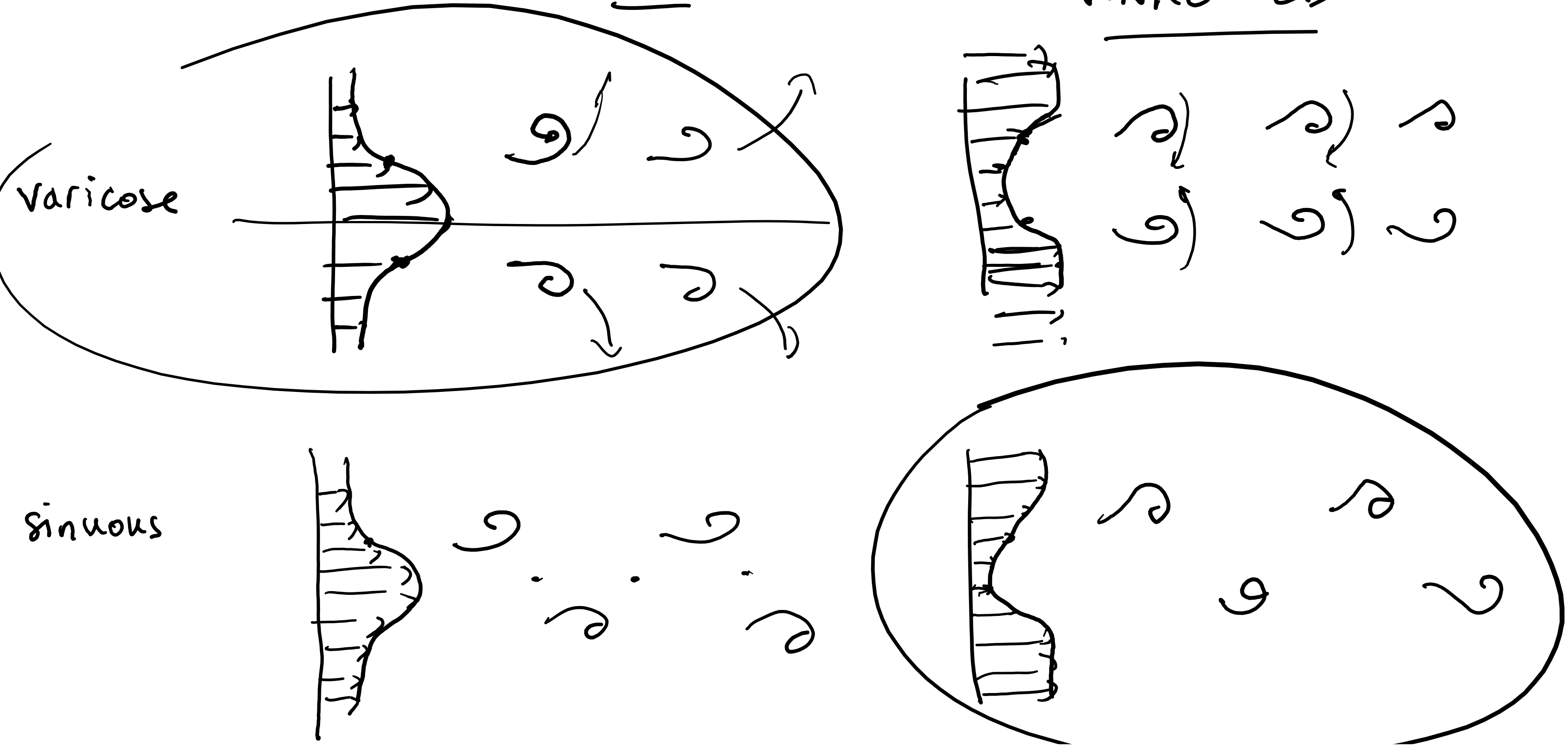
Re=10000
Eclt turbulent

Couche limite turbulente

C. L. turb. décoléé

JET 2D

WAKE 2D

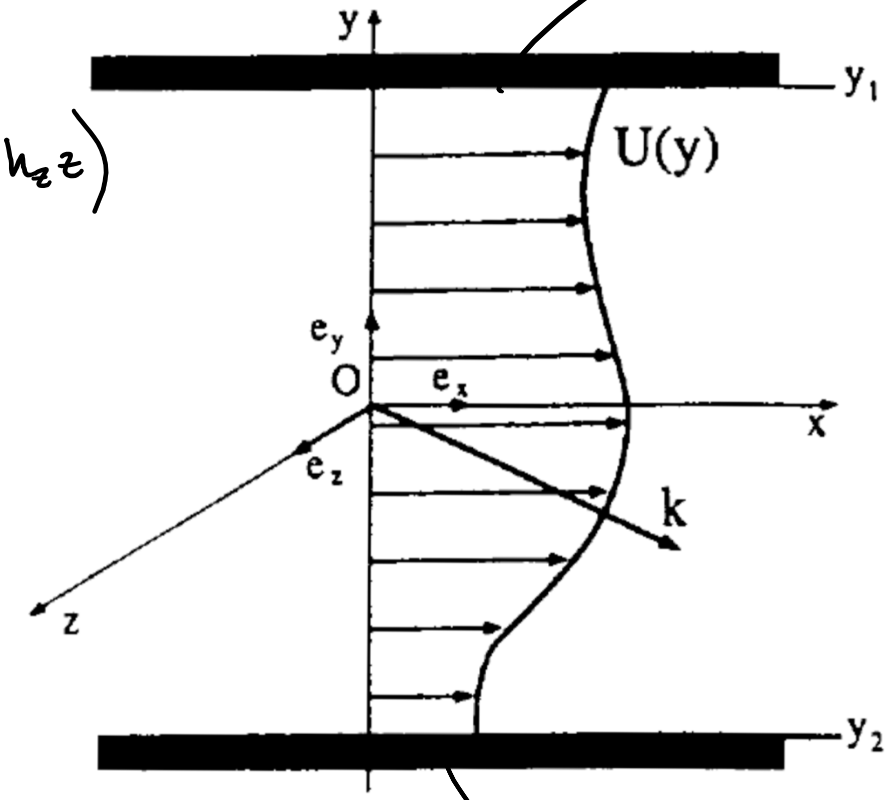


PARALLEL FLOW CONCEPTS

Inviscid 3D instabilities

Basic flow

SLIP



SLIP

$e^{i(k_x x - \omega t + k_z z)}$

BASE FLOW

$V = W = 0$

UNIDIRECTIONAL

STEADY $\frac{\partial}{\partial t} = 0$

parallel $\frac{\partial}{\partial x} = 0$

INVISCID

PARALLEL FLOW CONCEPTS

Inviscid 3D instabilities

3D Euler equations

$$\nabla \cdot \mathrm{U} = 0$$

$$\frac{\partial \mathrm{U}}{\partial t} + (\mathrm{U} \cdot \nabla)\mathrm{U} = -\nabla P$$

Basic flow + perturbation

$$\mathbf{U}(\mathbf{x}, t) = U(y)\mathbf{e}_x + \mathbf{u}(\mathbf{x}, t)$$

$$P(\mathbf{x}, t) = P_0 + p(\mathbf{x}, t)$$

Linear Euler equations

$$\nabla \cdot \mathbf{u} = 0 \qquad \left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\mathbf{u} + U'(y)\, v\, \mathbf{e}_x = -\nabla p$$

PARALLEL FLOW CONCEPTS

Inviscid 3D instabilities

Normal mode decomposition

$$\mathbf{u}(\mathbf{x},t) = \mathcal{R}e\left\{\hat{\mathbf{u}}(y)\exp[i(k_x x + k_z z - \omega t)]\right\}$$

$$p(\mathbf{x},t) = \mathcal{R}e\left\{\hat{p}(y)\exp[i(k_x x + k_z z - \omega t)]\right\}$$

$$\mathbf{k} = k_x\mathbf{e}_x + k_z\mathbf{e}_z \qquad\qquad c = \omega/k_x$$

Linear o.d.e.'s

$$ik_x\hat{u} + ik_z\hat{w} + \frac{d\hat{v}}{dy} = 0$$

$$ik_x\left[U(y) - c\right]\hat{u} + U'(y)\hat{v} = -ik_x\hat{p}$$

$$ik_x\left[U(y) - c\right]\hat{v} = -\frac{d\hat{p}}{dy},$$

$$ik_x\left[U(y) - c\right]\hat{w} = -ik_z\hat{p}$$

$$\hat{v}(y_1) = \hat{v}(y_2) = 0$$

3D dispersion relation

$$D(\mathbf{k},\omega) = 0$$

PARALLEL FLOW CONCEPTS

Inviscid 3D instabilities

Squire's transformation

$$\tilde{k}^2 = k_x^2 + k_z^2 , \qquad \tilde{c} = c ,$$

$$\tilde{k}\tilde{u} = k_x\hat{u} + k_z\hat{w} , \qquad \tilde{v} = \hat{v} , \qquad \tilde{p}/\tilde{k} = \hat{p}/k_x .$$

Linear o.d.e.'s

$$i\tilde{k}\tilde{u} + \frac{d\tilde{v}}{dy} = 0 ,$$

$$i\tilde{k}\left[U(y) - \tilde{c}\right]\tilde{u} + U'(y)\tilde{v} = -i\tilde{k}\tilde{p} ,$$

$$i\tilde{k}\left[U(y) - \tilde{c}\right]\tilde{v} = -\frac{d\tilde{p}}{dy} ,$$

$$\tilde{v}(y_1) = \tilde{v}(y_2) = 0 ,$$

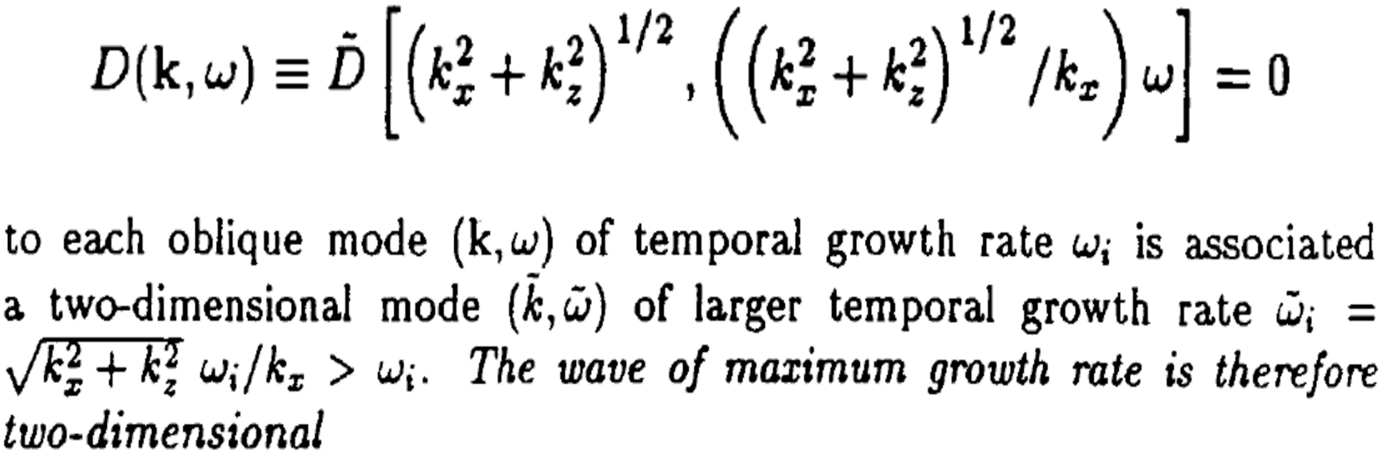
2D dispersion relation

$$\tilde{D}(\tilde{k}, \tilde{\omega}) = 0$$

PARALLEL FLOW CONCEPTS

Inviscid 3D instabilities

Squire’s transformation



$$D(\mathbf{k},\omega) \equiv \tilde{D}\left[\left(k_x^2+k_z^2\right)^{1/2}, \left(\left(k_x^2+k_z^2\right)^{1/2}/k_x\right)\omega\right] = 0$$

to each oblique mode $(\mathbf{k},\omega)$ of temporal growth rate ω_i is associated a two-dimensional mode $(\tilde{k},\tilde{\omega})$ of larger temporal growth rate $\tilde{\omega}_i = \sqrt{k_x^2+k_z^2}\,\omega_i/k_x > \omega_i$. *The wave of maximum growth rate is therefore two-dimensional*

2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi = 0$$

Basic flow + perturbation

$$\Psi(x,t)=\int U(y)dy+\psi(x,y,t)$$

$$u=\partial_y\psi \text{ et } v=-\partial_x\psi$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial \psi}{\partial x} = 0$$

Rayleigh equation

Dispersion relation

Normal mode decomposition

$$\psi(x,y,t) = \mathcal{R}e\left\{\phi(y)\, e^{i(kx-\omega t)}\right\}$$

$$\boxed{[U(y) - c]\,[\phi'' - k^2\phi] - U''(y)\,\phi = 0}$$

$\phi(y)$ ⇨0 at $y = \pm\infty$

Dispersion relation

$$D(k,\omega) = 0$$

PARALLEL FLOW CONCEPTS

Inviscid instabilities

Rayleigh's inflection point criterion

In order for the basic flow U(y) to be unstable, it should have an inflection point, say at $y = y_s$, such that $U''(y_s) = 0$

or, in other words Ω(y) has an extremum

vorticity $\Omega = -\frac{dU}{dy}$

$$\left.\frac{d^2U}{dy^2}\right|_{y_s} = 0 \implies \left.\frac{d\Omega}{dy}\right|_{y_s} = 0$$

Rayleigh theorem (1916)

$$(U-c)(\frac{d^2\psi}{dy^2}-k^2\psi)-U''(y)\psi=0$$

$$\int_{y_1}^{y_2}\left((\frac{d^2\psi}{dy^2}-k^2\psi)\psi^*-\frac{U''(y)\psi\psi^*}{U-c}\right)dy=0$$

$$\int_{y_1}^{y_2}\left((\frac{d^2\psi}{dy^2}-k^2\psi)\psi^*-\frac{U''(y)\psi\psi^*}{|U-c|^2}(U-c^*)\right)dy=0$$

$$[\frac{d\psi}{dy}\psi^*]_{y_1}^{y_2}+\int_{y_1}^{y_2}\left(-\frac{d\psi}{dy}\frac{d\psi^*}{dy}-k^2\psi\psi^*-\frac{U''(y)\psi\psi^*}{|U-c|^2}(U-c^*)\right)dy=0$$

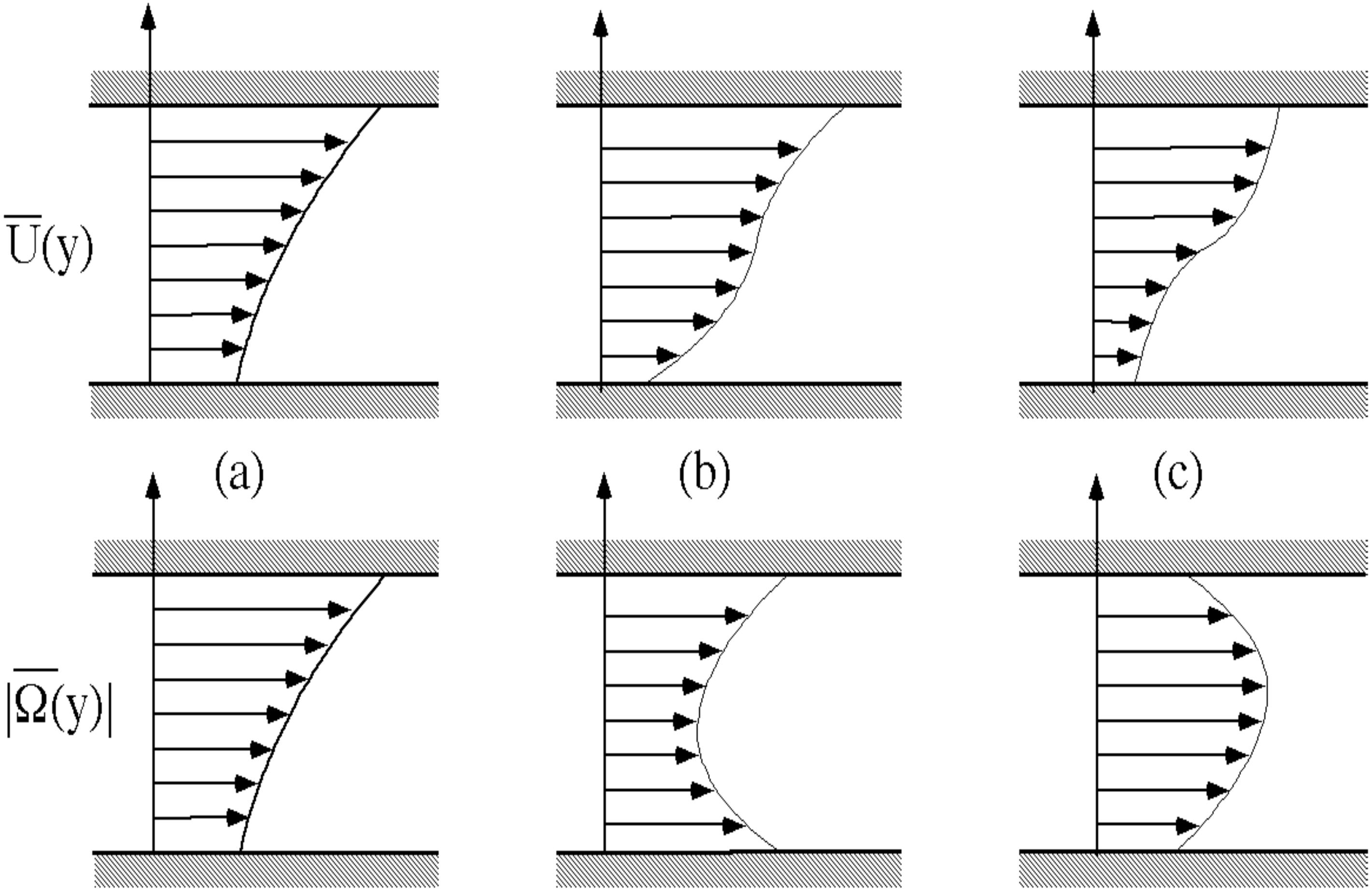
Rayleigh theorem

$$[\cancelto{0}{\frac{d\psi}{dy}}\psi^*]_{y_1}^{y_2} + \int_{y_1}^{y_2}\left(-\underbrace{\frac{d\psi}{dy}\frac{d\psi^*}{dy}}_{\text{real}} - \underbrace{k^2\psi\psi^*}_{\text{real}} - \frac{U''(y)\psi\psi^*}{\underbrace{|U-c|^2}_{\text{real}}}(U-c^*)\right)dy = 0$$

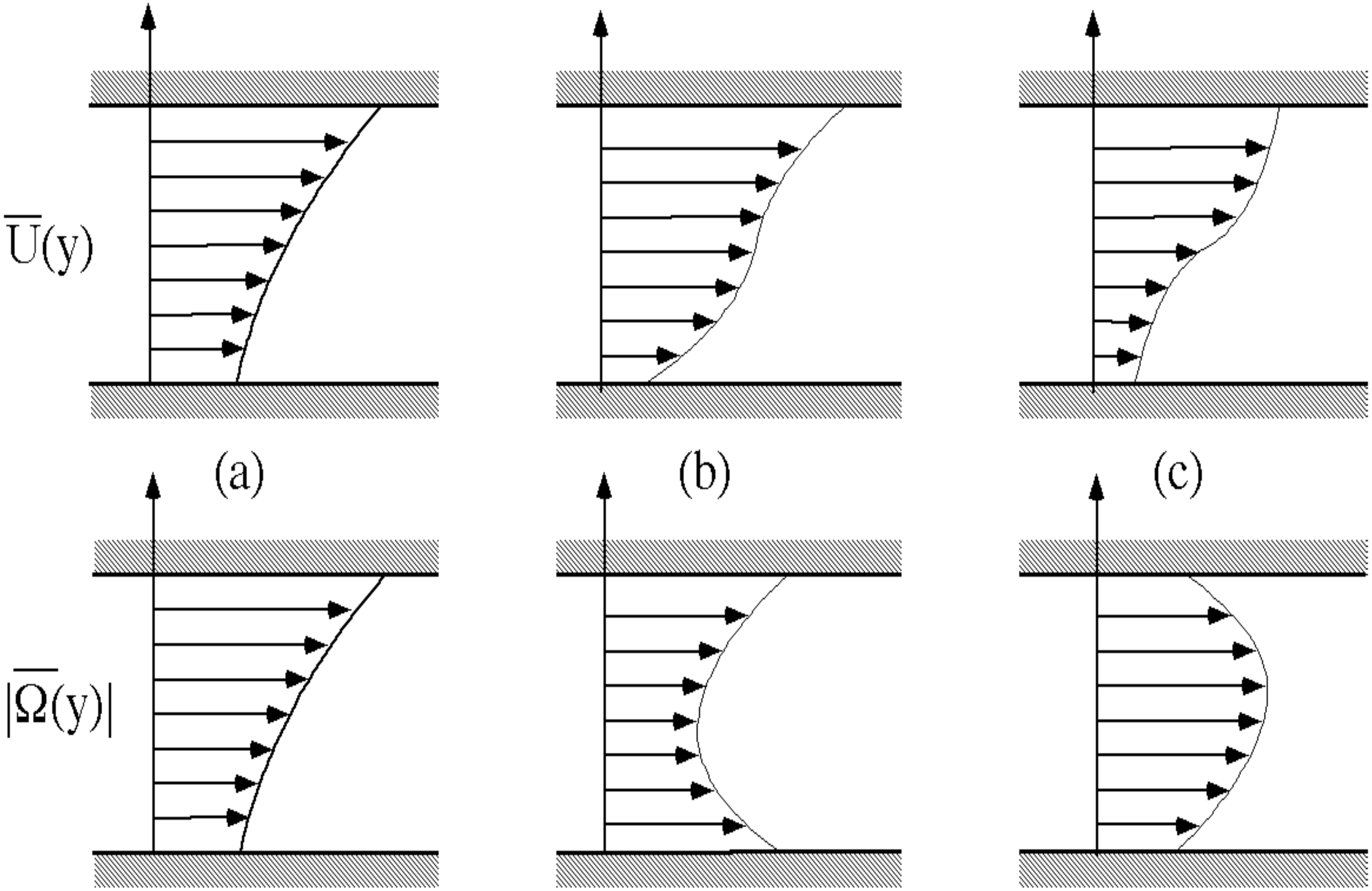
take the imaginary part

$$\int_{y_1}^{y_2}\frac{U''(y)|\psi|^2}{|U-c|^2}c_i dy = 0$$

Quiz



Quiz



stable

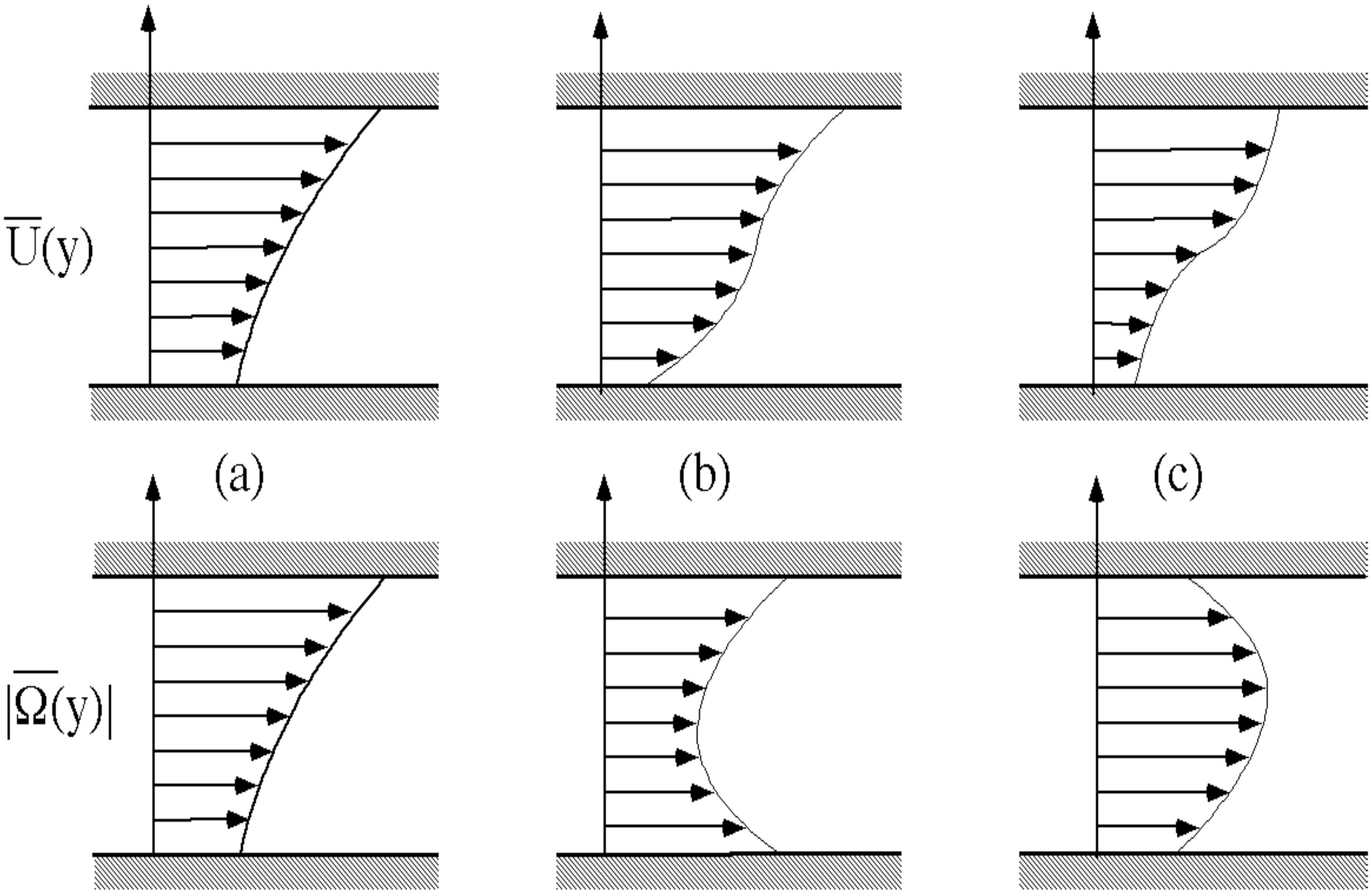
Unstable?

Unstable?

Fjorthoft criterion (1950)

For monotonous velocity profiles with only one inflection point, $|\Omega(y)|$ should have a maximum for instability to be possible

Quiz

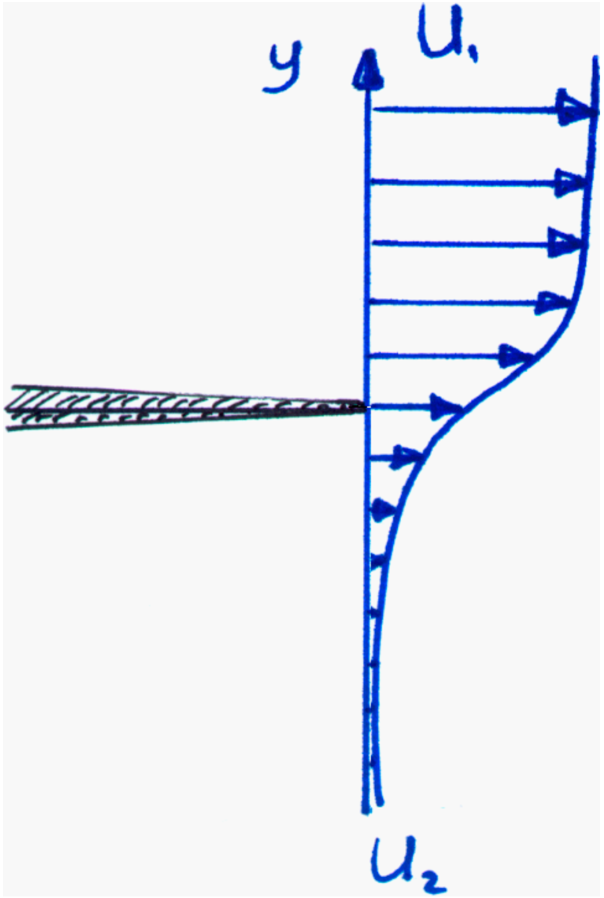


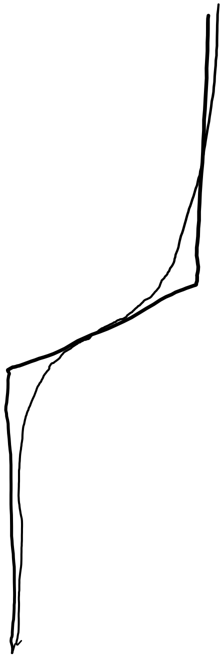
stable

Stable

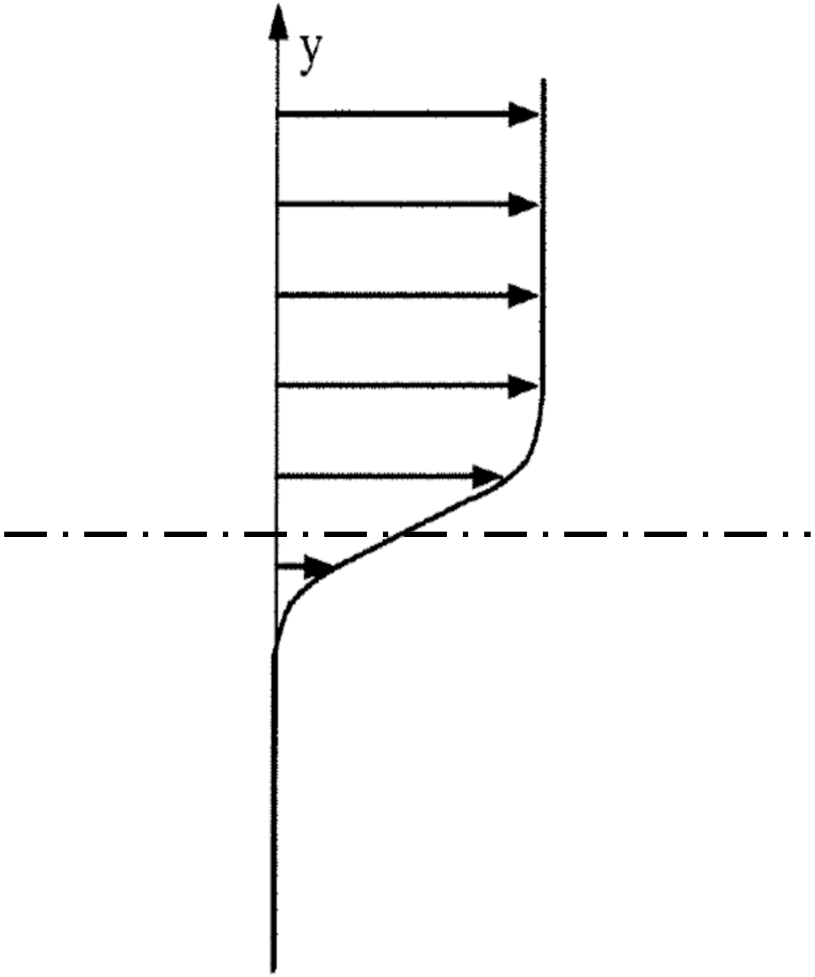
Unstable?

Tangent hyperbolic mixing layer



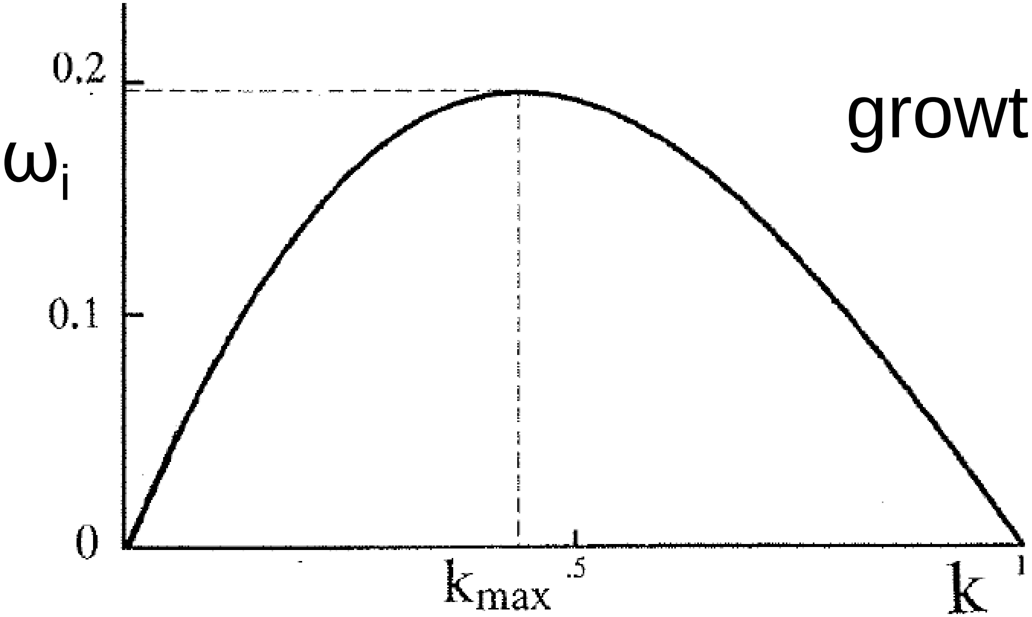


Tangent hyperbolic shear layer

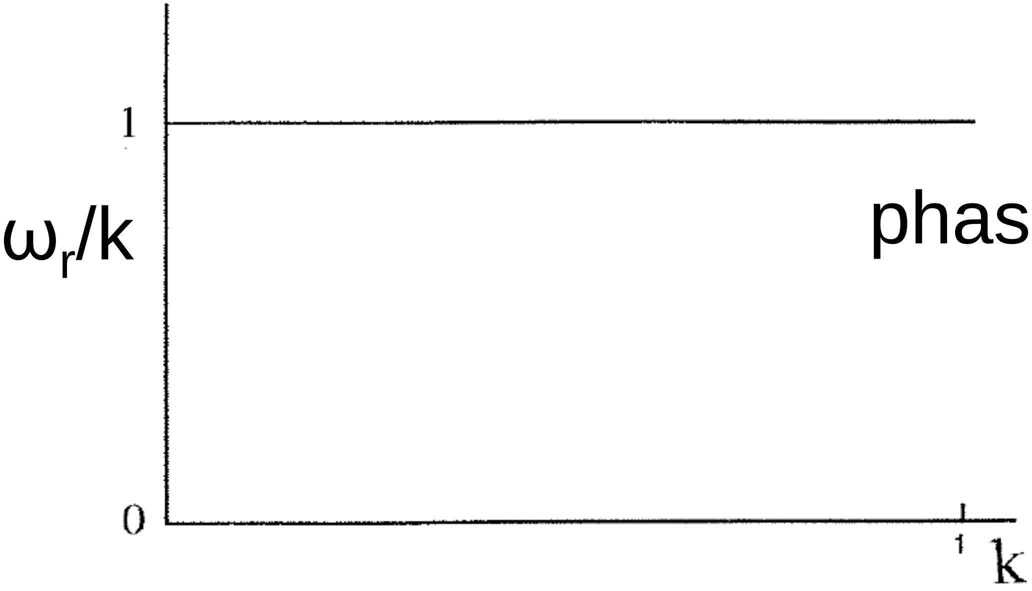


$$U_B(y) = \tanh y$$

Tangent hyperbolic shear layer



growth-rate



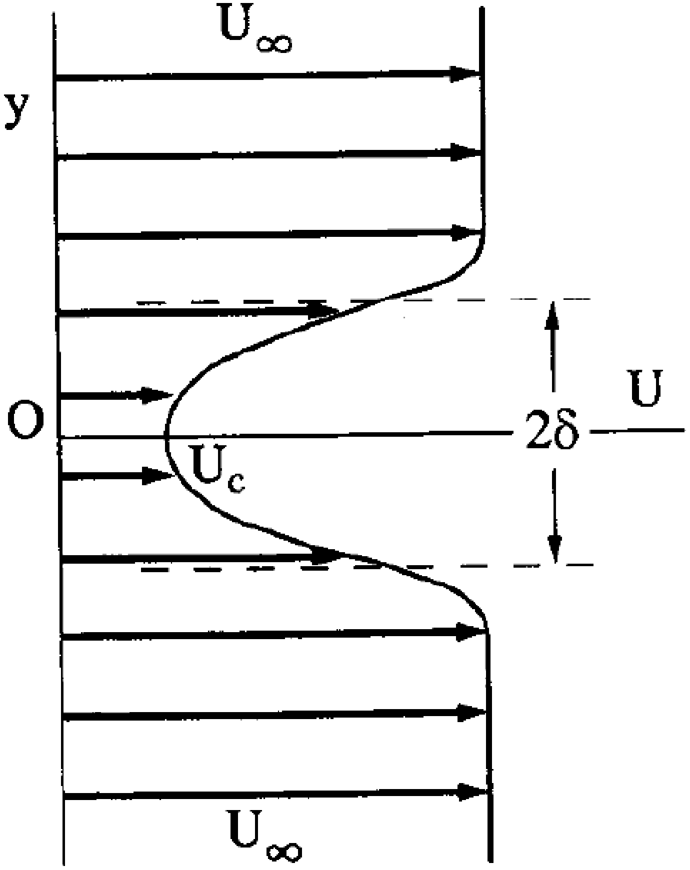
phase velocity

Wake instability

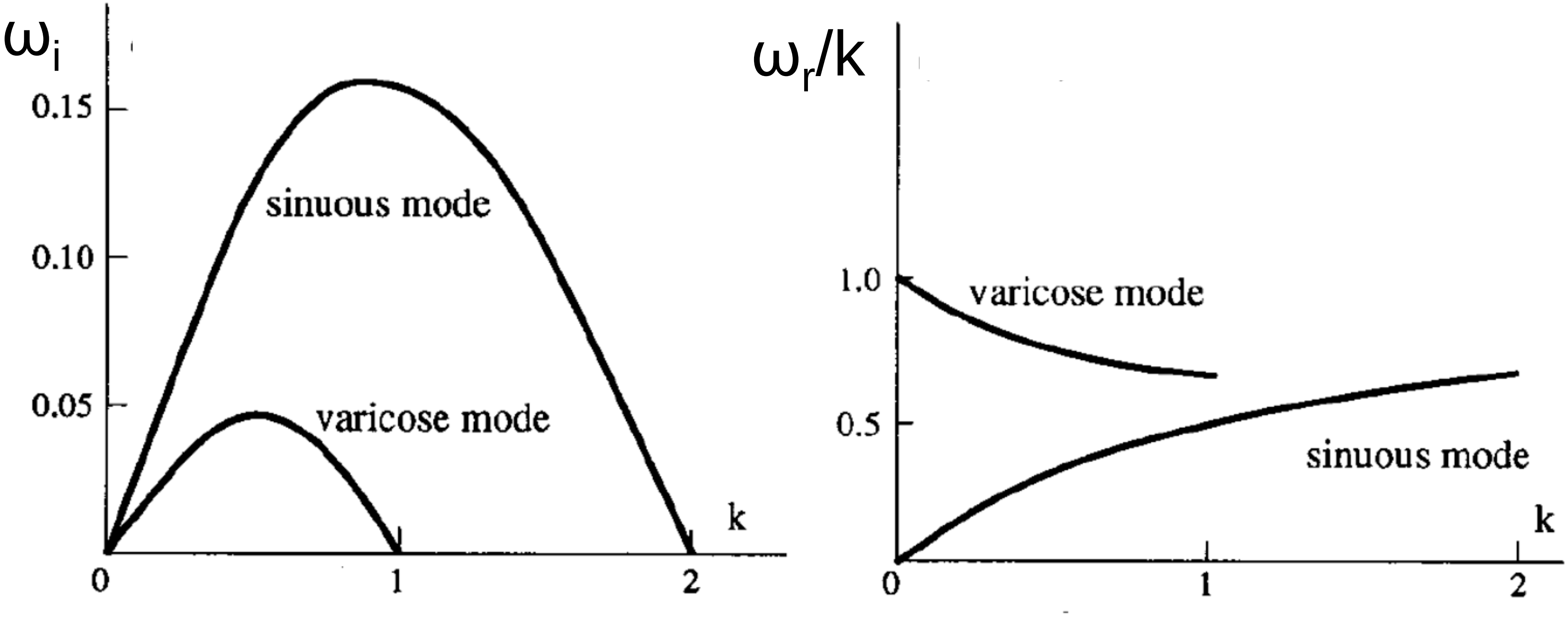
$$U(y) = U_\infty + (U_c - U_\infty)\,\mathrm{sech}^2\frac{y}{\delta}$$

Velocity ratio

$$R \equiv (U_c - U_\infty)/(U_c + U_\infty)$$



Wake instability



Betchov & Criminale (1966)

How to solve Rayleigh equation?

We fix k, we need to find all c and ψ such that

$$\left(U \left(\frac{d^2}{dy^2} - k^2 \right) - U''(y) \right) \psi = c \left(\frac{d^2}{dy^2} - k^2 \right) \psi$$

$$\psi(-L) = \psi(L) = 0$$

Formally,

$$\mathcal{A}\psi = c\,\mathcal{E}\psi$$

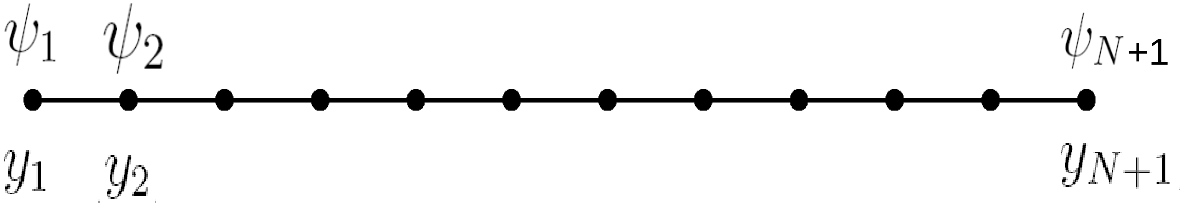
Discretize

$$\mathbf{A}\Psi = c\mathbf{E}\Psi$$

Generalized eigenvalue problem

How to solve Rayleigh equation?

Method 1: Finite differences of order 1.



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \\ \psi(y_{N+1}) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \psi_{N+1} \end{pmatrix} \qquad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \\ \psi''(y_{N+1}) \end{pmatrix}$$

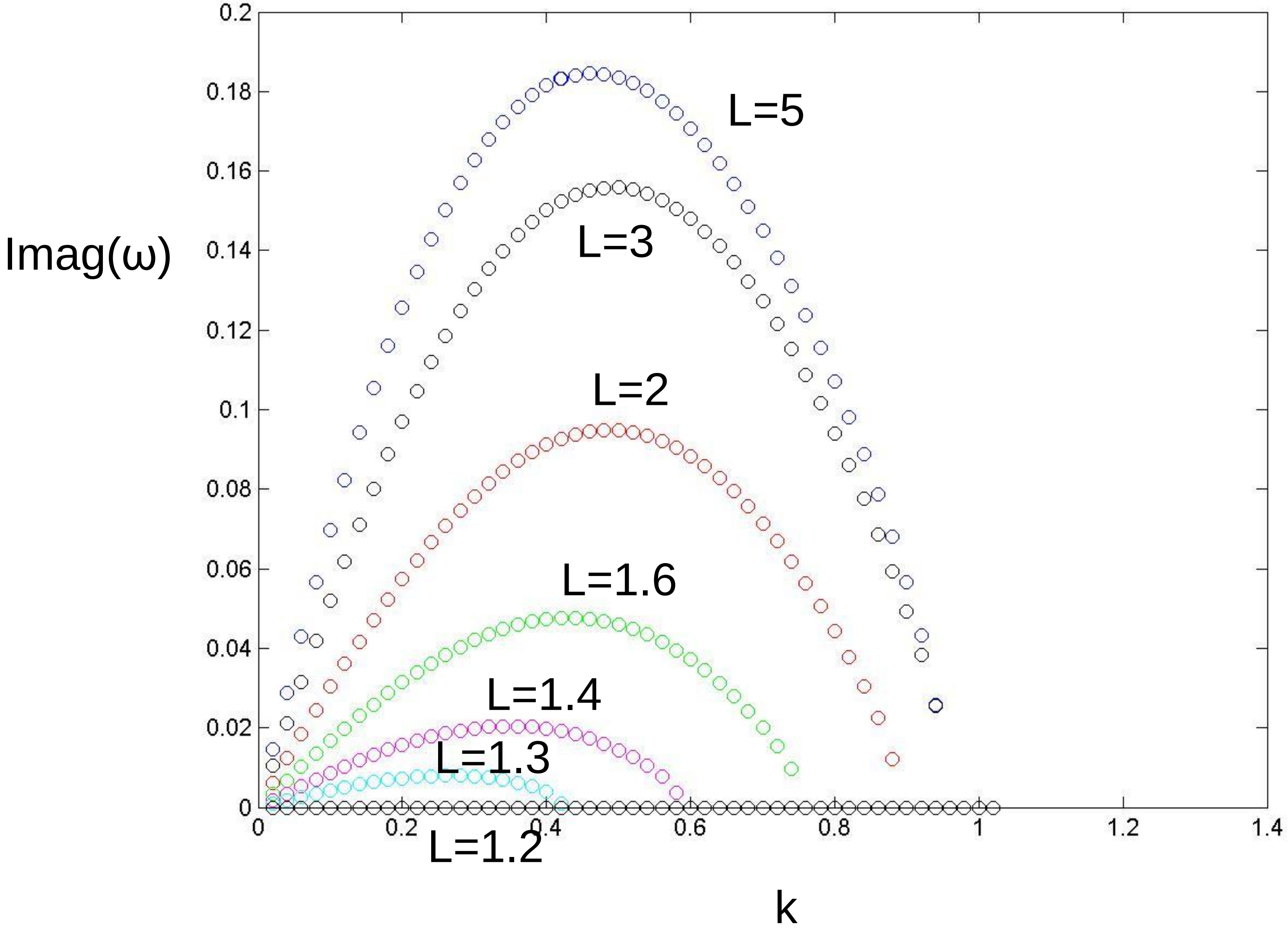
How to solve Rayleigh equation?

Method 1: Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

Influence of confinement



PARALLEL FLOW CONCEPTS

Viscous 3D instabilities

3D Navier - Stokes equations

$$\nabla \cdot \mathbf{U} = 0\,,$$

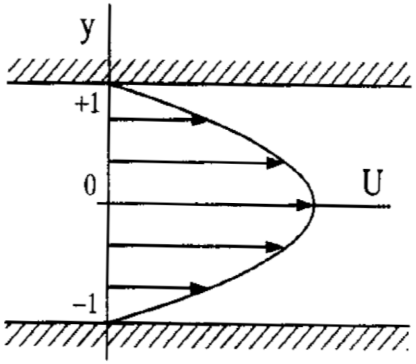
$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla P + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{U}$$

Basic flow + perturbation

$$\mathbf{U}(\mathbf{x},t) = U(y)\mathbf{e}_x + \mathbf{u}(\mathbf{x},t)$$

$$P(\mathbf{x},t) = P_0(x) + p(\mathbf{x},t)$$

Basic flow



PARALLEL FLOW CONCEPTS

Viscous 3D instabilities

Squire's transformation

$$\tilde{k}^2 = k_x^2 + k_z^2 \,, \qquad \tilde{c} = c \,,$$

$$\tilde{k}\tilde{u} = k_x \hat{u} + k_z \hat{w} \,, \qquad \tilde{v} = \hat{v} \,, \qquad \tilde{p}/\tilde{k} = \hat{p}/k_x \,.$$

$$\tilde{k}\widetilde{\mathrm{Re}} = k_x \, \mathrm{Re}$$

3D dispersion relation

$$D(\mathbf{k}, \omega; \mathrm{Re}) \equiv \tilde{D}\left[(k_x^2 + k_z^2)^{1/2}, \frac{(k_x^2 + k_z^2)^{1/2}}{k_x} \omega; \frac{k_x}{(k_x^2 + k_z^2)^{1/2}} \mathrm{Re} \right] = 0$$

To each oblique mode $(\mathbf{k}, \omega)$ of temporal growth rate ω_i, at Reynolds number Re, corresponds a two-dimensional mode $(\tilde{k}, \tilde{\omega})$ of larger growth rate $\tilde{\omega}_i = \omega_i \sqrt{k_x^2 + k_z^2}/k_x$, at a *lower* Reynolds number $\widetilde{\mathrm{Re}} = \mathrm{Re}\, k_x / \sqrt{k_x^2 + k_z^2}$.

2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi=\frac{1}{Re}\nabla^4\Psi$$

Basic flow + perturbation

$$\Psi(x,t)=\int U(y)dy+\psi(x,y,t)$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial \psi}{\partial x}=\frac{1}{Re}\nabla^4\psi$$

2D PARALLEL FLOW CONCEPTS

Dispersion relation

Normal mode decomposition

$$\psi(x,y,t) = \mathcal{R}e\left\{\phi(y)\, e^{i(kx-\omega t)}\right\}$$

Orr-Sommerfeld equation

$$[U(y)-c][\phi'' - k^2\phi] - U''(y)\,\phi = \frac{1}{i\,k\,Re}\left(\frac{d^2}{dy^2} - k^2\right)^2 \phi$$

$\phi(y) \Rightarrow 0$ at $y = \pm\infty$

Dispersion relation

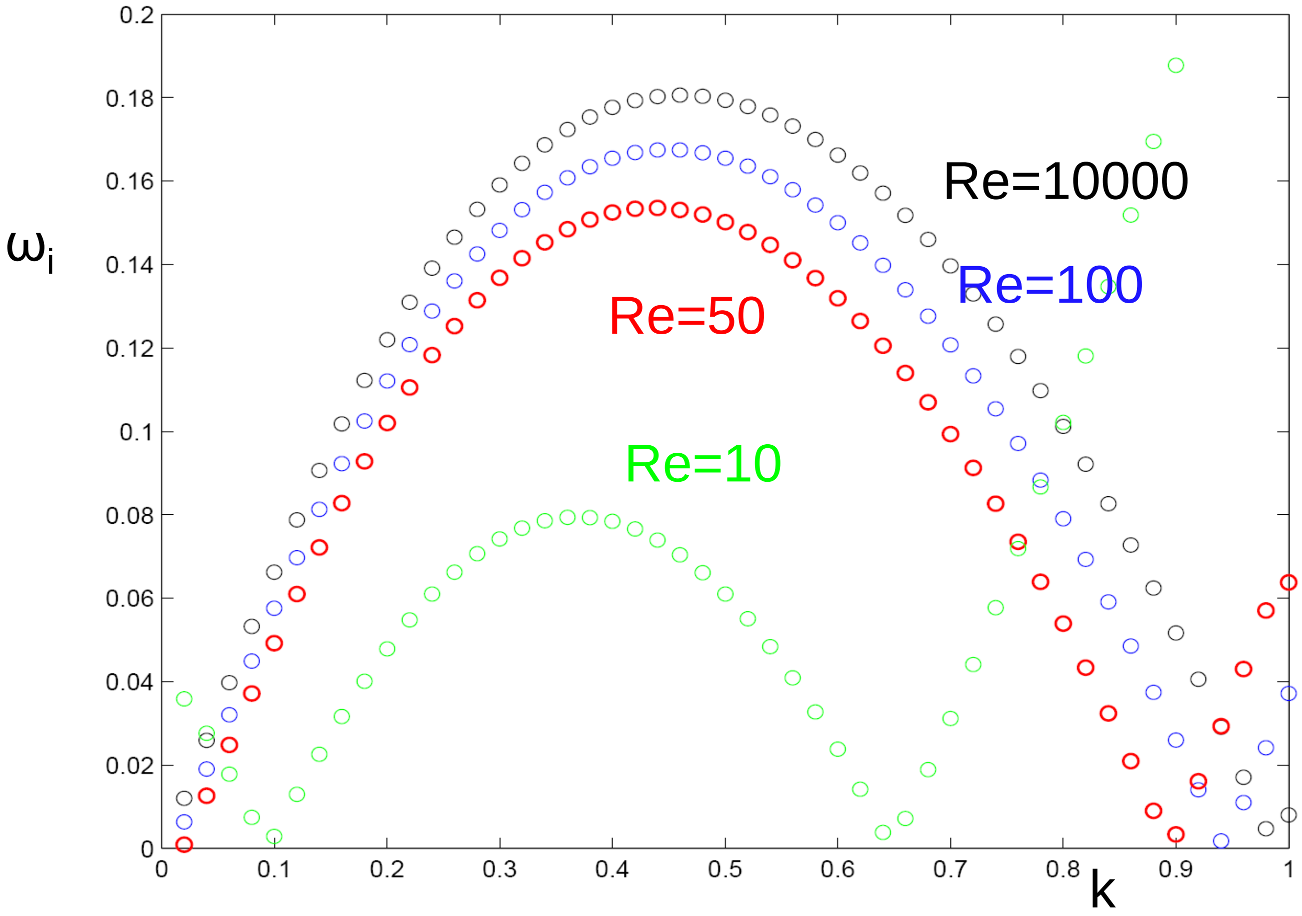
$$D(k,\omega;Re) = 0$$

Viscosity has limited stabilizing influence on K-H instability

$$Re := \Delta U\,\delta/\nu$$

$$\omega_{i,max}\frac{\delta}{\Delta U} \approx \sqrt{\frac{0,2}{1+\underset{\text{constant}}{@}/0,2Re}}$$

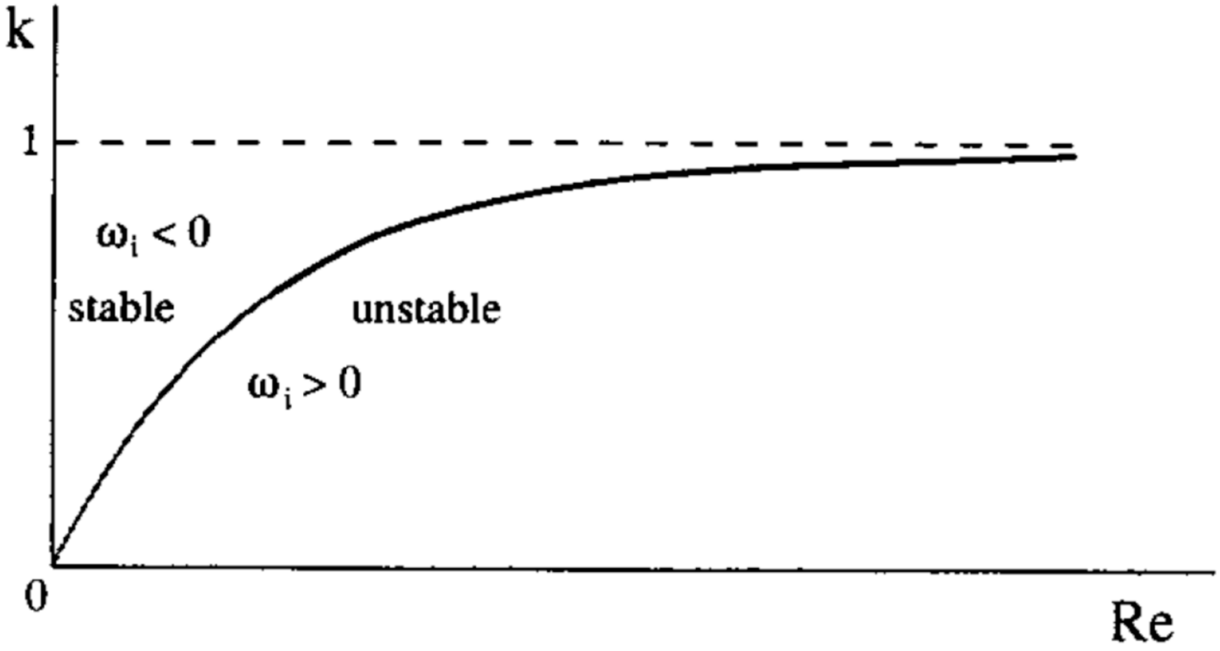
Viscosity has stabilizing influence on K-H instability



PARALLEL FLOW CONCEPTS

Viscous instabilities

Hyperbolic tangent mixing layer



What about stable flows (no inflexion point)?