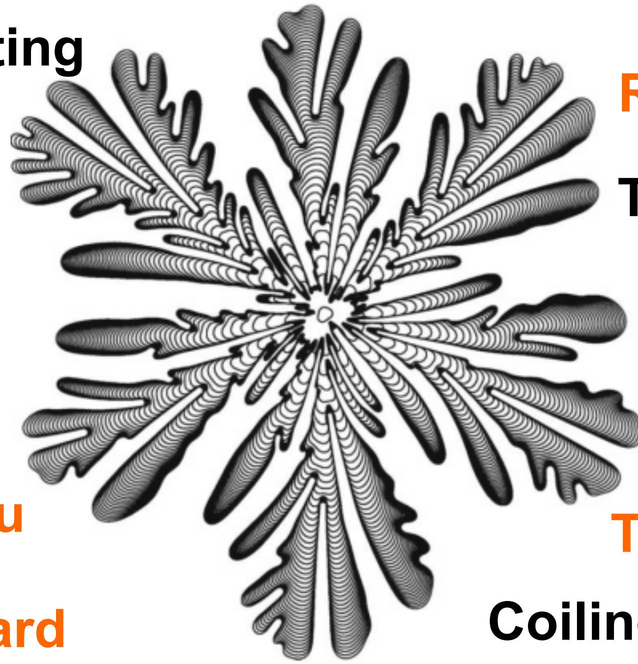
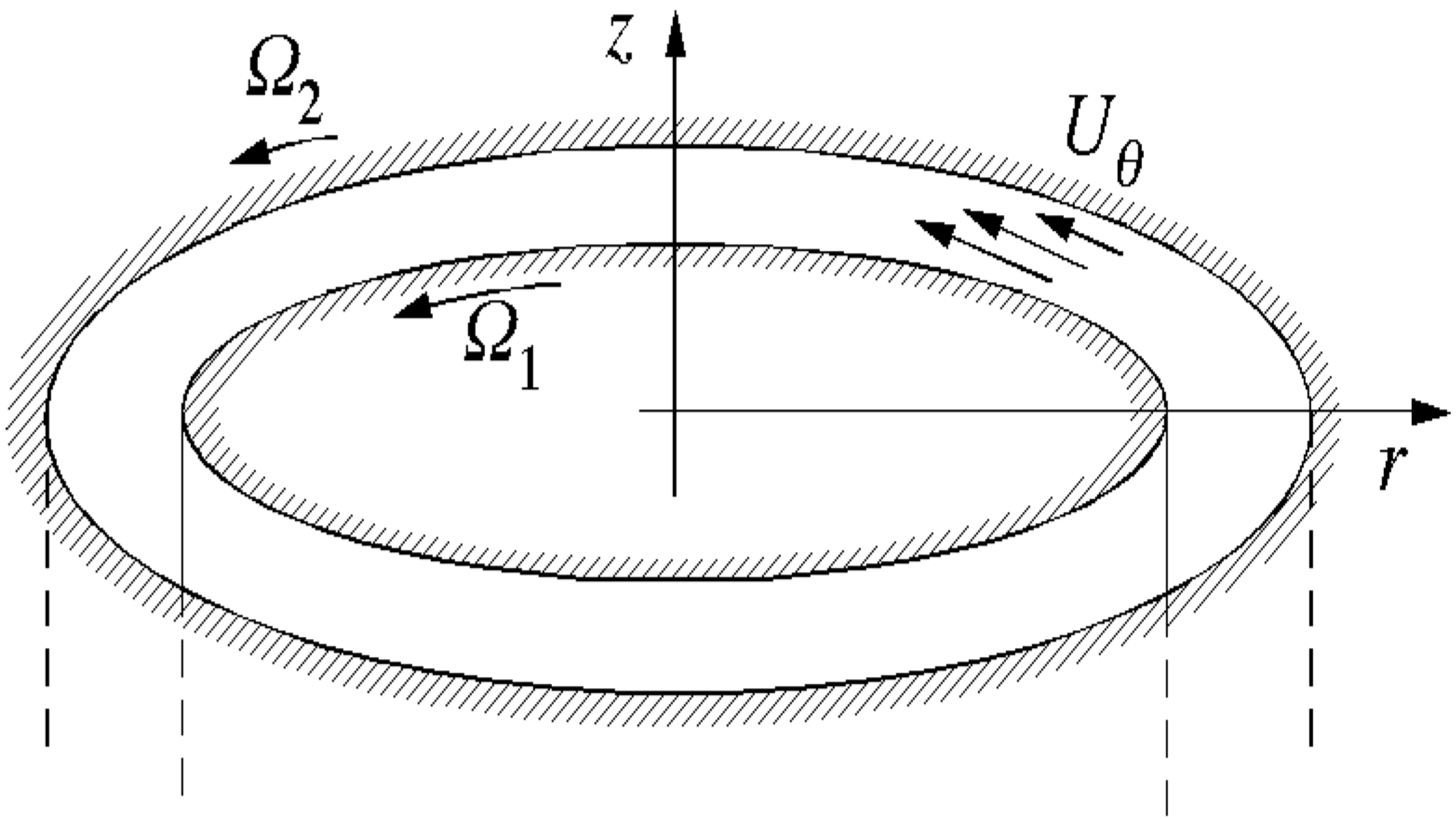
Most flows are unstable...

Vortex shedding
Saffman-Taylor
Flow separation
Tollmien-Schlichting
Rayleigh-Taylor



Lift-up and Streaks
Traffic waves
Meandering instability
Gravito-capillary waves
Taylor-Couette
Rayleigh-Plateau
Tearing instability
Rayleigh-Benard
Coiling instability
Benard-Marangoni
Kelvin-Helmholtz

Taylor Couette

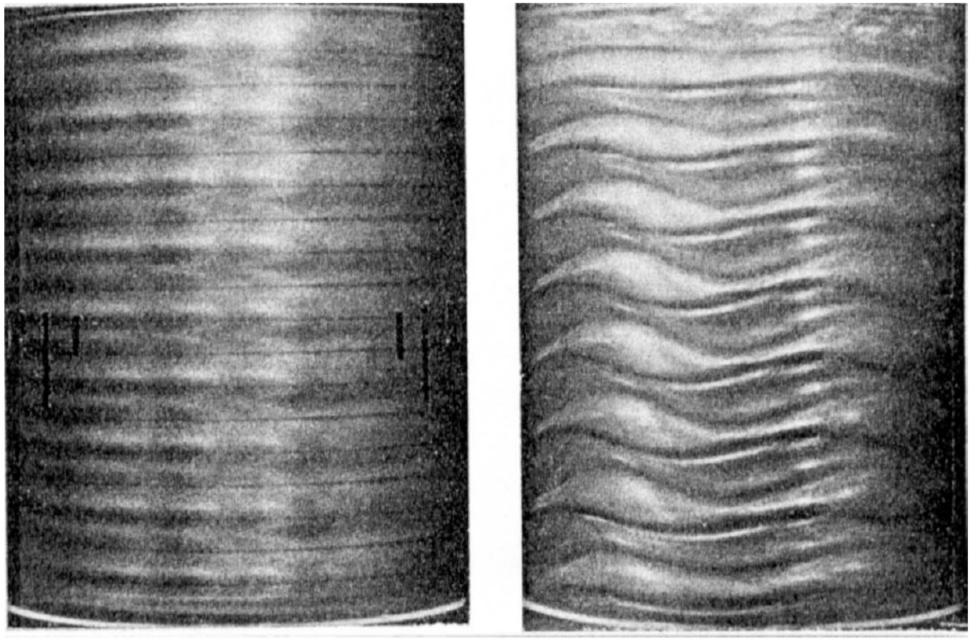


Taylor Couette

Movie by Garcia, Chomaz, Huerre, LadHyX, France

Taylor Couette

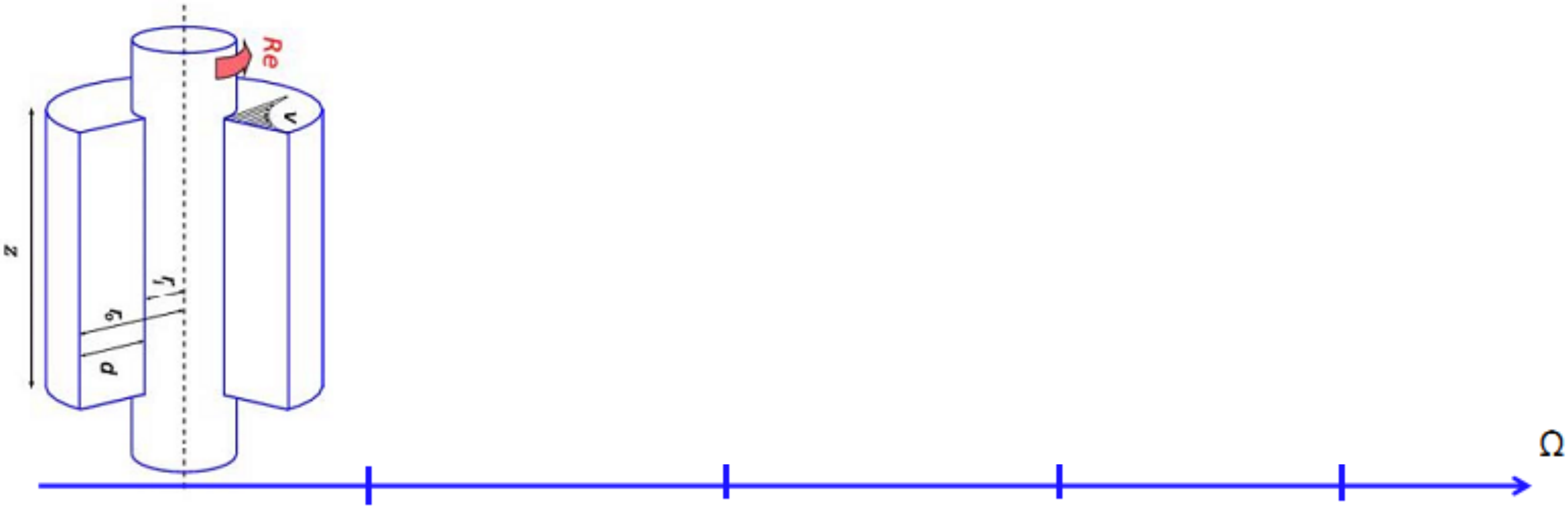
Movie by Garcia, Chomaz, Huerre, LadHyX, France



– *Rouleaux annulaires de Taylor. (a)* $Ta/Ta_c = 1.1$; *(b)* $Ta/Ta_c = 6.0$, *rouleaux ondulants apparus suite à une instablité secondaire* $(\lambda = 2\pi R/4)$. *(Fenstermacher, Swinney & Gollub 1979).*

Taylor Couette

Cascade of instabilities

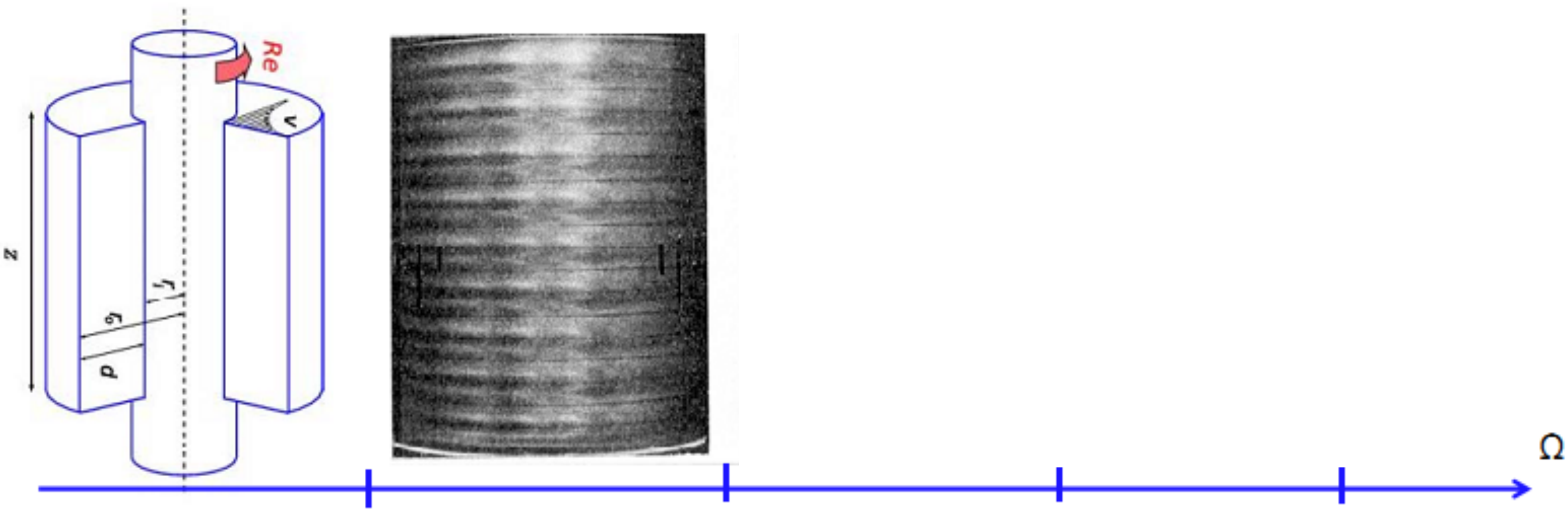


d/dt=0; d/dθ=0; d/dz=0;

Taylor Couette

Cascade of instabilities

Taylor vortices

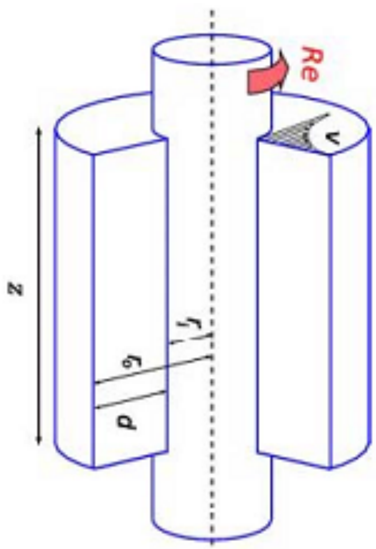


d/dt=0; d/dθ=0; d/dz=0;

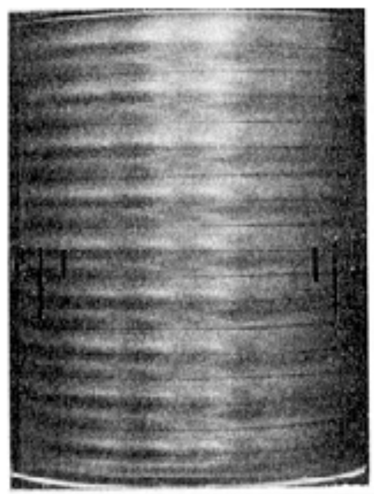
d/dt=0; d/dθ=0; d/d~~z~~=0;

Taylor Couette

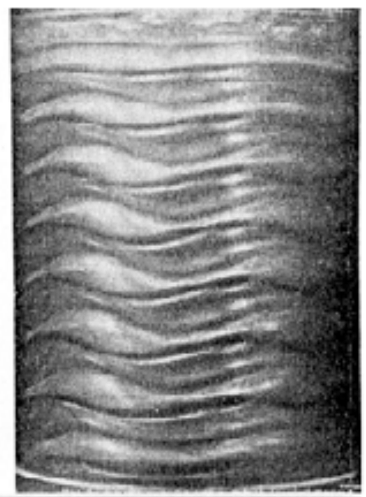
Cascade of instabilities



Taylor vortices



Ondulated vortices



Ω

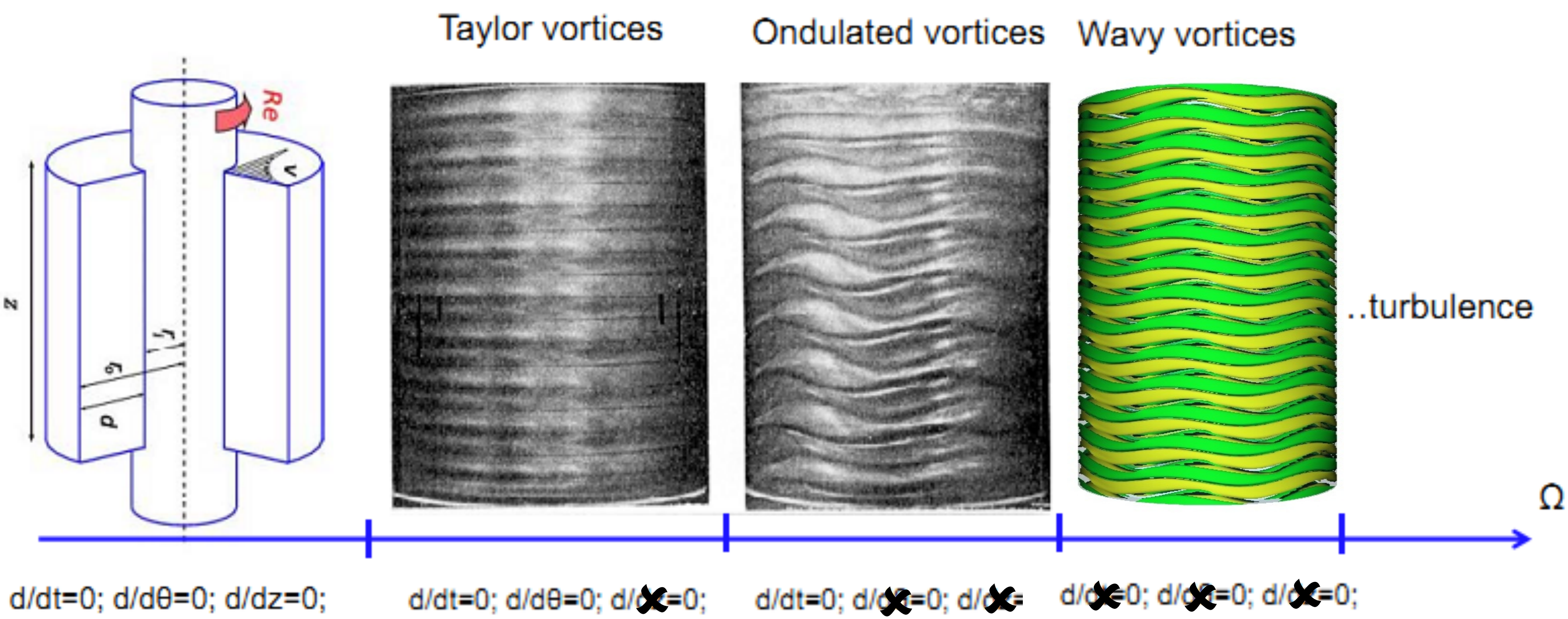
d/dt=0; d/dθ=0; d/dz=0;

d/dt=0; d/dθ=0; d/d✗=0;

d/dt=0; d/d✗=0; d/d✗=0;

Taylor Couette

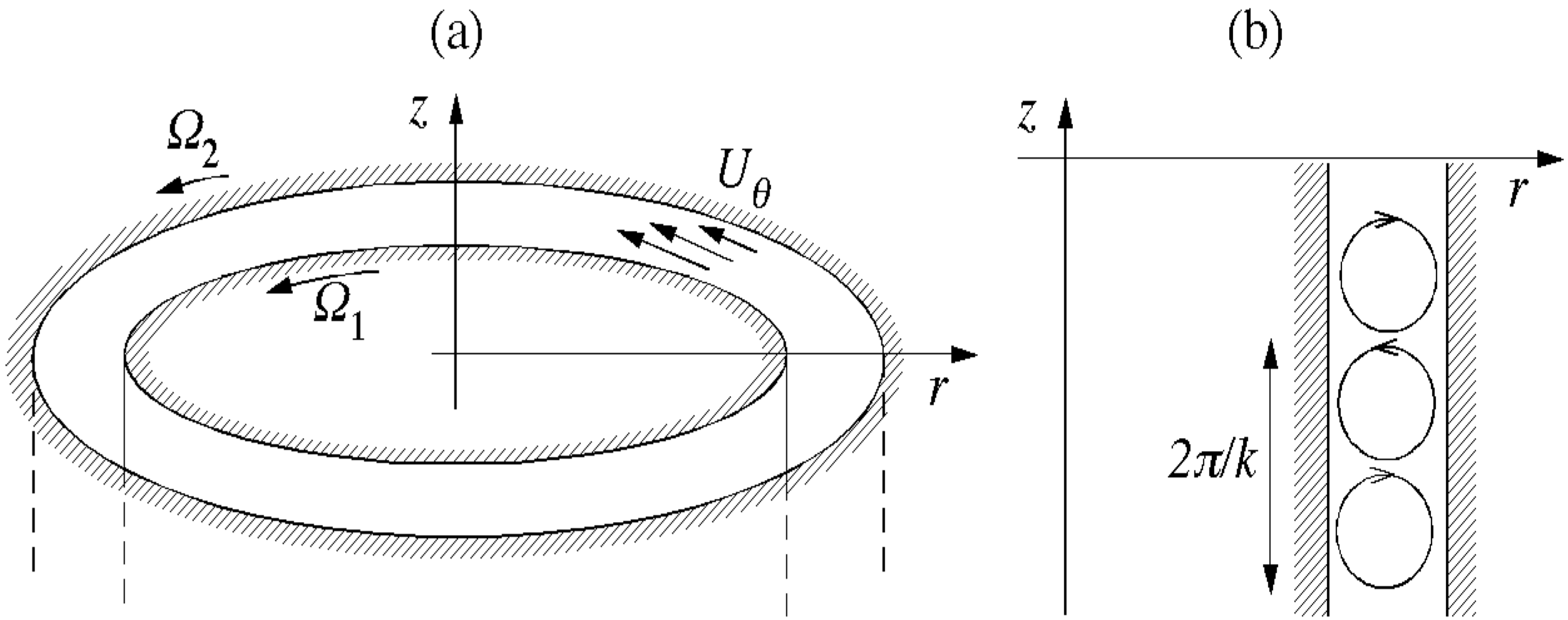
Cascade of instabilities



Instability analysis:

1. **Physical mechanism**
2. **Equations and boundary conditions**
3. **Base state**
4. **Linearized equations**
5. **Normal mode expansion**
6. **Dispersion relation**
7. **Analysis of the dispersion relation**

Taylor Couette



Navier-Stokes

$$\rho\left(\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right)$$

$$\rho\left(\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 u_z$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}\right)$$ convective derivative

$$\nabla^2 = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)$$ Laplacian

Navier-Stokes

inertia — pressure stress — viscous stress

$$\rho\left(\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right)$$

$$\rho\left(\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 u_z$$

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$$\nabla^2 = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)$$ Laplacian

Base flow

$$\rho \frac{U_\theta^2}{r} = \frac{\partial P}{\partial r}$$

$$\frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} = 0$$

Boundary conditions

$$U_\theta(R_1) = R_1\Omega_1$$

$$U_\theta(R_2) = R_2\Omega_2$$

General solution

$$U_\theta(r) = Ar + B/r$$

Boundary conditions

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad ; \quad B = \frac{(\Omega_1 - \Omega_2)\, R_1^2 R_2^2}{R_2^2 - R_1^2}$$

Rayleigh criterion

Rayleigh criterion: A necessary and sufficient condition for stability of the $\{O, U_\theta(r), 0\}$ basic flow ($r \in]R_1, R_2[$) to inviscid axisymmetric perturbations, is that $\partial (rU_\theta)^2 / \partial r > 0$ everywhere in $[R_1, R_2]$.

Inviscid flow

steady flow

axisymmetric flow

parallel flow

unidirectional flow

Rayleigh criterion

Rayleigh criterion: A necessary and sufficient condition for stability of the $\{O, U_\theta(r), 0\}$ basic flow ($r \in]R_1, R_2[$) to inviscid axisymmetric perturbations, is that $\partial \left(rU_\theta\right)^2 / \partial r > 0$ everywhere in $[R_1, R_2]$.

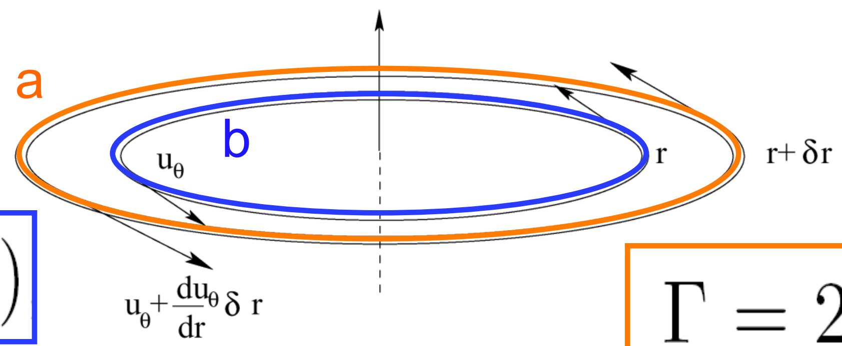
Inviscid flow	$\mu = 0$
steady flow	$\partial / \partial t = 0$
axisymmetric flow	$\partial / \partial \theta = 0$
parallel flow	$\partial / \partial z = 0$
unidirectional flow	$\{0, U_\theta(r), 0\}$

$$\rho U_\theta^2 / r = \partial P / \partial r$$

Mechanism for Rayleigh Criterion
Let us move an annulus of flow at r_b to another location $r_a > r_b$

Velocity profile before moving annulus

$$\Gamma = 2\pi r_b U_\theta(r_b)$$



$$\Gamma = 2\pi r_a u_\theta(r_a)$$

Velocity after moving annulus

Kelvin's theorem:

$$u_\theta(r_a) = (r_b/r_a)U_\theta(r_b)$$

Mechanism for Rayleigh Criterion
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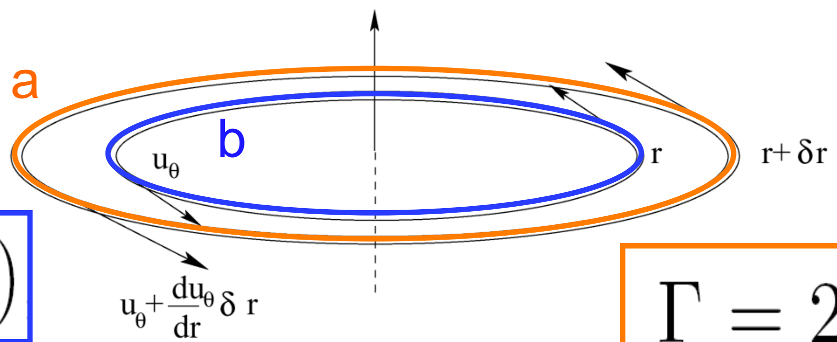


$$\Gamma = 2\pi r_b U_\theta(r_b)$$

$$\Gamma = 2\pi r_a u_\theta(r_a)$$

Velocity after moving annulus

Kelvin's theorem:

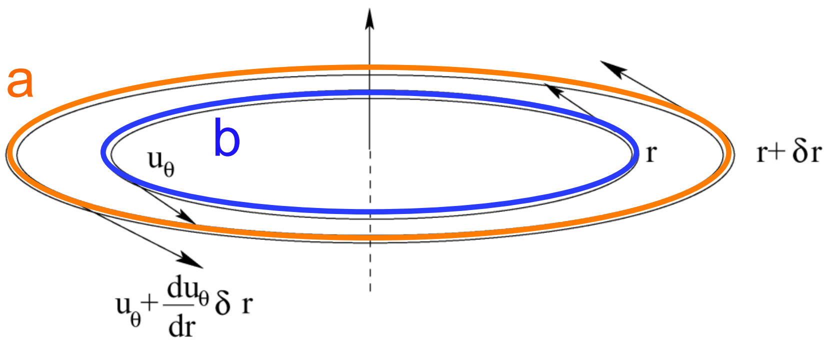
$$u_\theta(r_a) = (r_b/r_a)U_\theta(r_b)$$

Centrifugal force/ Pressure balance

$$\rho U_\theta^2/r = \partial P/\partial r$$

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



new velocity

if $$\overbrace{\rho u_\theta^2}(r_a)/r_a > (dP/dr)(r_a) = \rho U_\theta^2(r_a)/r_a$$

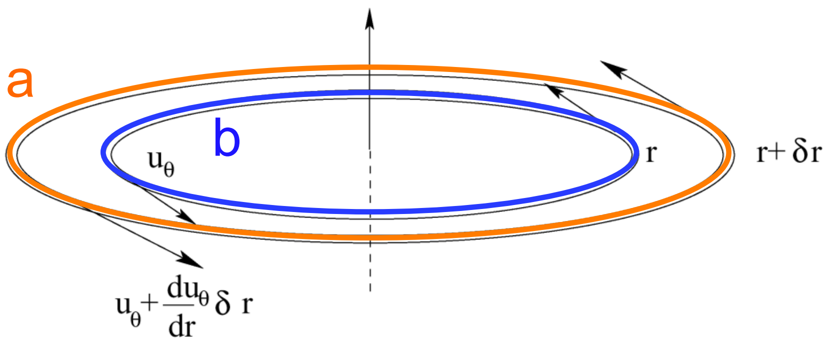
centrifugal force > pressure gradient

⇨ The annulus further escapes towards high r

⇨ UNSTABLE

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



new velocity

if $\overbrace{\rho u_\theta^2(r_a)}^{\text{new velocity}}/r_a < (dP/dr)(r_a) = \rho U_\theta^2(r_a)/r_a$

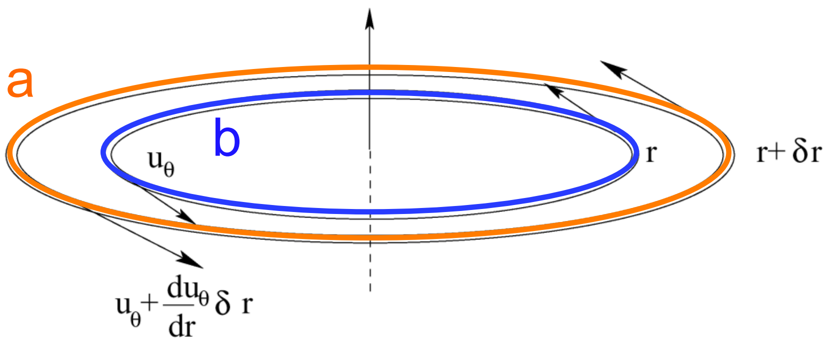
centrifugal force < pressure gradient

⇨ The annulus is brought back to its initial position

⇨ STABLE

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



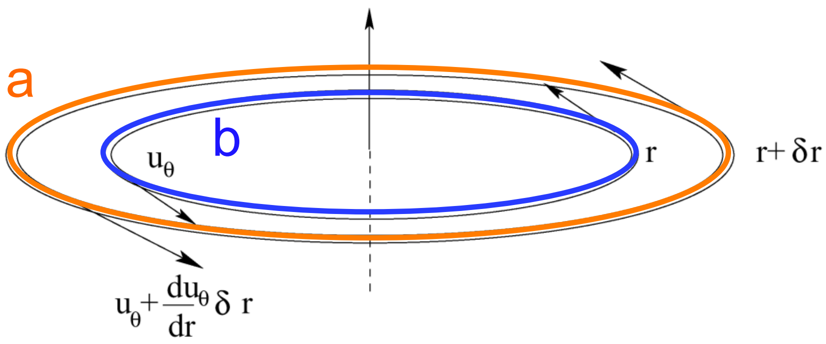
Condition for stability

$$U_\theta^2(r_b)r_b^2 < U_\theta^2(r_a)r_a^2$$

$$d\left(rU_\theta\right)^2/dr > 0$$

Centrifugal instabilities

Mechanism and Rayleigh criterion 1916



Condition for stability

$$U_\theta^2(r_b) r_b^2 < U_\theta^2(r_a) r_a^2$$

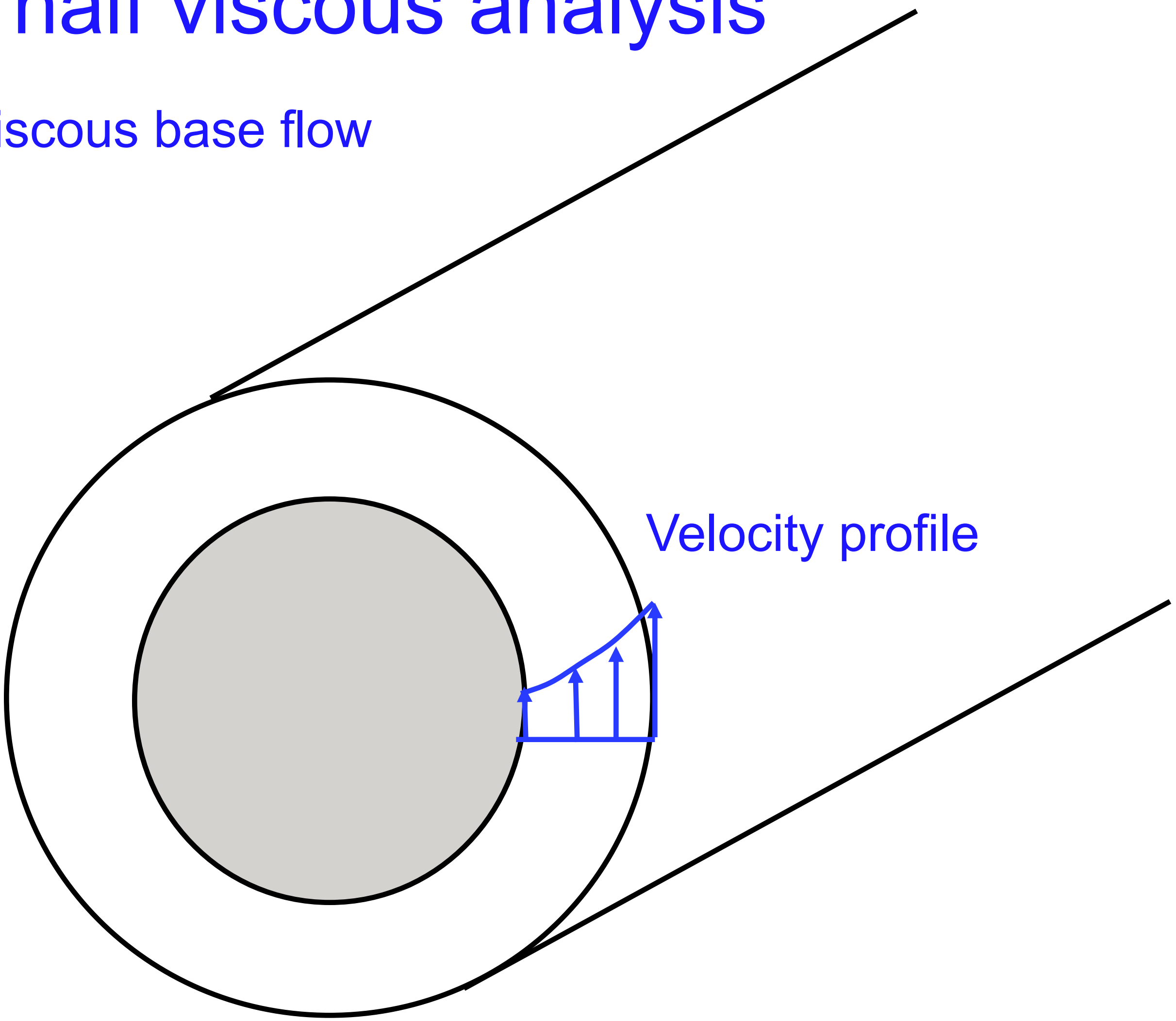
$$d\left(rU_\theta\right)^2/dr > 0$$

$U_\theta \varsigma > 0$

axial vorticity
$\varsigma = 1/r\ d(ru_\theta)/dr$

The half viscous analysis

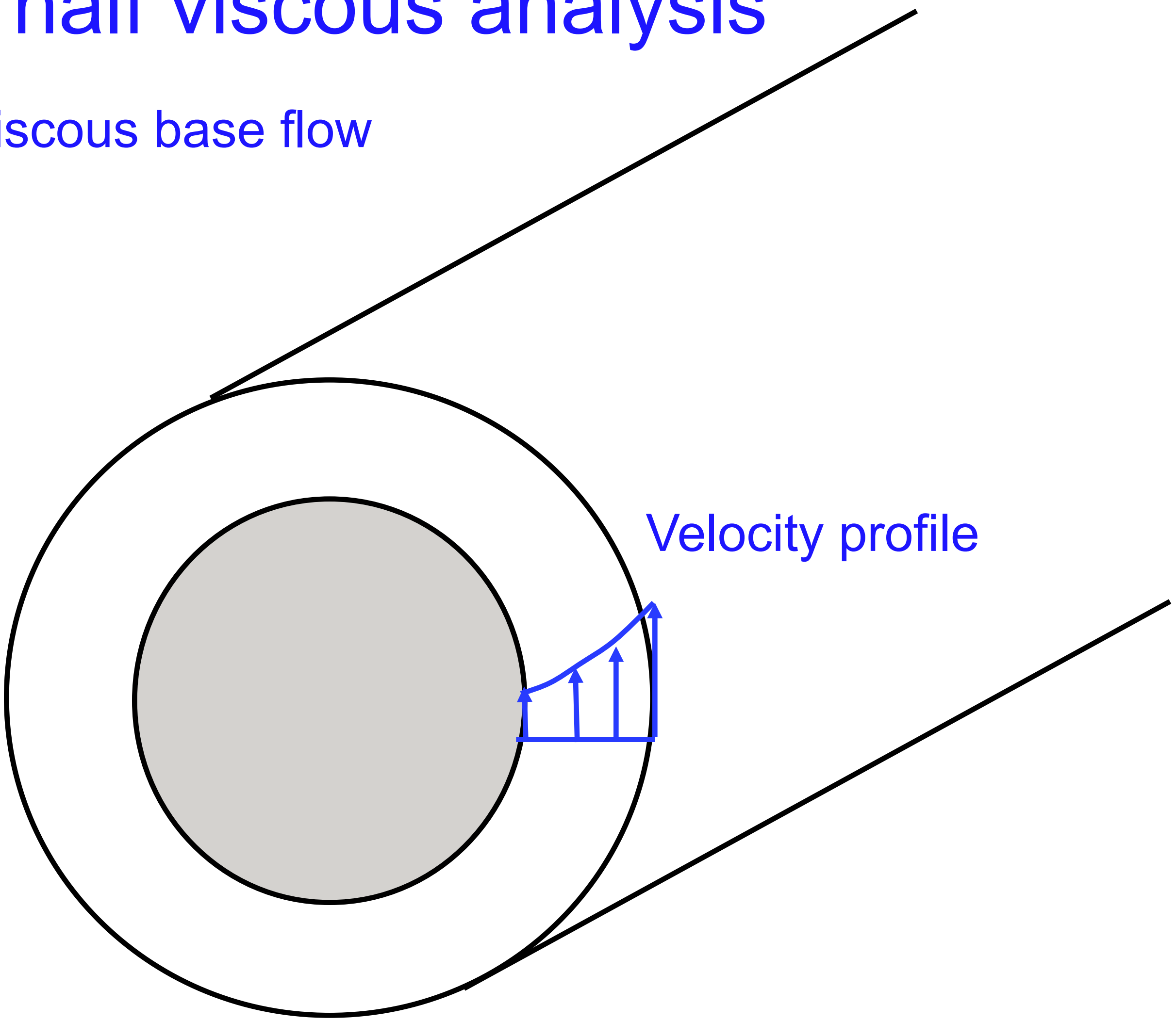
Inviscid analysis of a viscous base flow



The base flow velocity profile is only selected by viscosity!

The half viscous analysis

Inviscid analysis of a viscous base flow



if stable, viscosity is expected to be further stabilizing

if unstable, we expect a critical Reynolds number

The half viscous analysis

Inviscid analysis of a viscous base flow?

viscous time?

convective time?

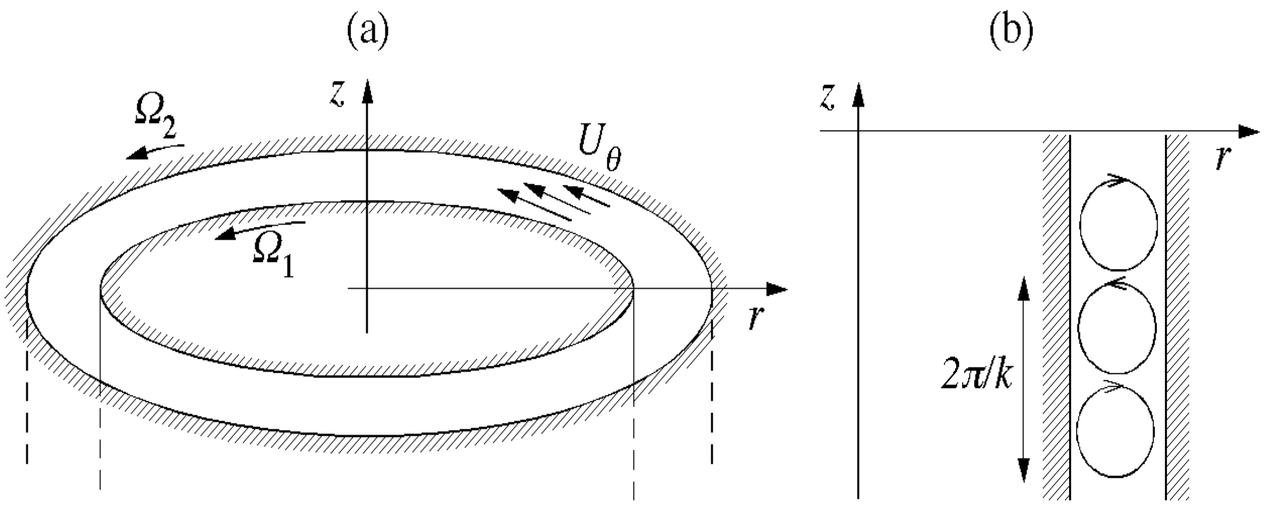
The half viscous analysis

Inviscid analysis of a viscous base flow

$$\tau_\nu = (R_2 - R_1)^2/\nu$$ viscous time

$$\tau_{U_2} = (R_2 - R_1)/(\Omega_2 R_2)$$ convective time

$$\tau_\nu/\tau_{U_2} = (R_2 - R_1)\Omega_2 R_2/\nu = \mathrm{Re}_2$$ Reynolds number

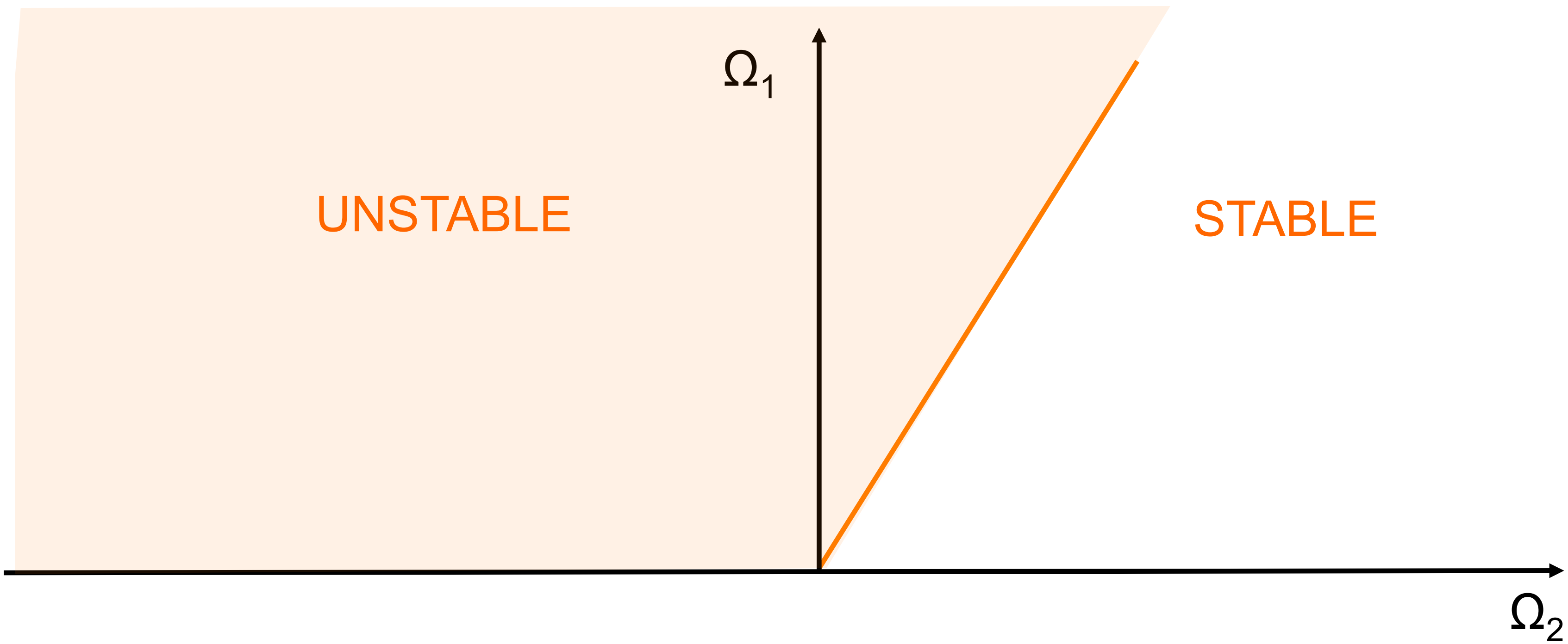


Rayleigh criterion

$$d(rU_\theta)^2/dr = 4Ar(Ar^2 + B)$$ <0

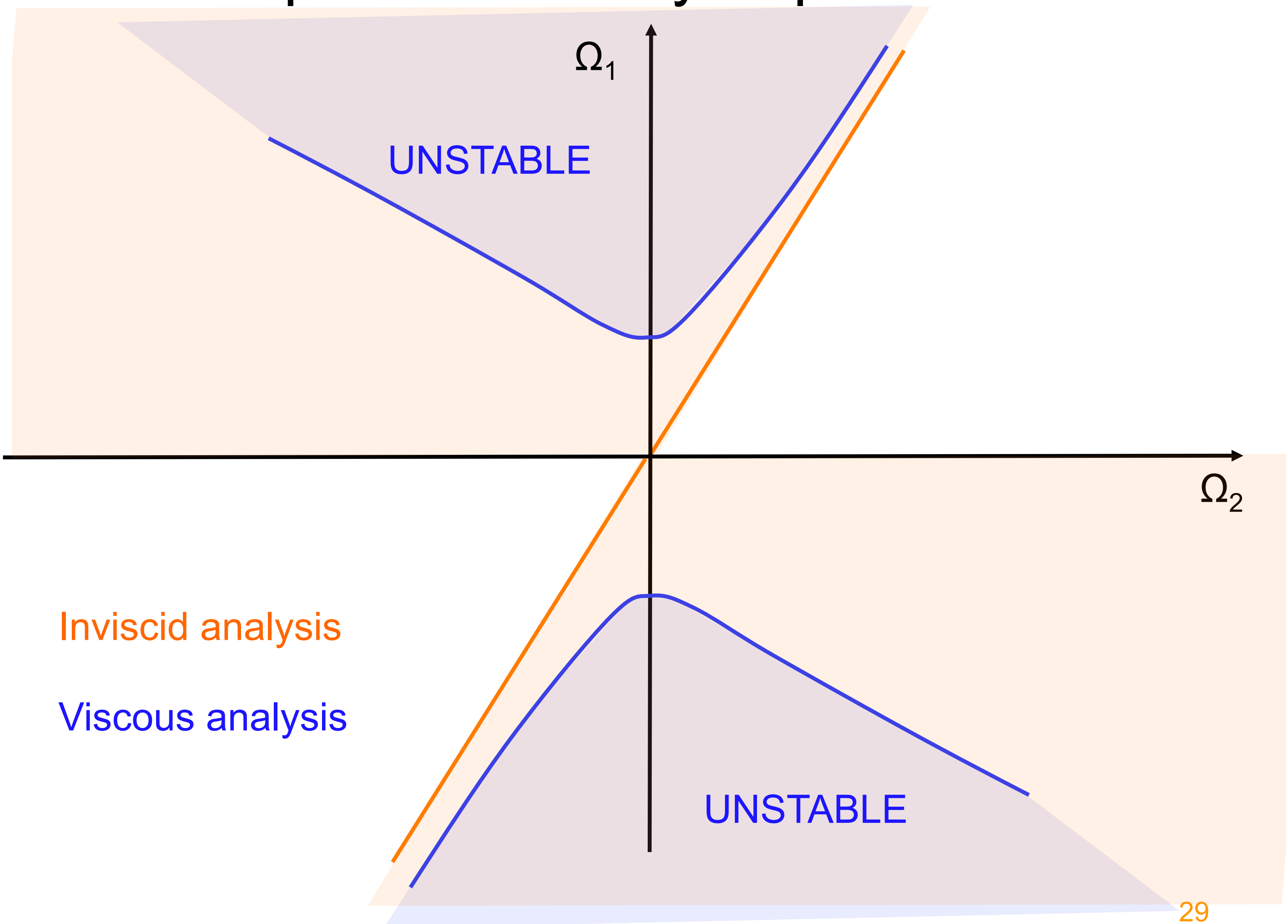
Unstable if $\Omega_2 r_2^2 < \Omega_1 r_1^2$.

Comparison Theory/Experiment

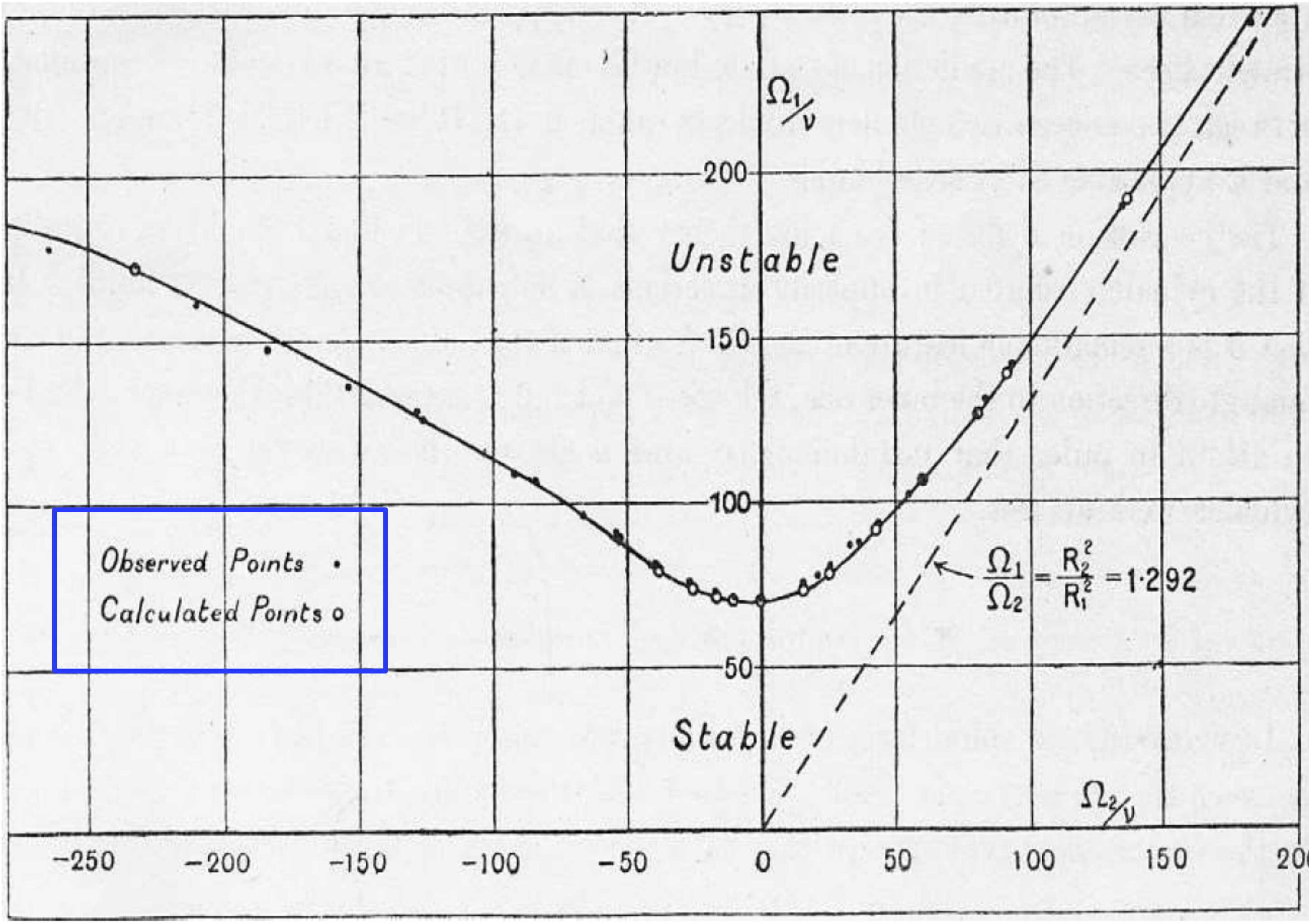


$$\Omega_2 r_2^2 < \Omega_1 r_1^2.$$

Comparison Theory/Experiment



Comparison Theory/Experiment



Comparison between observed and calculated speeds at which instability first appears in the case when $R_1 = 3.55$ cm, $R_2 = 4.035$ cm (from Taylor, 1923).