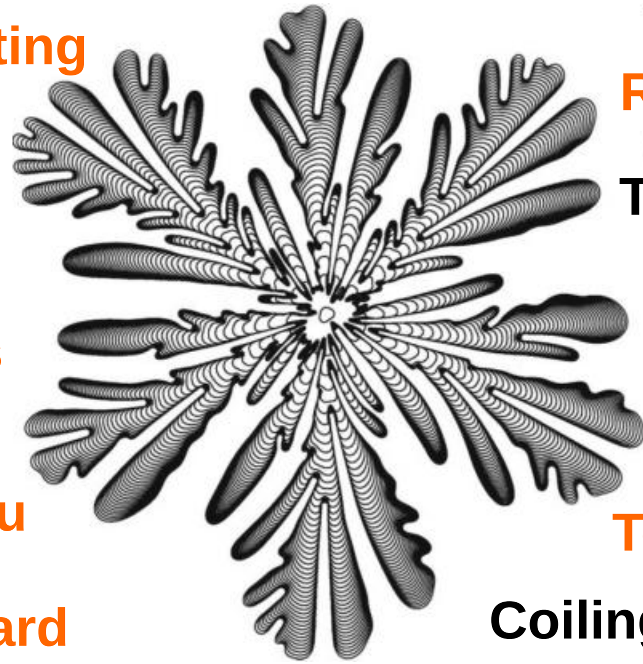
# Most flows are unstable...

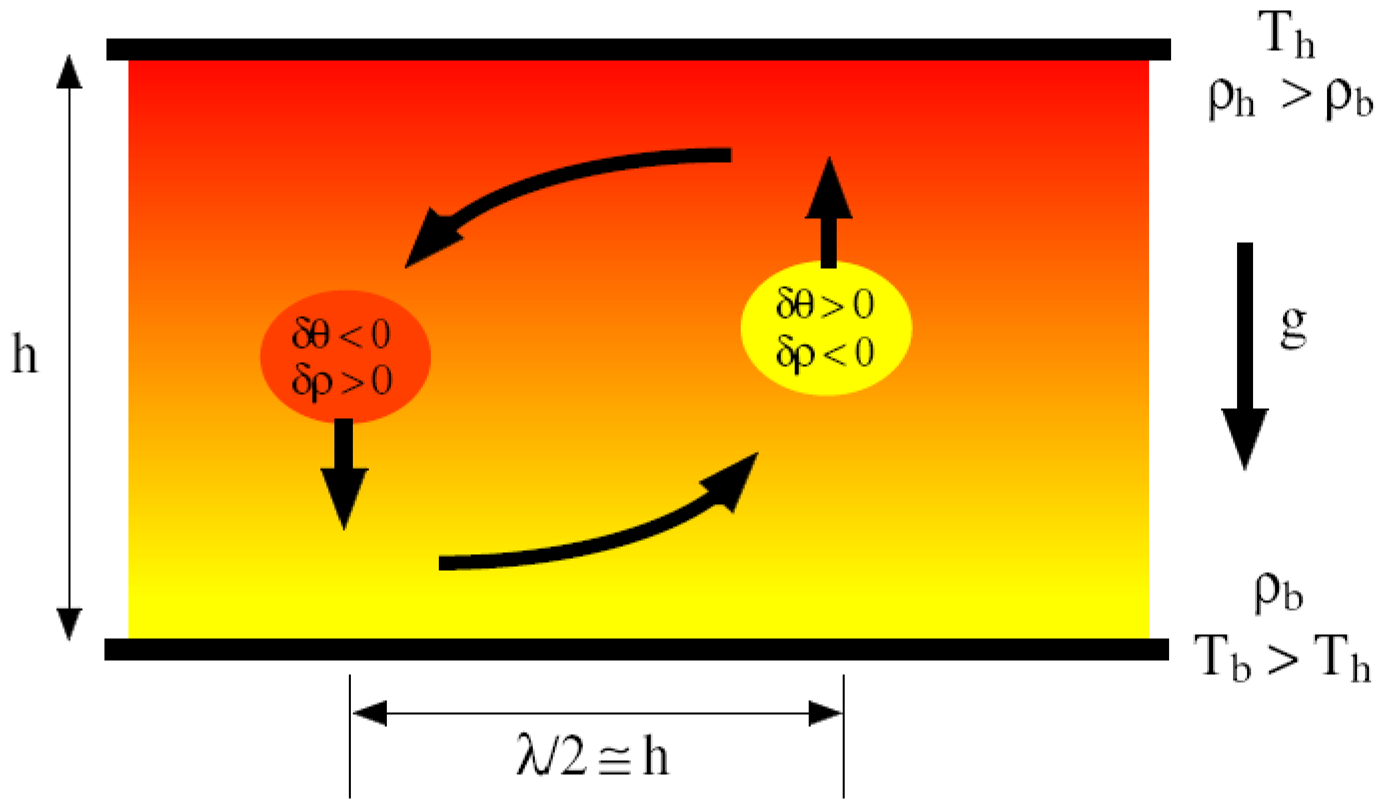
Vortex shedding
Saffman-Taylor
Flow separation
Tollmien-Schlichting
Rayleigh-Taylor
Lift-up and Streaks
Traffic waves
Meandering instability
Gravito-capillary waves



Taylor-Couette
Rayleigh-Plateau
Tearing instability
Rayleigh-Benard
Coiling instability
Benard-Marangoni
Kelvin-Helmholtz

## Chaos

# Rayleigh Bénard convection



# Lessons learned from linear stability analysis

1. The pure conductive solution becomes unstable above a threshold, which for fixed depth h and fluid properties is simply proportional to the temperature gradient.

2. We therefore introduced a control parameter which equals 0 at threshold

$$r = \frac{\Delta T - \Delta T_c}{\Delta T_c}$$

3. Close to threshold, when linearity can be assumed, the perturbation writes

$$v_z = X(t) f_v(z) \sin(kx)\,, \qquad k = 2\pi/\lambda\,.$$

4. Temperature and vertical velocity are in phase

5. The unstable wavelength at threshold is roughly 2h

# Heuristic equation for an unstable system

$$\frac{d}{dt}X = \sigma X$$

$$\sigma = r/\tau_0$$

1. We retrieve the critical slowing at threshold
2. The system is stable (damped in fact!) for $r<0$ and unstable for $r>0$

# Heuristic equation for an unstable system

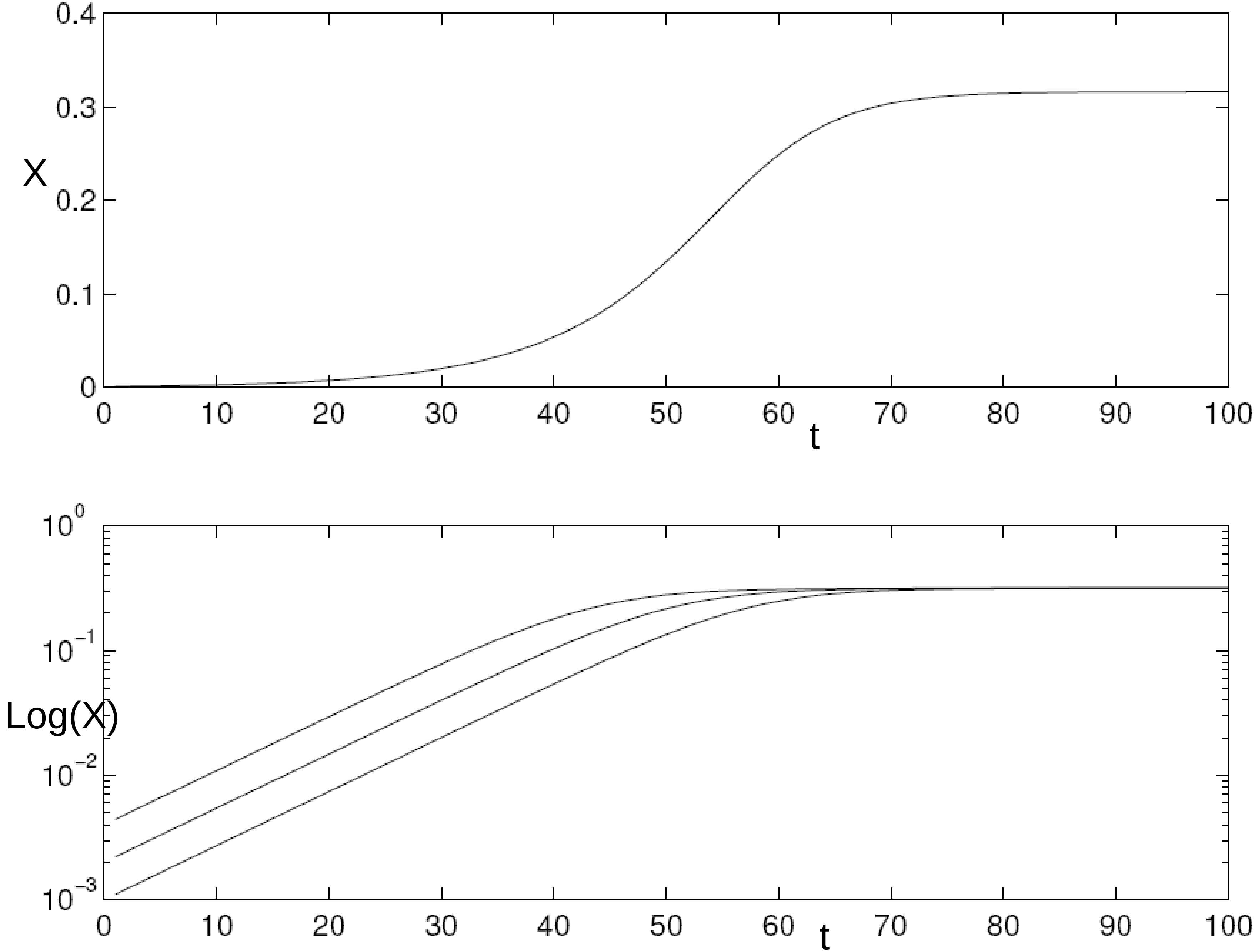
1. Since for r>0, there is exponential growth, X cannot assumed to remain small for long…
2. Nonlinear effects have to be introduced

$$\frac{d}{dt}X = \sigma_{\text{eff}} X$$

3. where $\sigma_{\text{eff}}$ is now a function of X!

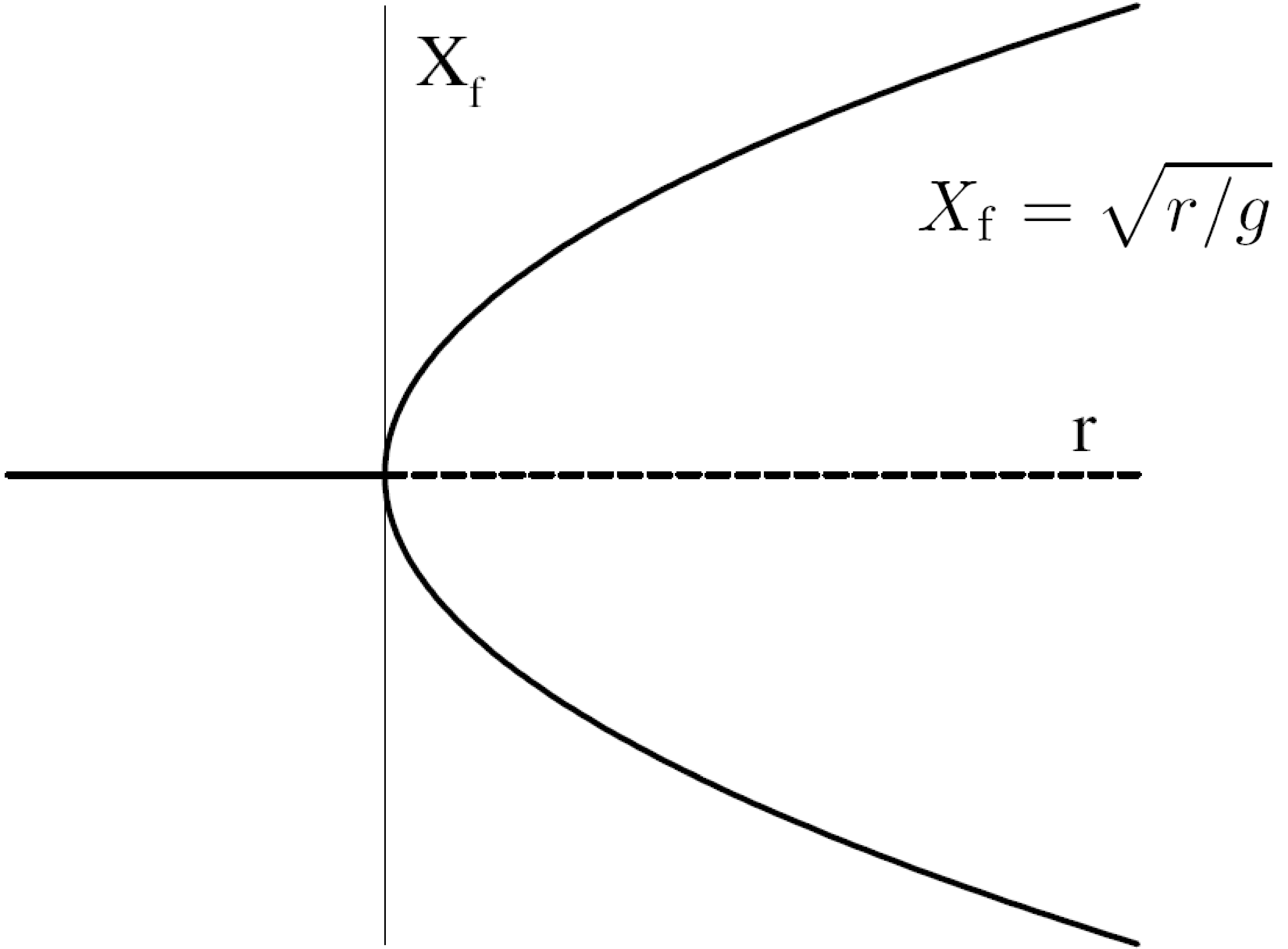
4. By symmetry, there is no reason that a positive velocity anomaly would behave differently than a negative one…

$$\tau_0 \sigma_{\text{eff}} = r - gX^2$$



X
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80
90
100
t
Log(X)
$10^0$
$10^{-1}$
$10^{-2}$
$10^{-3}$
0
10
20
30
40
50
60
70
80
90
100
t

$$\tau_0 \frac{d}{dt} X = F_r(X) = rX - gX^3$$


Xf
$X_{\rm f} = \sqrt{r/g}\,.$
r

# Stability of the bifurcated state?

$$\begin{aligned}\tau_0 \tfrac{d}{dt}(X_\mathrm{f} + X') &= \tau_0 \tfrac{d}{dt} X' = r(X_\mathrm{f} + X') - g(X_\mathrm{f} + X')^3 \\ &= \left[r X_\mathrm{f} - g X_\mathrm{f}^3\right] + \left(r - 3g X_\mathrm{f}^2\right) X' + \mathcal{O}\left({X'}^2\right)\end{aligned}$$

$$\tau_0 \frac{d}{dt} X' = -2rX'$$

Stable since r>0!

# Now we include more physics

X: vertical velocity
Y : temperature anomaly
P: Prandtl number

Driving force Archimedes → $Y$

Viscous resistance → $-X$

$$\frac{d}{dt}X = P(Y - X)$$

# Now we include more physics

X: vertical velocity
Y : temperature anomaly
P: Prandtl number
R: Rayleigh number

Driving force Archimedes → $Y$; Viscous resistance → $X$

$$\frac{d}{dt}X = P(Y - X)$$

$$\frac{d}{dt}Y = RX - Y$$

Heat transport → $RX$; Heat diffusion → $Y$

# Stability of equilibrium state (X=0,Y=0)

$$X = X_0 \exp(st) , \qquad Y = Y_0 \exp(st) ,$$

$$\begin{aligned} sX_0 &= P(Y_0 - X_0) , \\ sY_0 &= RX_0 - Y_0 , \end{aligned}$$

$$(s + P)(s + 1) - RP = 0$$

$$s^{(\pm)} = \frac{1}{2} \left( -(P + 1) \pm \left[ (P - 1)^2 + 4PR \right]^{1/2} \right)$$

# Stability of equilibrium state (X=0,Y=0)

$$X = X_0 \exp(st)\,, \qquad Y = Y_0 \exp(st)\,,$$

$$s^{(\pm)} = \tfrac{1}{2}\left(-(P+1) \pm \left[(P-1)^2 + 4PR\right]^{1/2}\right)$$

>0

Unstable if $\left[(P-1)^2 + 4PR\right] > (P+1)^2$

$$4PR > 4P$$

$$R > 1$$

There is also a mean temperature correction Z

X $\propto \sin(kx)$

Y $\propto \sin(kx)$

$$\frac{d}{dt}Z = XY - bZ$$

# Lorentz model

X: vertical velocity

Y : temperature anomaly

Z: mean temperature correction

Driving force Archimedes

Viscous resistance

$$\frac{d}{dt}X = P(Y - X)$$

Heat transport

$$\frac{d}{dt}Y = (R - Z)X - Y$$

Heat diffusion

$$\frac{d}{dt}Z = XY - bZ$$

Relaxation to linear temperature profile

Mean effect on temperature

# Limiting equation for R close to 1 and X=Y

$$Z = XY/b$$

$$\frac{d}{dt}X = (R-1)X - X^3/b$$

# Now consider Lorentz system at large R and b=8/3, P=10

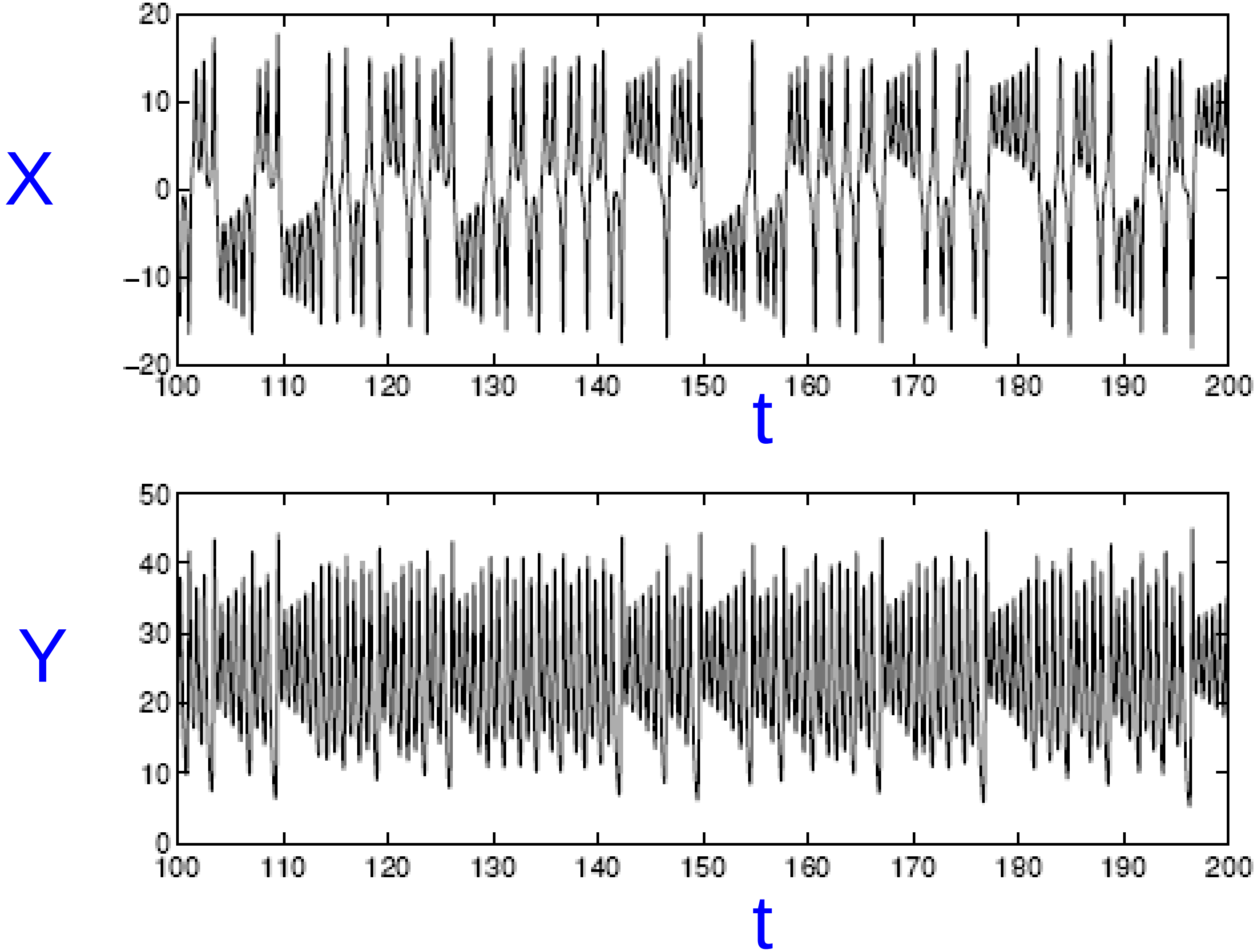
$$\frac{d}{dt}X = P(Y - X)$$

$$\frac{d}{dt}Y = (R - Z)X - Y$$

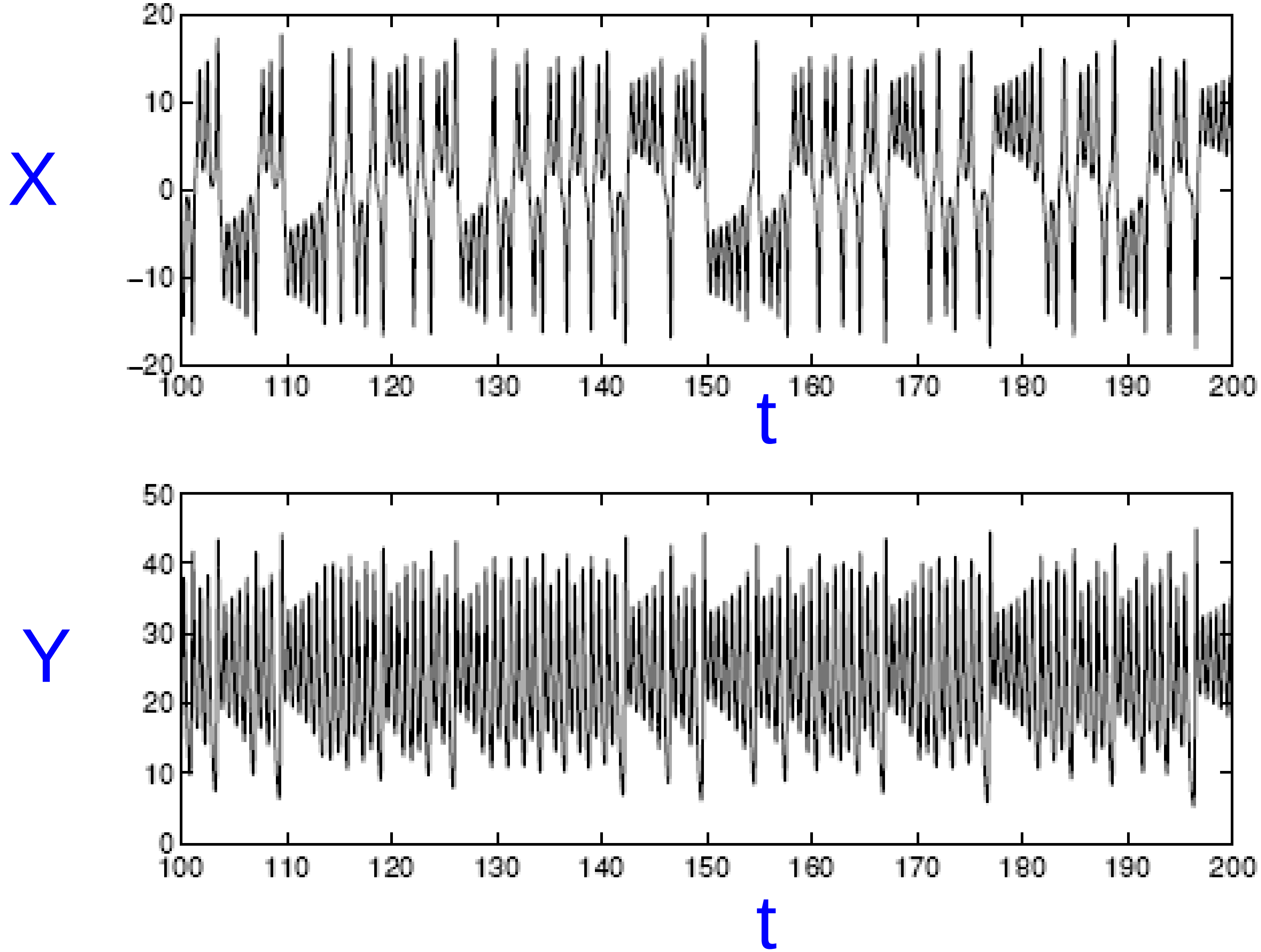
$$\frac{d}{dt}Z = XY - bZ$$

One can show that the steady convection roll solution becomes unstable for R>24.74

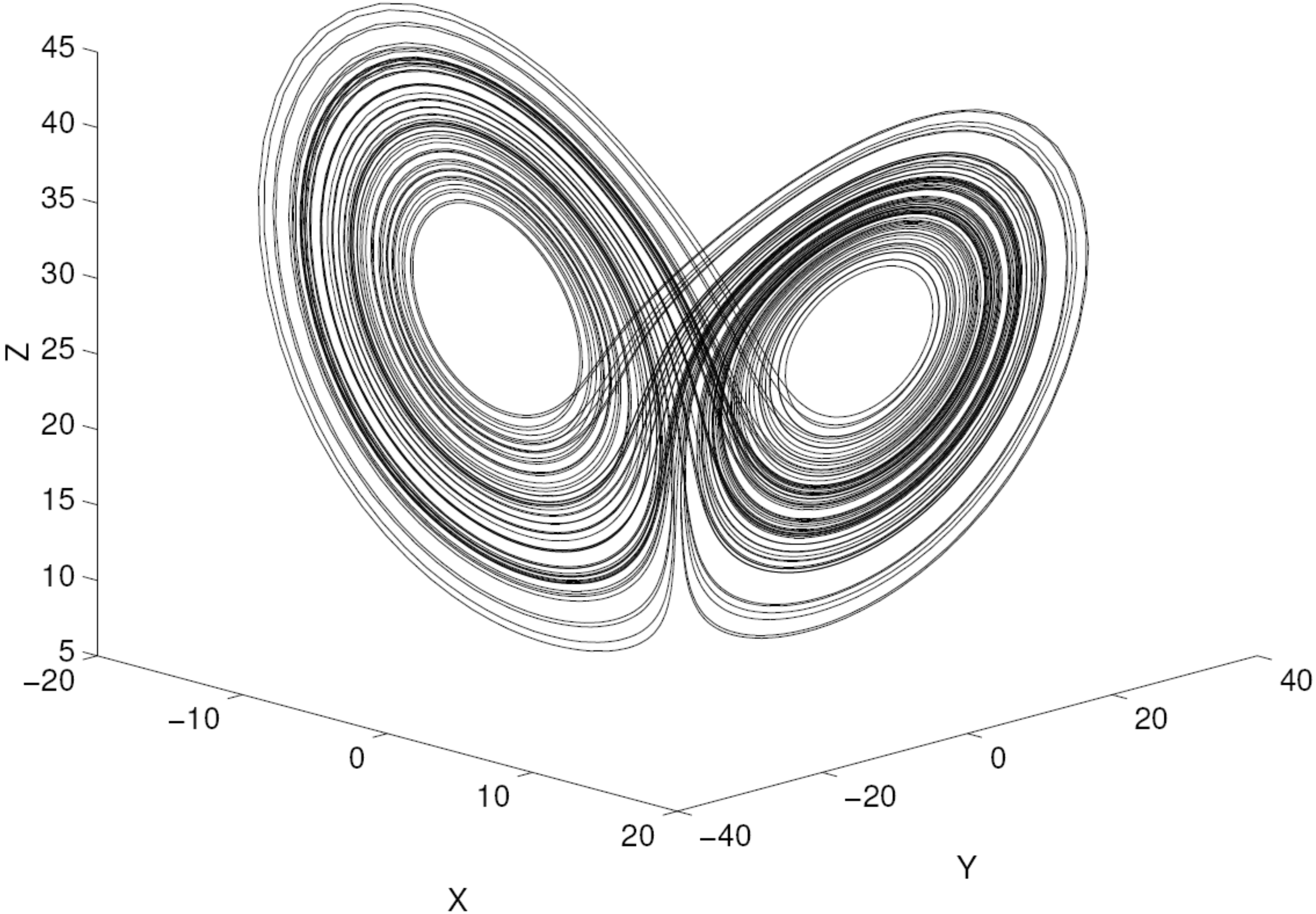
# Here R=28, b=8/3, P=10



# Here R=28, b=8/3, P=10
# This is chaotic!



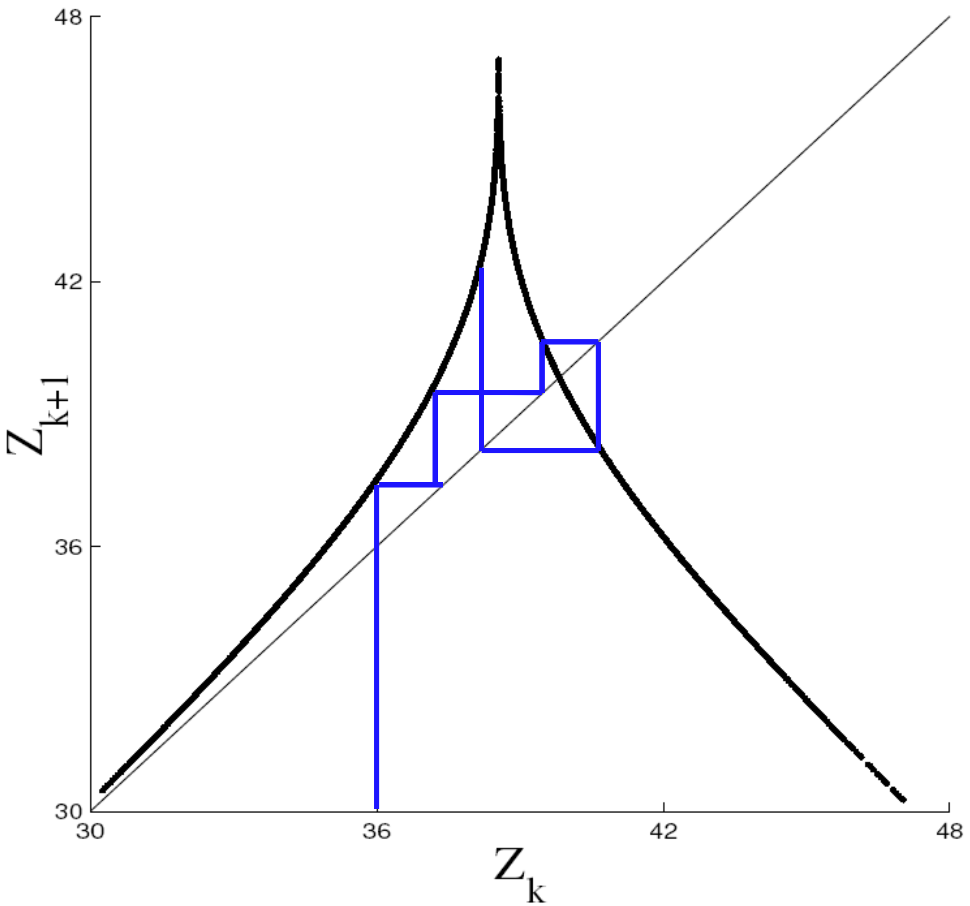
# Here R=28, b=8/3, P=10
# This is chaotic!



# What happens?

## Lorenz plot the successive maxima of Z, $Z_k$

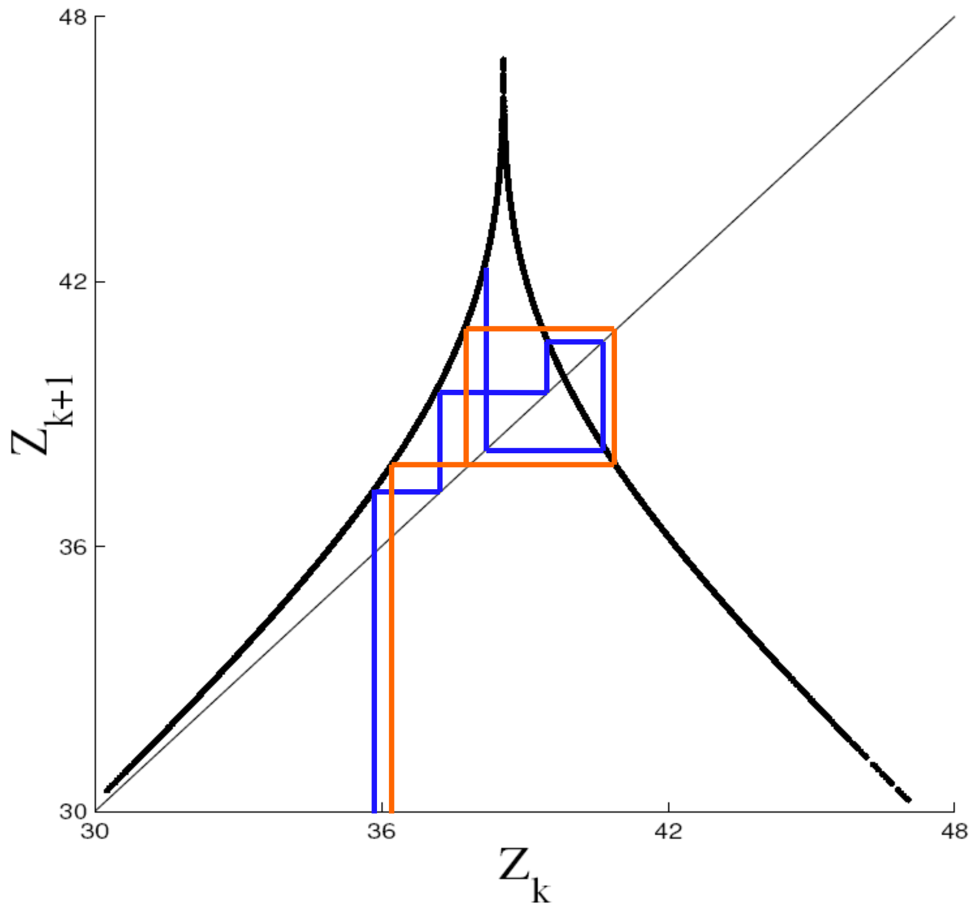
## In fact he plots $Z_{k+1}$ as a function of $Z_k$



# What happens?

Lorentz plot the successive maxima of Z, $Z_k$

In fact he plots $Z_{k+1}$ as a function of $Z_k$



Discrete time iterations

$$Z_{k+1}=f(Z_k)$$

Consider $Z_k'=Z_k+\delta Z$

$$Z_{k+1}'=Z_{k+1}+df/dZ(Z_k)\delta Z$$

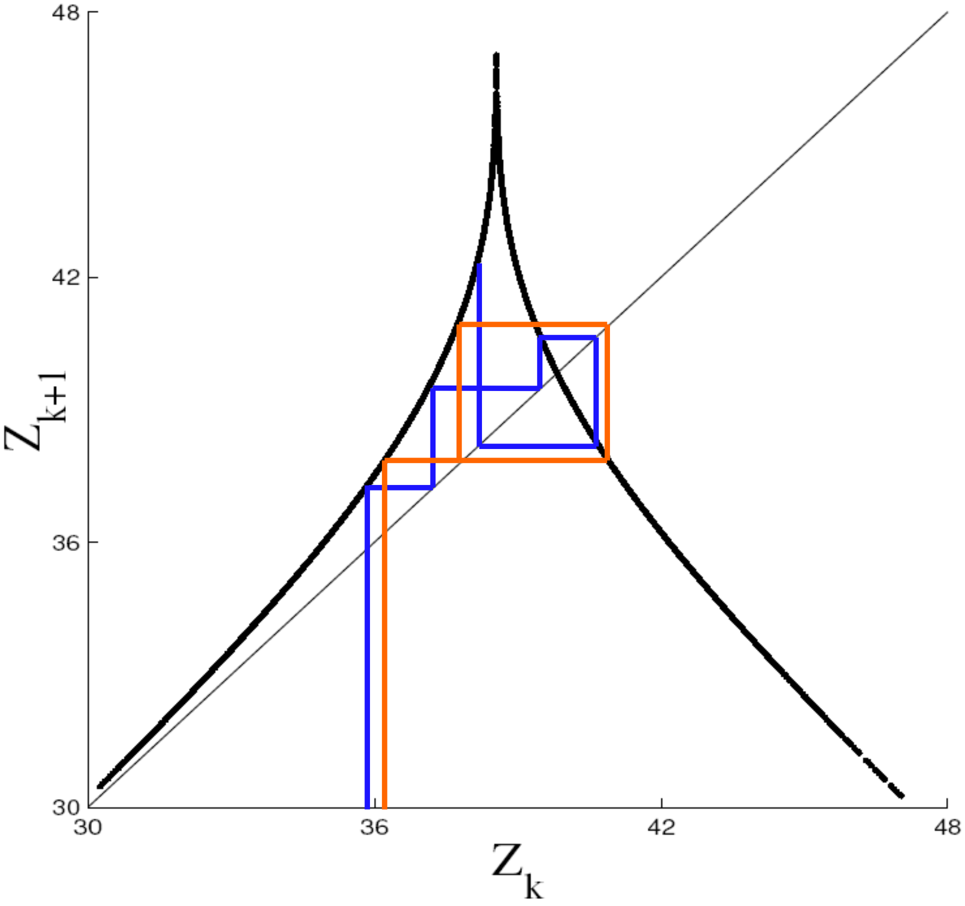
The divergence depends on the the sign of $|df/dZ|-1$

Here |df/dZ|>1 everywhere!

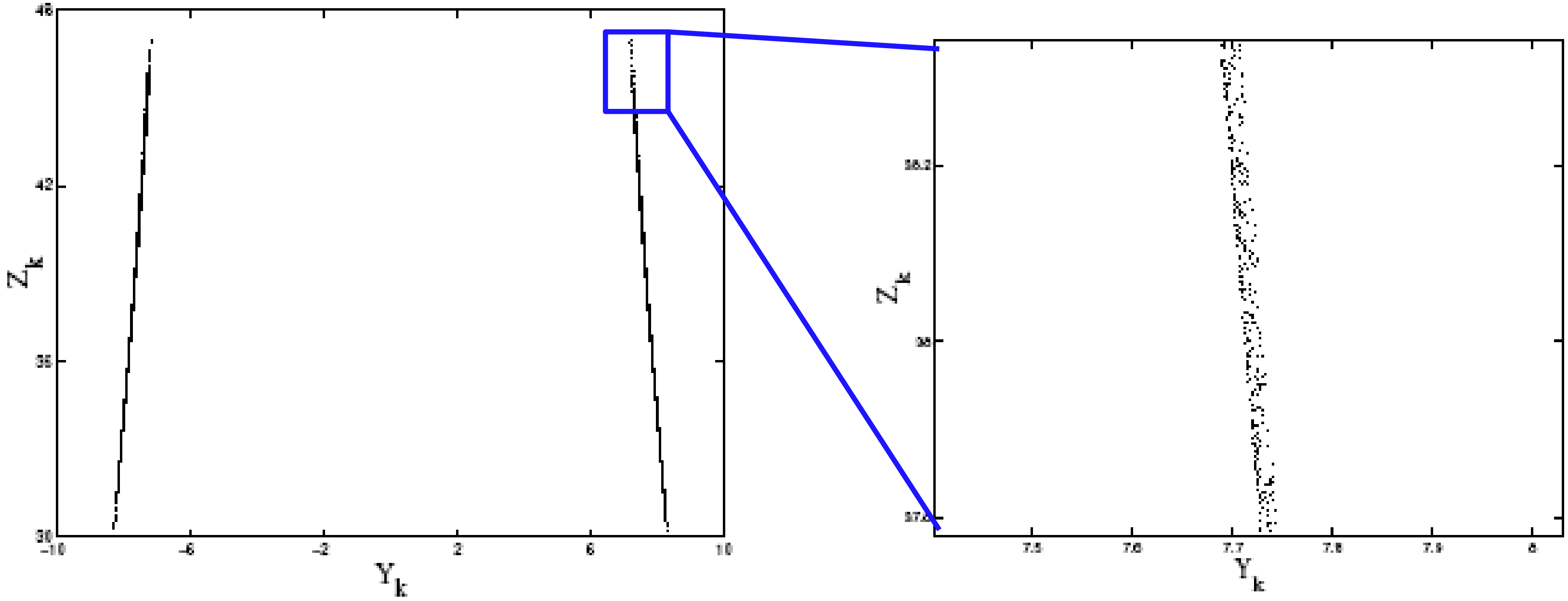
Small initial differences get exponentially amplified

Loss of memory of initial condition

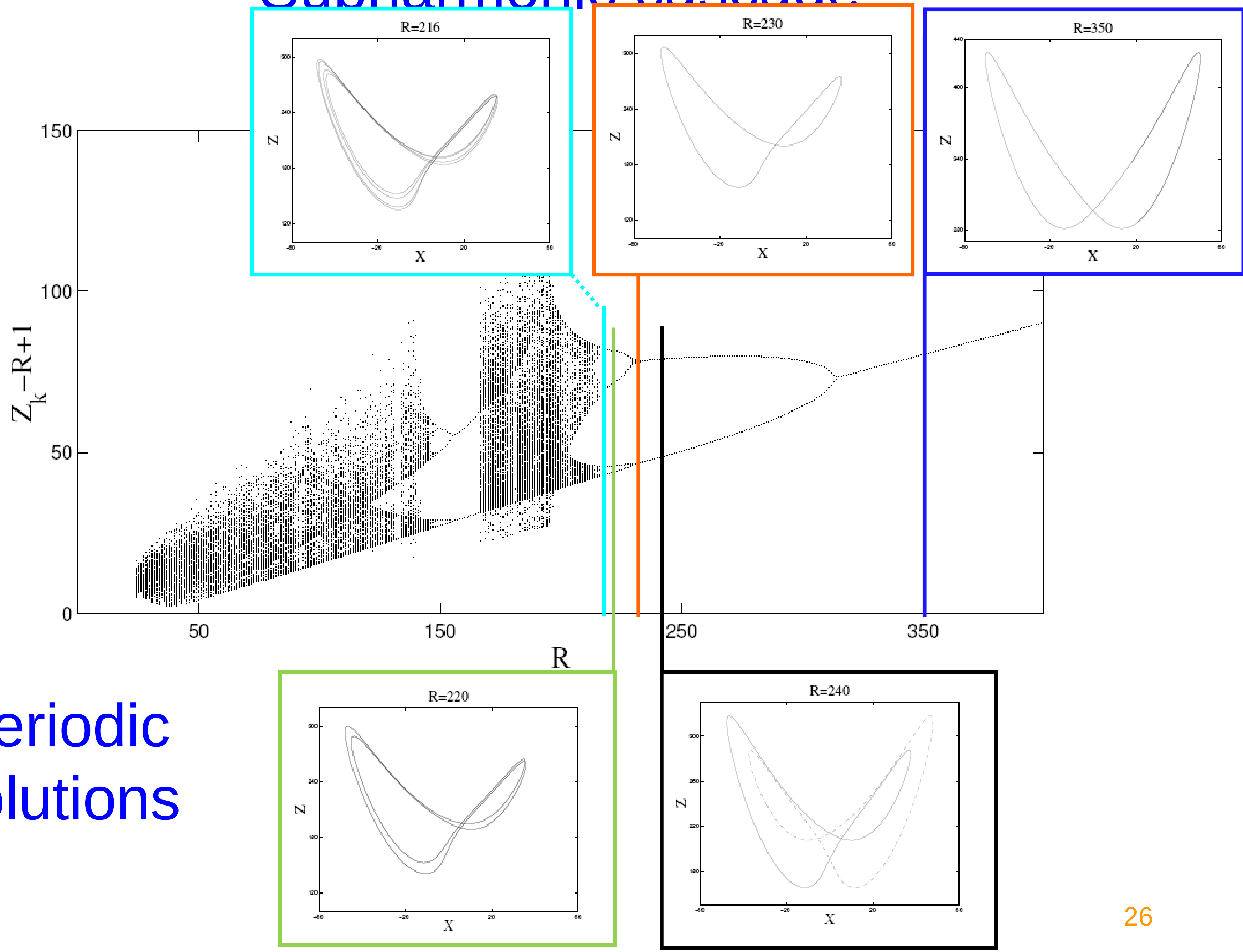
Butterfly effect



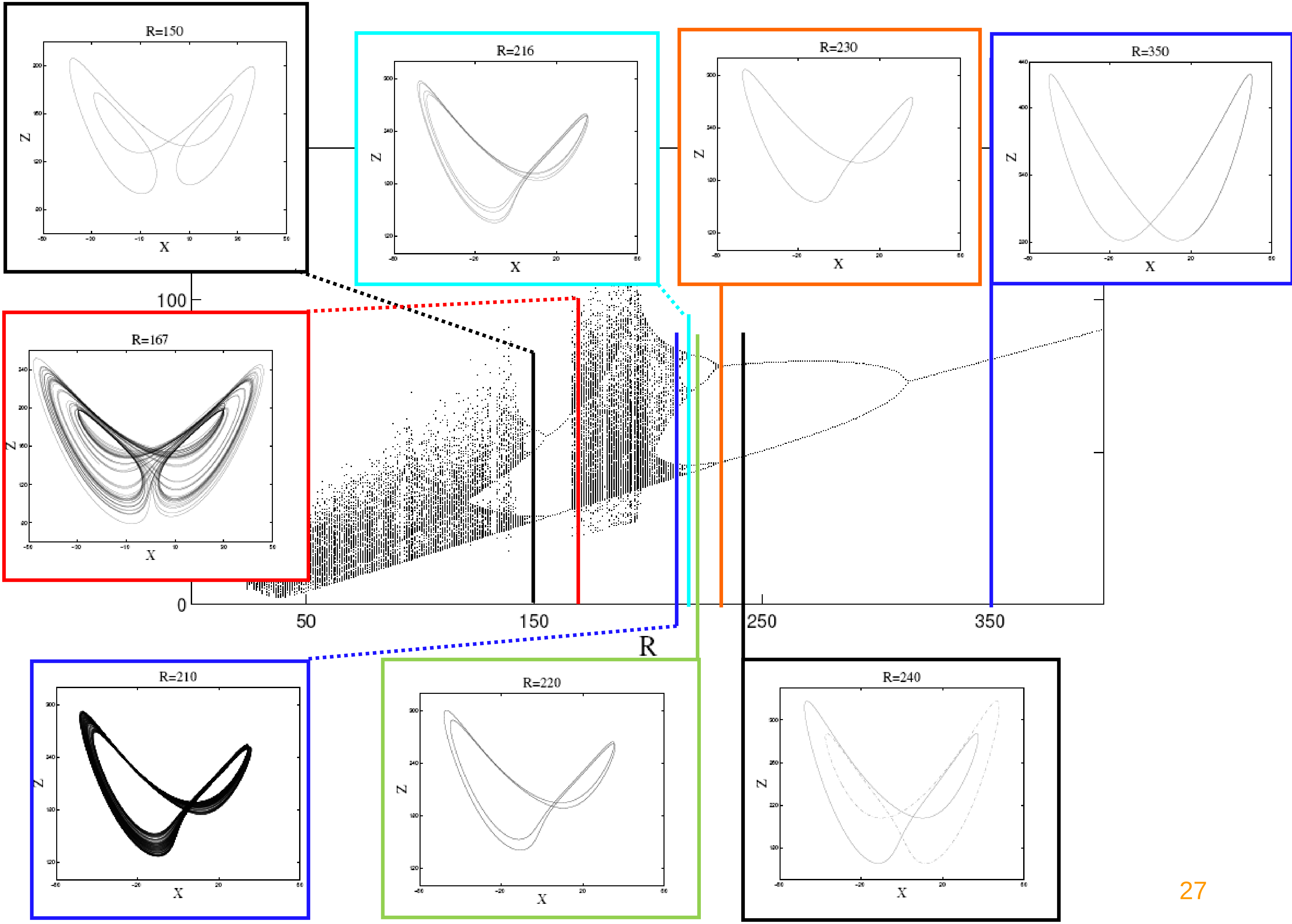
# Beware, this is not a function



# Subharmonic cascade




Periodic solutions


R=150
R=216
R=230
R=350
R=167
R=210
R=220
R=240
Z
X
100
0
50
150
250
350
R

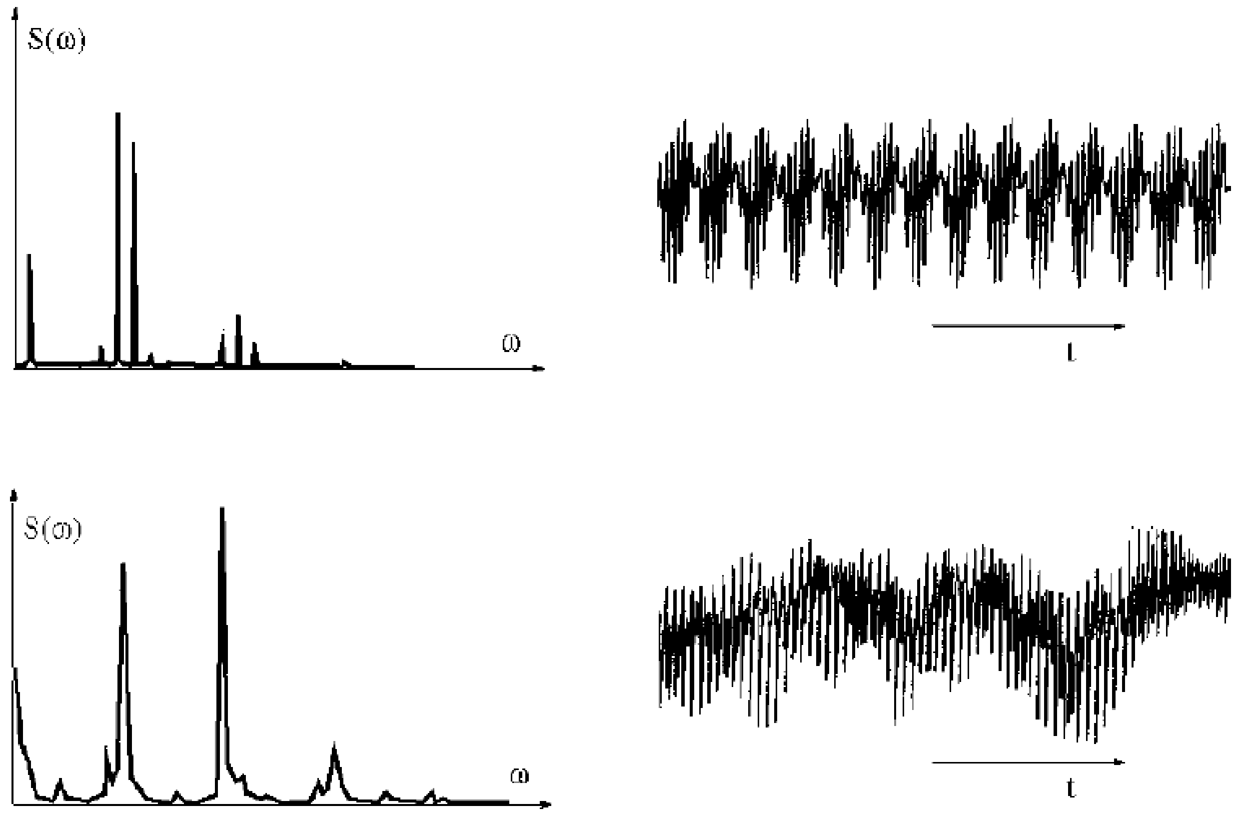
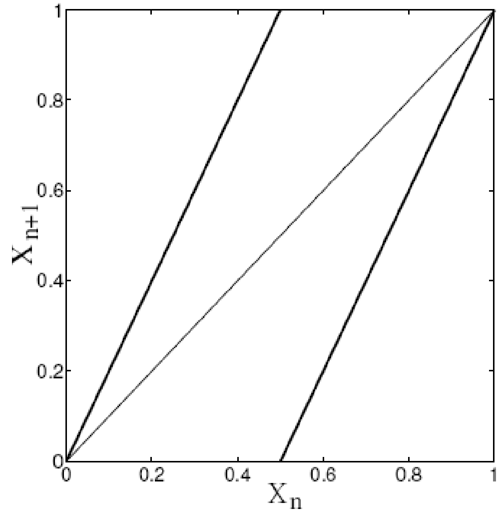


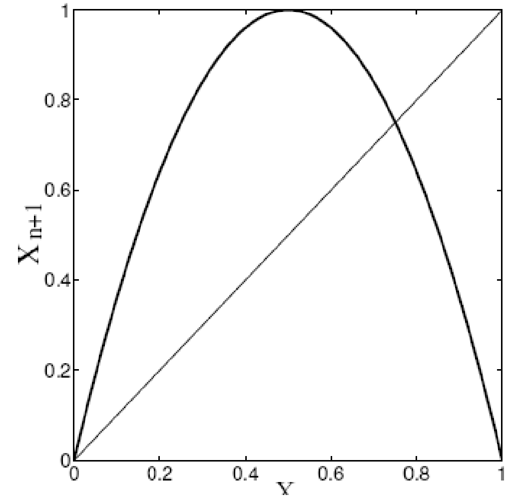

*Figure 6.14: Spectres du signal de vitesse mesuré en un point de la cellule. En régime bipériodique (en haut), le spectre est composé de deux raies principales et de leurs combinaisons simples. Le régime chaotique est caractérisé par des raies au pied élargi et par la présence de bruit de basse fréquence ($\omega \approx 0$) dans le spectre (d'après Bergé et Dubois, 1981).*

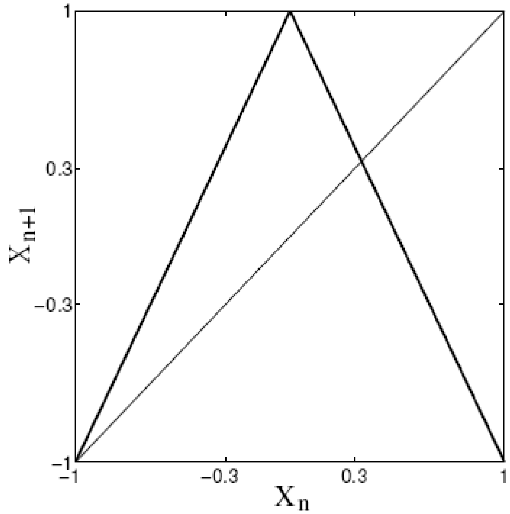
# Different maps

## dyadic

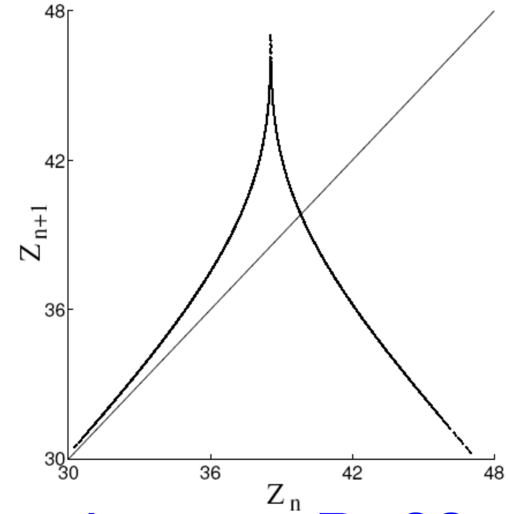



## logistic



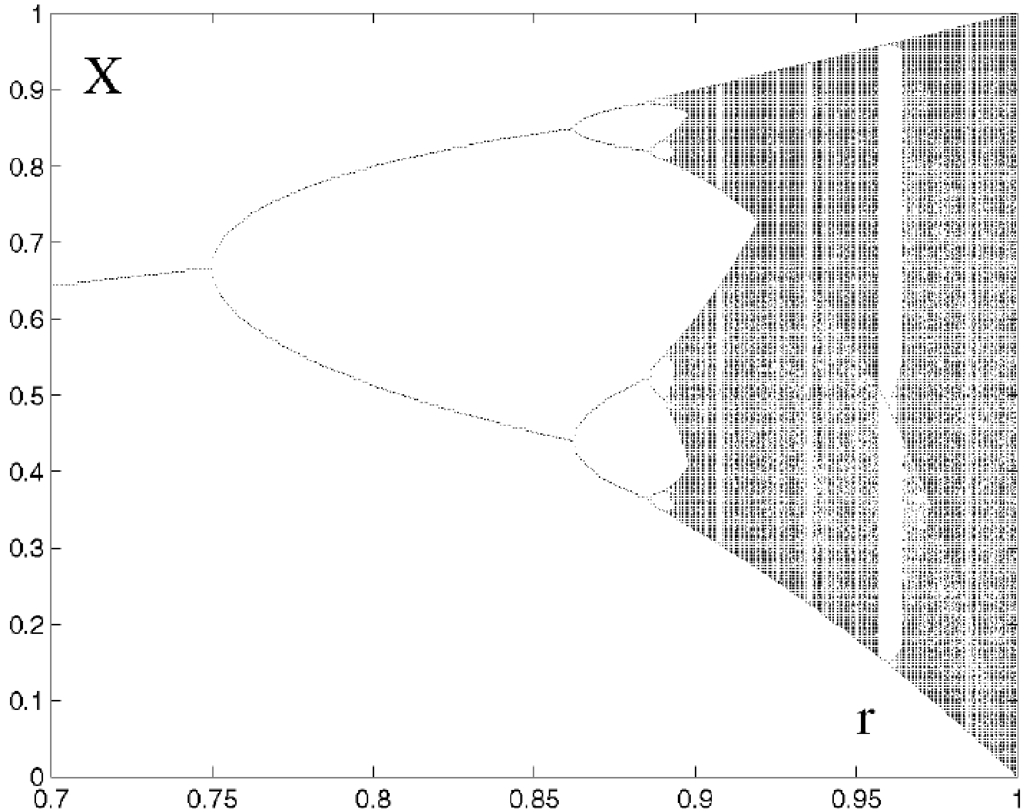

accent circonflexe




Lorentz R=28

# Logistic map

$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



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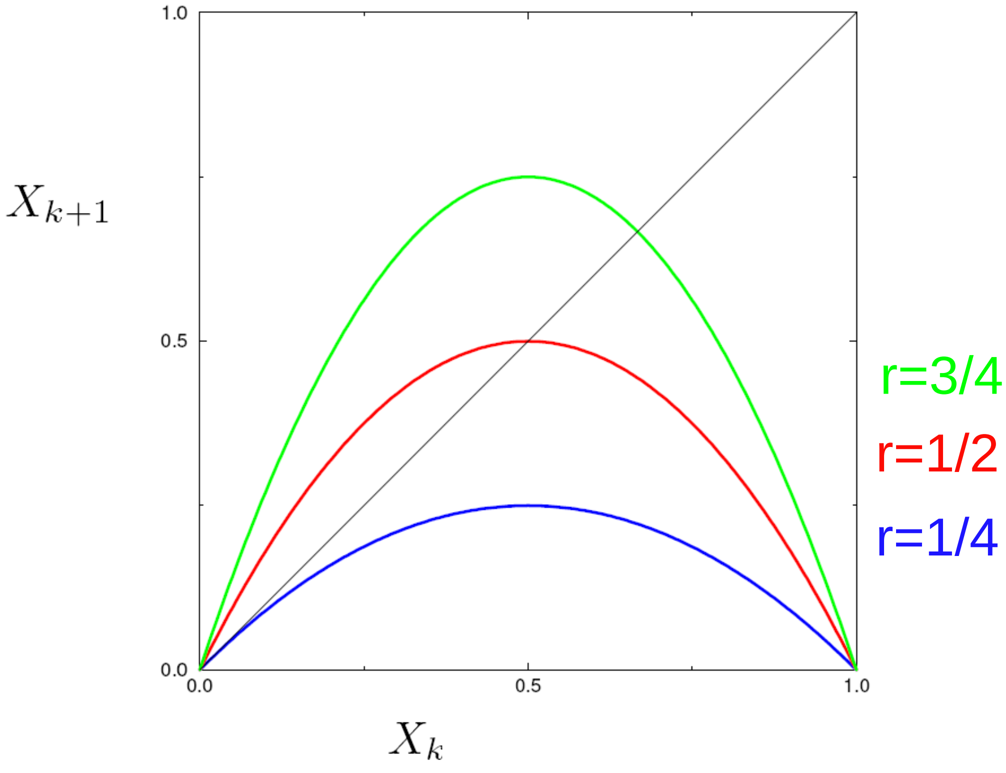

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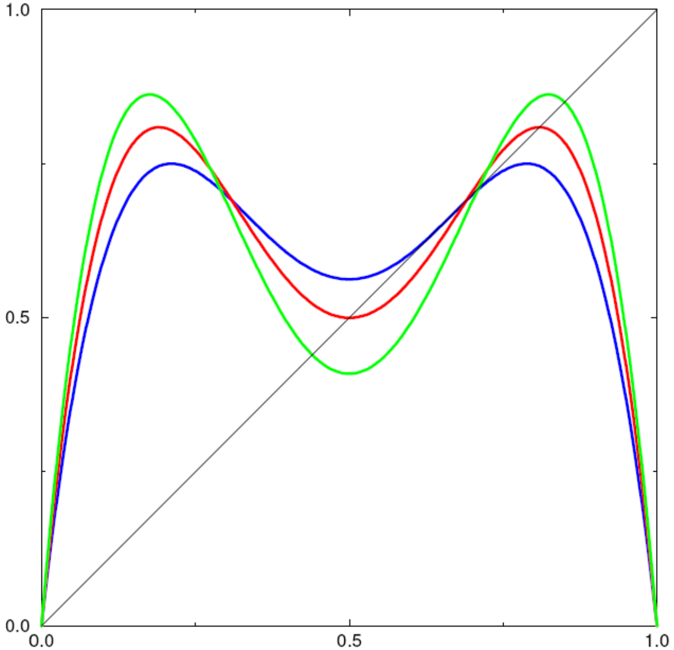


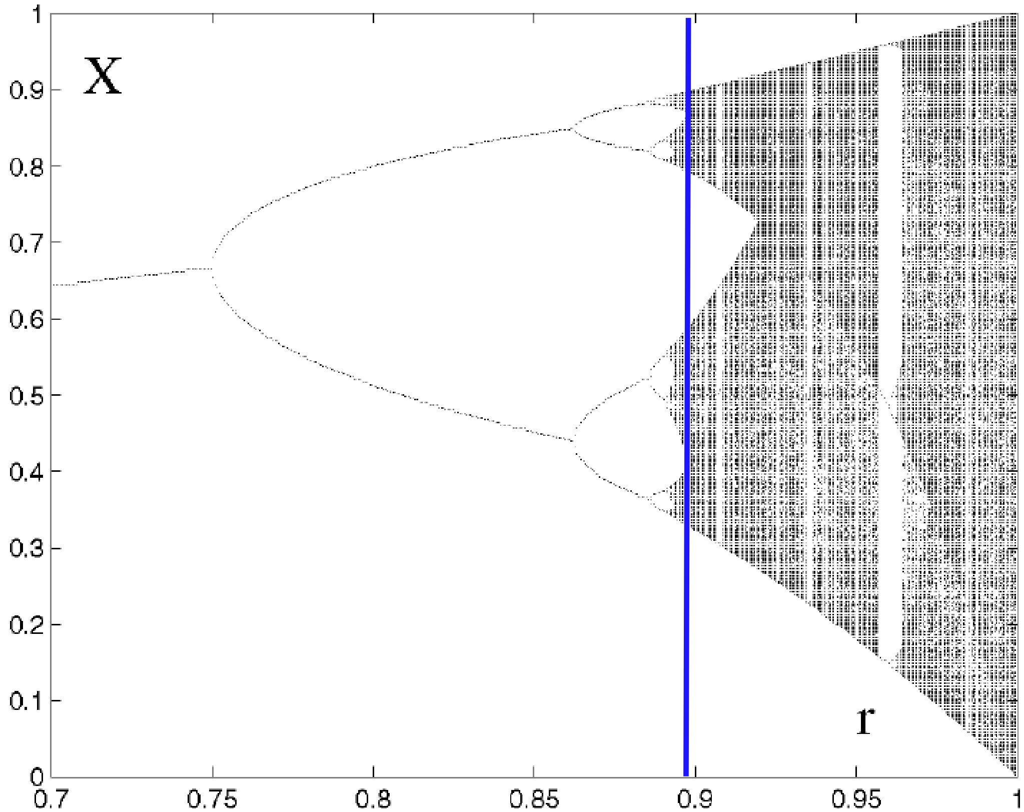
# Logistic map

## critical bifrucation parameters for period doubling

| $n$ | $T_{(n)}$ | $r_{(n)}$ | $r^{\text{ss}}_{(n)}$ |
|---|---|---|---|
| 0 | 1 | 1/4 | 1/2 |
| 1 | 2 | 3/4 | 0,80901... |
| 2 | 4 | 0,86237... | 0,87464... |
| 3 | 8 | 0,88602... | 0,88866... |
| 4 | 16 | 0,89218... | ... |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $\infty$ | $\infty$ | 0,89248... | 0,89248... |

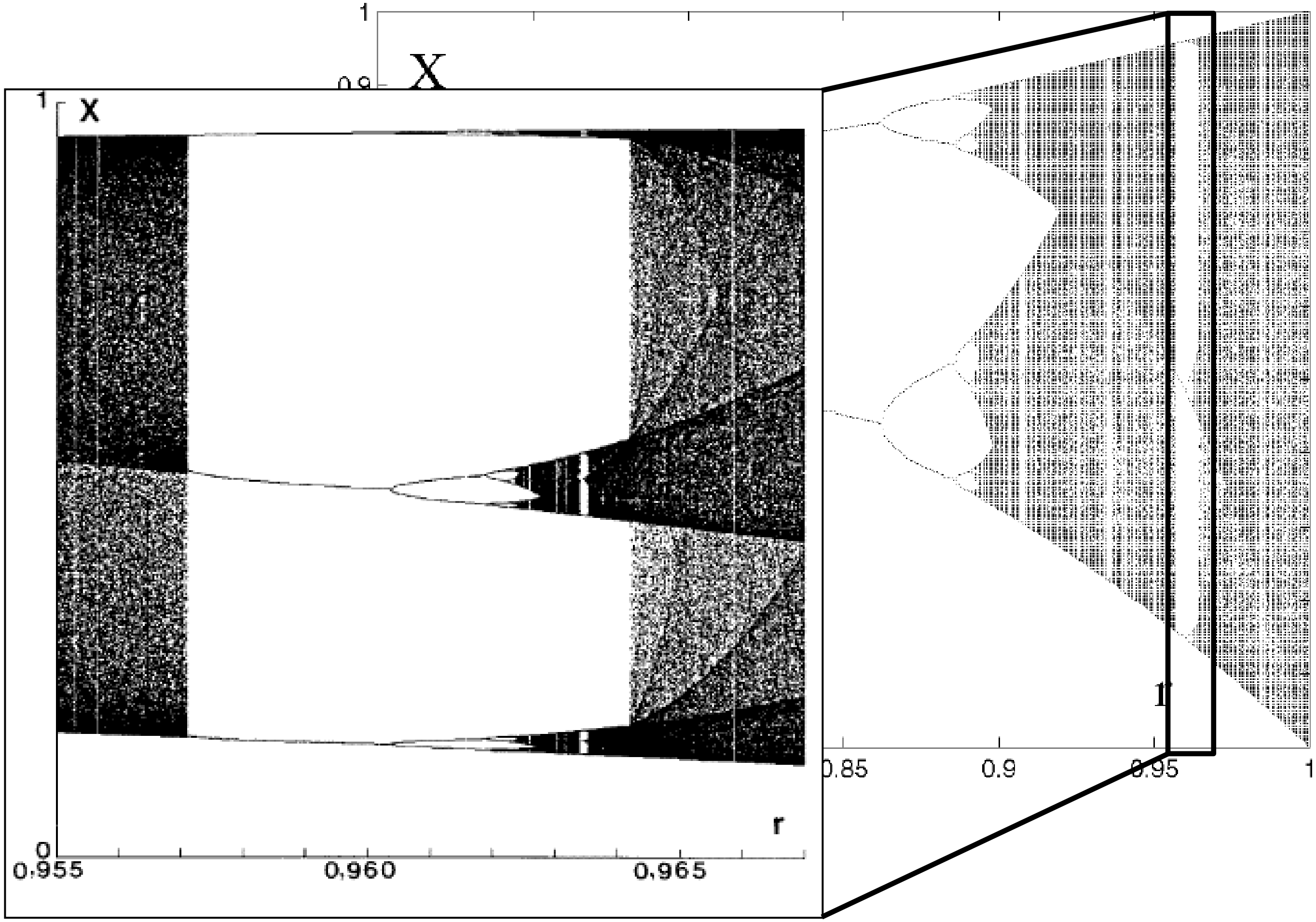
# Periodic window

$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$

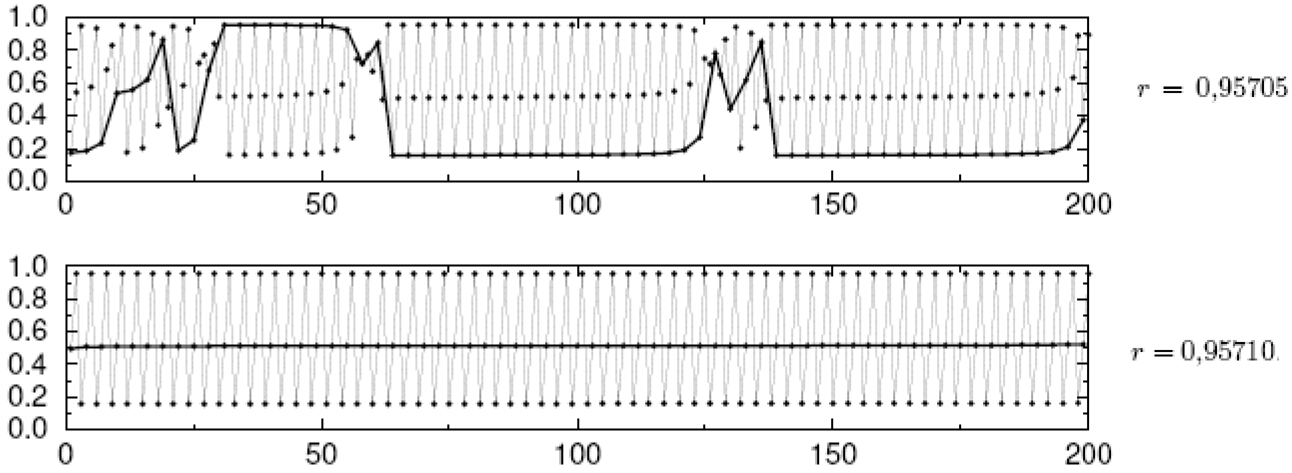


# Logistic map
# Periodic window

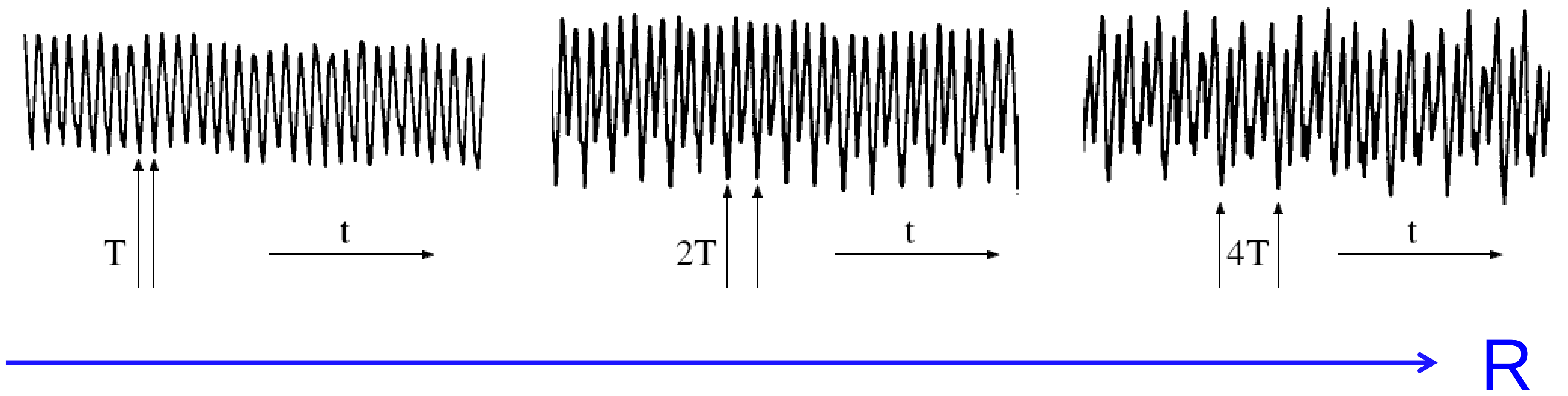
$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



# Very very sensitive system

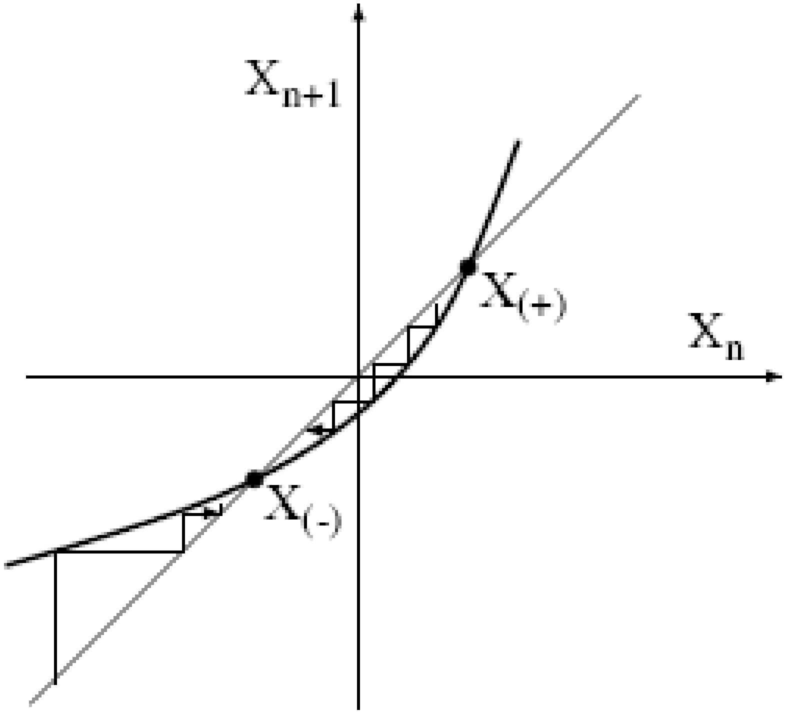


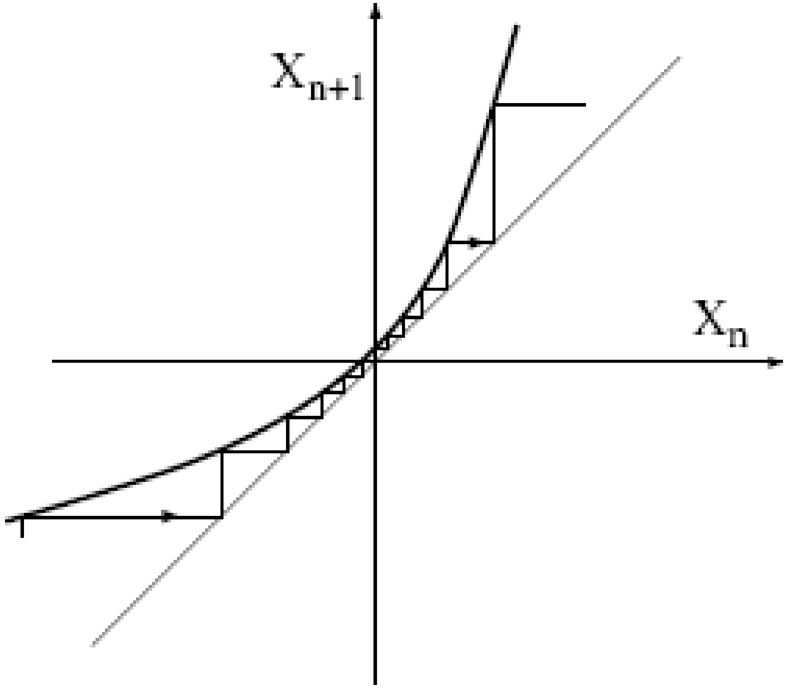
# Subharmonic cascade in convection




Libchaber and Maurer ‘80
Liquid Helium

# Intermittency

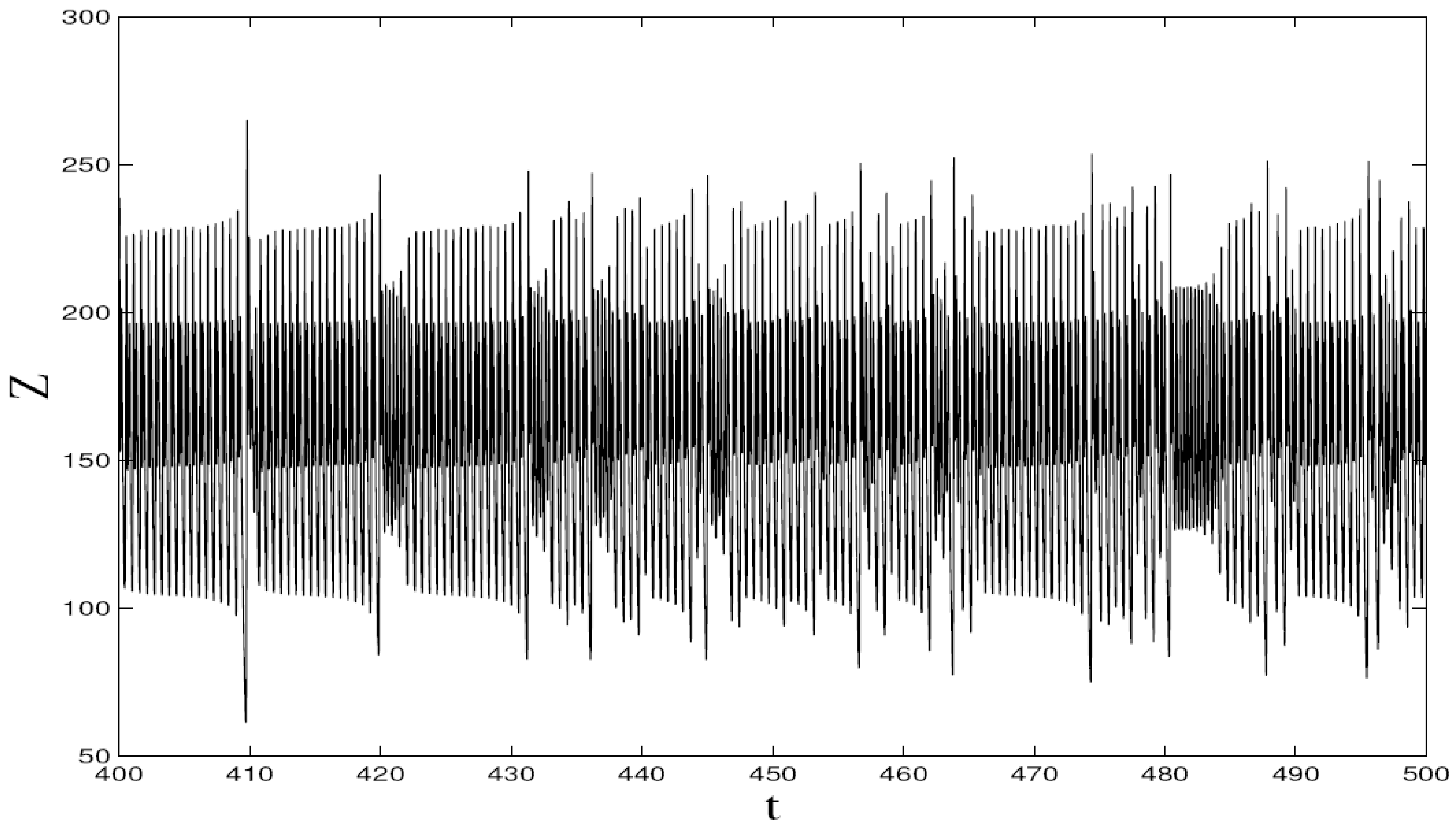



Stable

No fixed point-
Long attracting periods
Intermittent regime

# Intermittency




# Lorentz at R=166