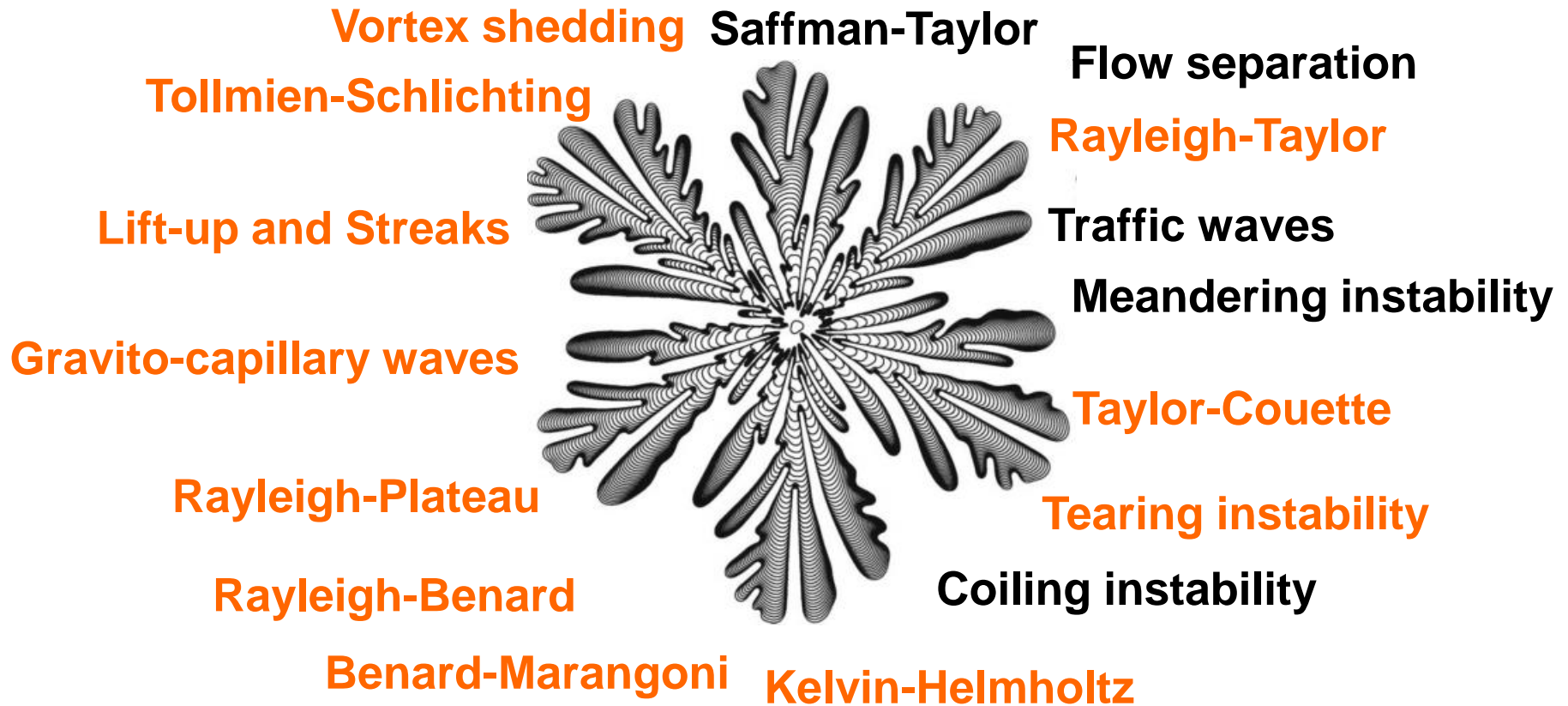
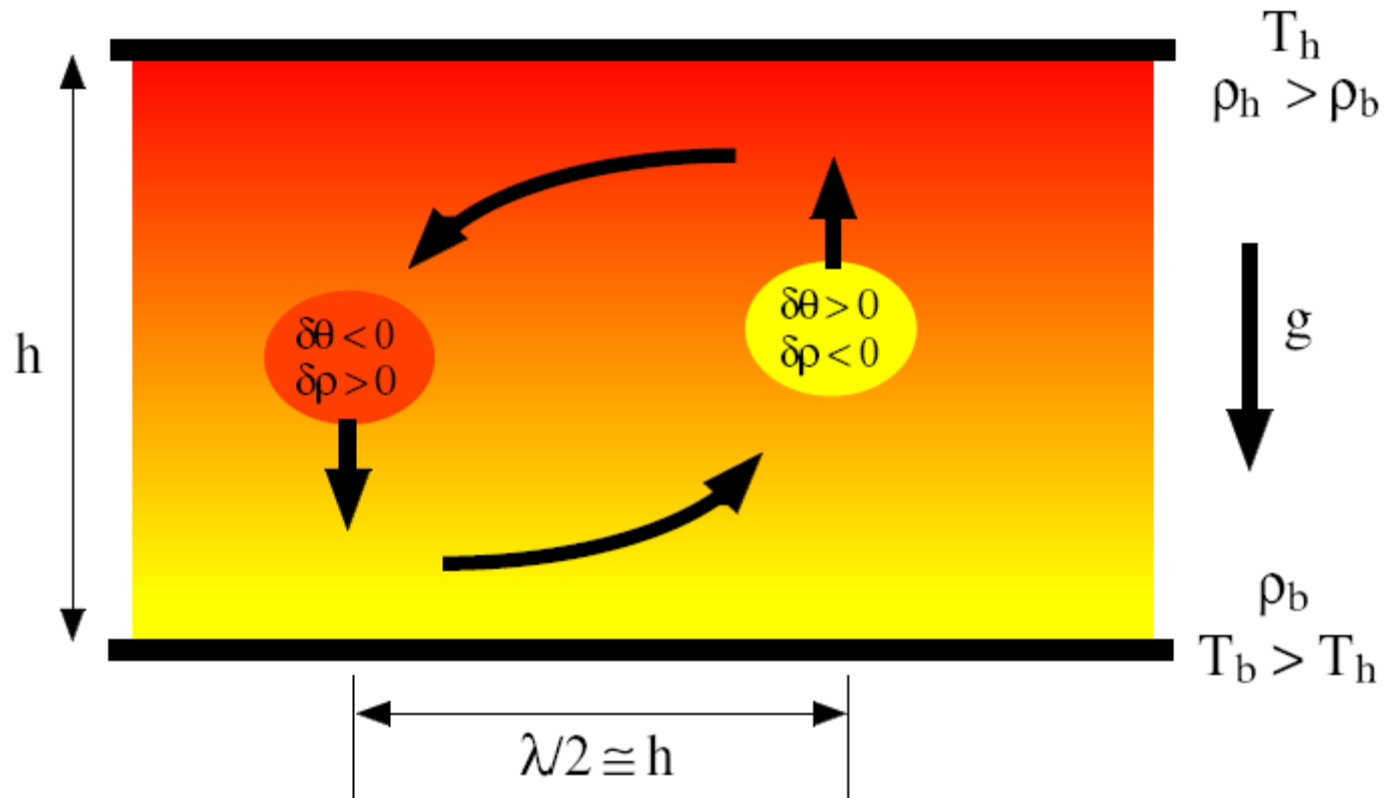


Most flows are unstable...



Chaos

Rayleigh Bénard convection



Lessons learned from linear stability analysis

1. The pure conductive solution becomes unstable above a threshold, which for fixed depth h and fluid properties is simply proportional to the temperature gradient.

2. We therefore introduced a control parameter which equals 0 at threshold

$$r = \frac{\Delta T - \Delta T_c}{\Delta T_c}$$

3. Close to threshold, when linearity can be assumed, the perturbation writes

$$v_z = X(t) f_v(z) \sin(kx), \quad k = 2\pi/\lambda.$$

4. Temperature and vertical velocity are in phase

5. The unstable wavelength at threshold is roughly $2h$

Heuristic equation for an unstable system

$$\frac{d}{dt}X = \sigma X$$

$$\sigma = r / \tau_0$$

1. We retrieve the critical slowing at threshold
2. The system is stable (damped in fact!) for $r < 0$ and unstable for $r > 0$

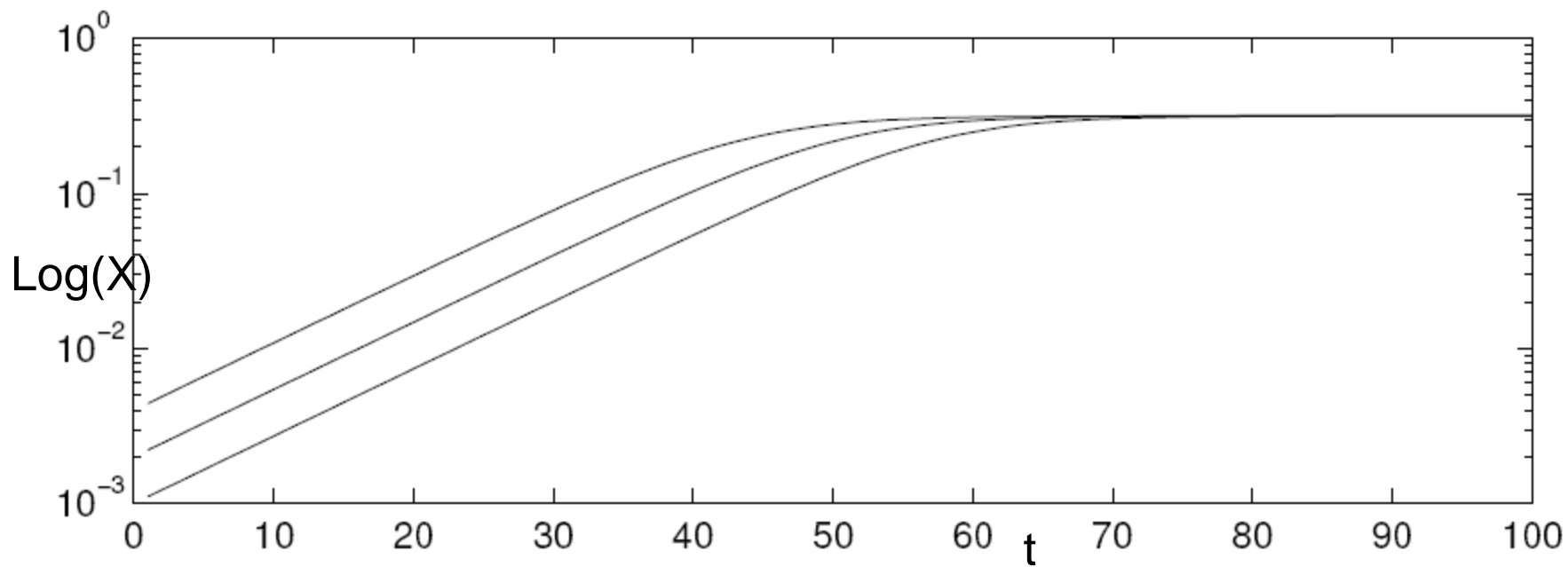
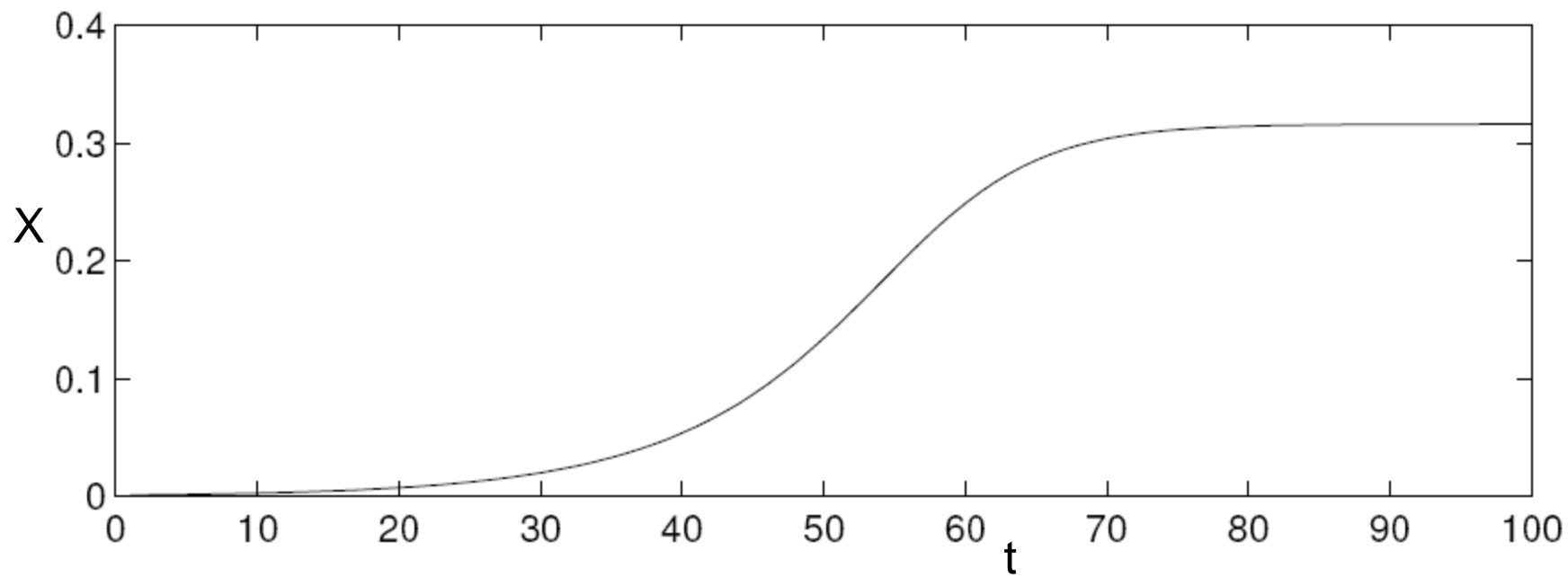
Heuristic equation for an unstable system

1. Since for $r > 0$, there is exponential growth, X cannot be assumed to remain small for long...
2. Nonlinear effects have to be introduced

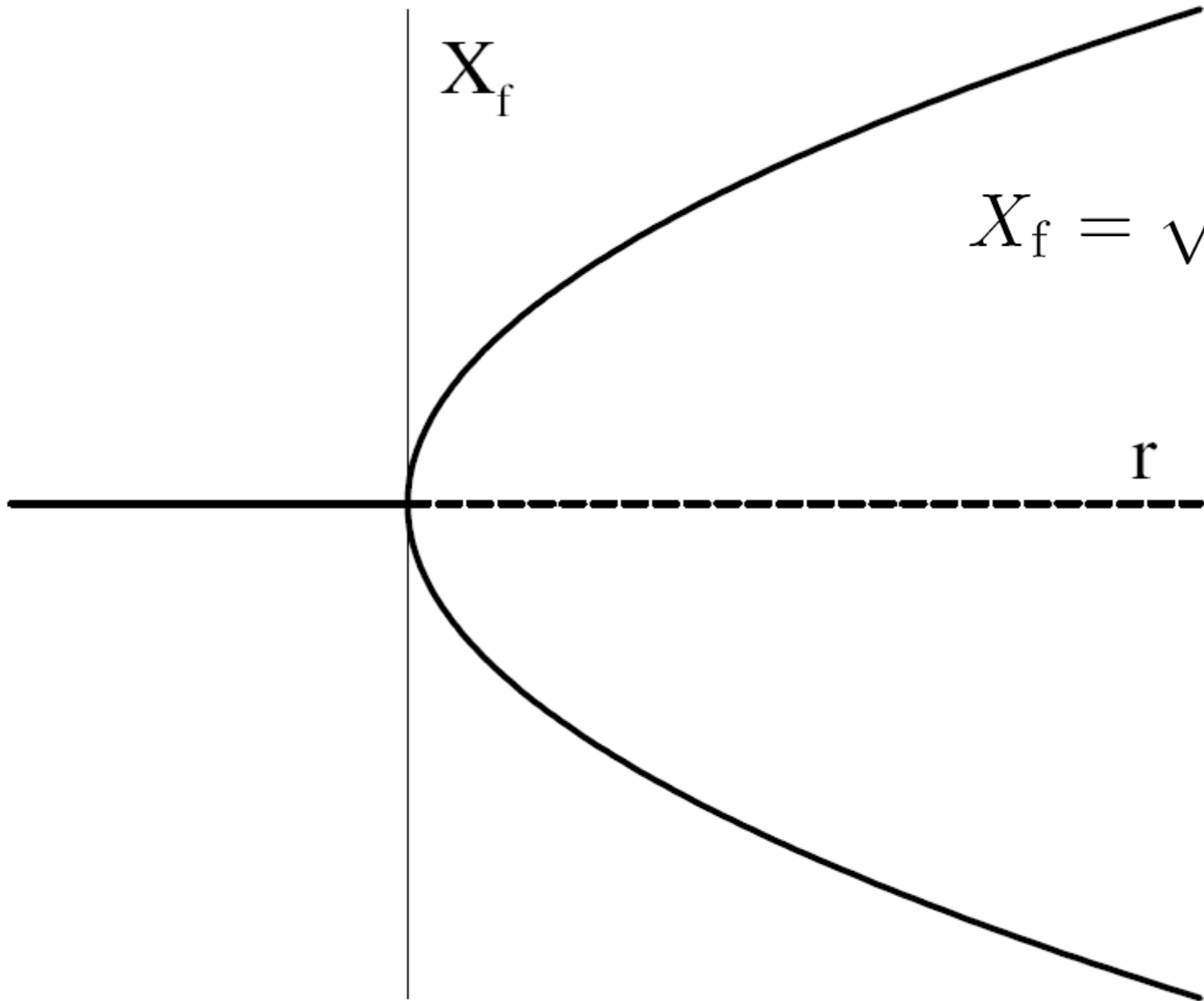
$$\frac{d}{dt} X = \sigma_{\text{eff}} X$$

3. where σ_{eff} is now a function of X !
4. By symmetry, there is no reason that a positive velocity anomaly would behave differently than a negative one...

$$\tau_0 \sigma_{\text{eff}} = r - gX^2$$



$$\tau_0 \frac{d}{dt} X = F_r(X) = rX - gX^3$$



$$X_f = \sqrt{r/g}.$$

Stability of the bifurcated state?

$$\begin{aligned}\tau_0 \frac{d}{dt}(X_f + X') &= \tau_0 \frac{d}{dt} X' = r(X_f + X') - g(X_f + X')^3 \\ &= [rX_f - gX_f^3] + (r - 3gX_f^2) X' + \mathcal{O}(X'^2)\end{aligned}$$

$$\tau_0 \frac{d}{dt} X' = -2r X'$$

Stable since $r > 0$!

Now we include more physics

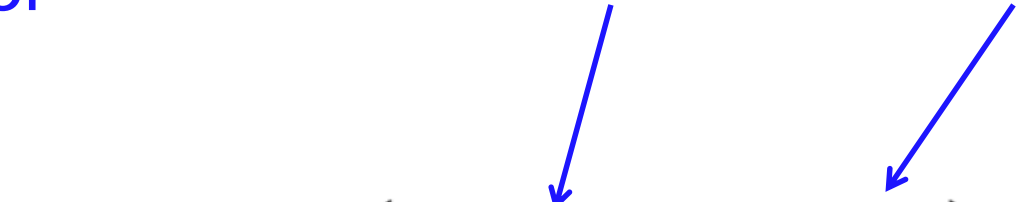
X: vertical velocity

Y : temperature anomaly

P: Prandtl number

Driving force
Archimedes

Viscous
resistance

$$\frac{d}{dt} X = P (Y - X)$$


Now we include more physics

X: vertical velocity


Y : temperature anomaly

P: Prandtl number

R: Rayleigh number

Driving force
Archimedes


Viscous
resistance

$$\frac{d}{dt} X = P (Y - X)$$


$$\frac{d}{dt} Y = R X - Y$$

Heat transport

Heat diffusion



Stability of equilibrium state ($X=0, Y=0$)

$$X = X_0 \exp(st), \quad Y = Y_0 \exp(st),$$

$$sX_0 = P(Y_0 - X_0),$$

$$sY_0 = RX_0 - Y_0,$$

$$(s + P)(s + 1) - RP = 0$$

$$s^{(\pm)} = \frac{1}{2} \left(-(P + 1) \pm [(P - 1)^2 + 4PR]^{1/2} \right)$$

Stability of equilibrium state ($X=0, Y=0$)

$$X = X_0 \exp(st), \quad Y = Y_0 \exp(st)$$

$$s^{(\pm)} = \frac{1}{2} \left(-(P + 1) \pm [(P - 1)^2 + 4PR]^{1/2} \right)$$

> 0

Unstable if $[(P - 1)^2 + 4PR] > (P + 1)^2$

$$4PR > 4P$$

$$R > 1$$

There is also a mean temperature correction Z

$$X \propto \sin(kx)$$

$$Y \propto \sin(kx)$$

$$\frac{d}{dt}Z = XY - bZ$$

Lorentz model

X: vertical velocity

Y : temperature anomaly

Z: mean temperature correction

Driving force
Archimedes

Viscous
resistance

$$\frac{d}{dt} X = P(Y - X)$$

Heat
transport

$$\frac{d}{dt} Y = (R - Z)X - Y$$

Heat
diffusion

$$\frac{d}{dt} Z = XY - bZ$$

Relaxation to linear
temperature profile

Mean effect on temperature

Limiting equation for R close to 1 and X=Y

$$Z = XY/b$$

$$\frac{d}{dt}X = (R - 1)X - X^3/b$$

Now consider Lorentz system at large R
and $b=8/3$, $P=10$

$$\frac{d}{dt}X = P(Y - X)$$

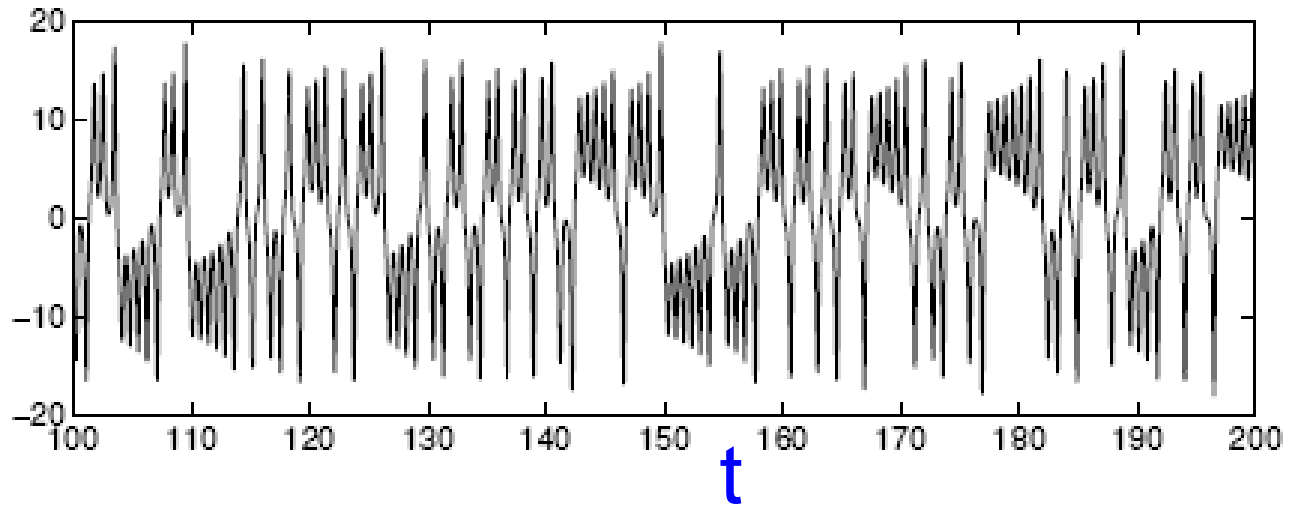
$$\frac{d}{dt}Y = (R - Z)X - Y$$

$$\frac{d}{dt}Z = XY - bZ$$

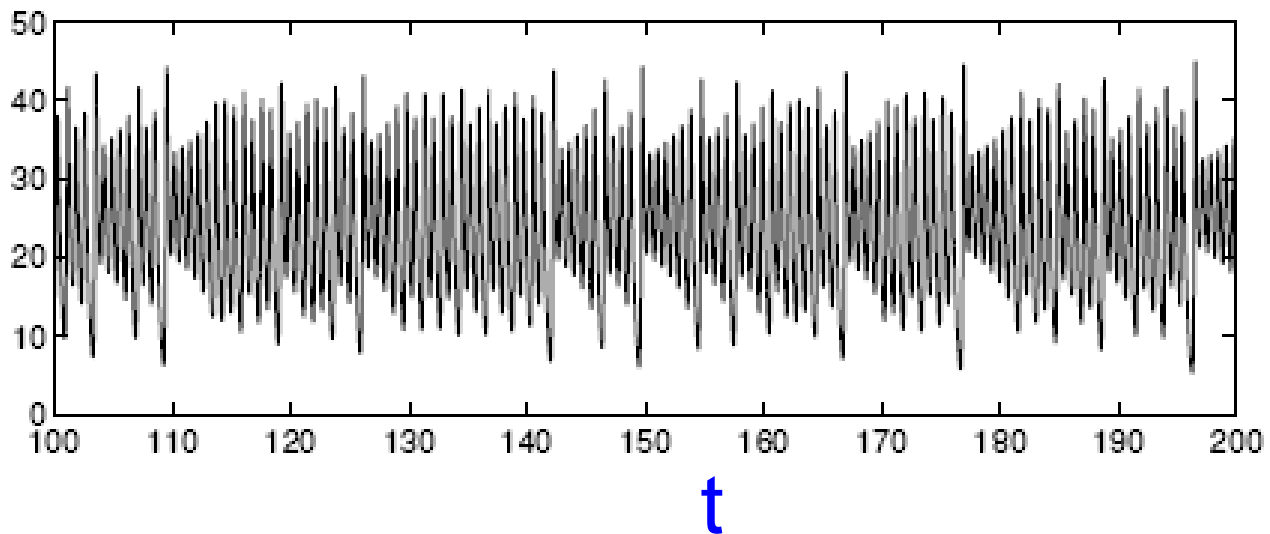
One can show that the steady convection
roll solution becomes unstable for $R > 24.74$

Here $R=28$, $b=8/3$, $P=10$

X

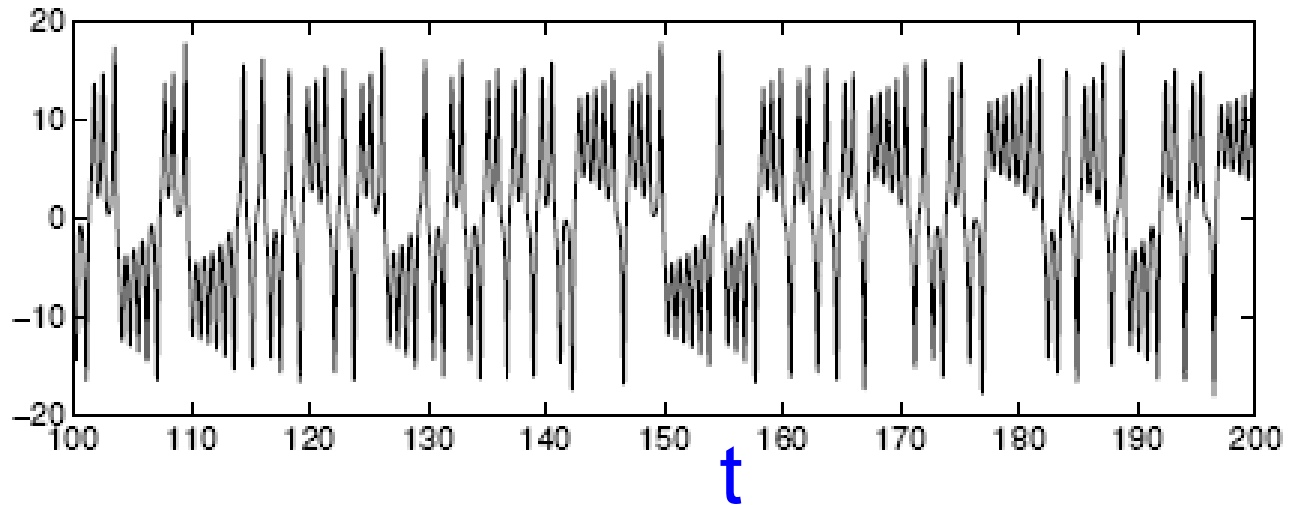


Y

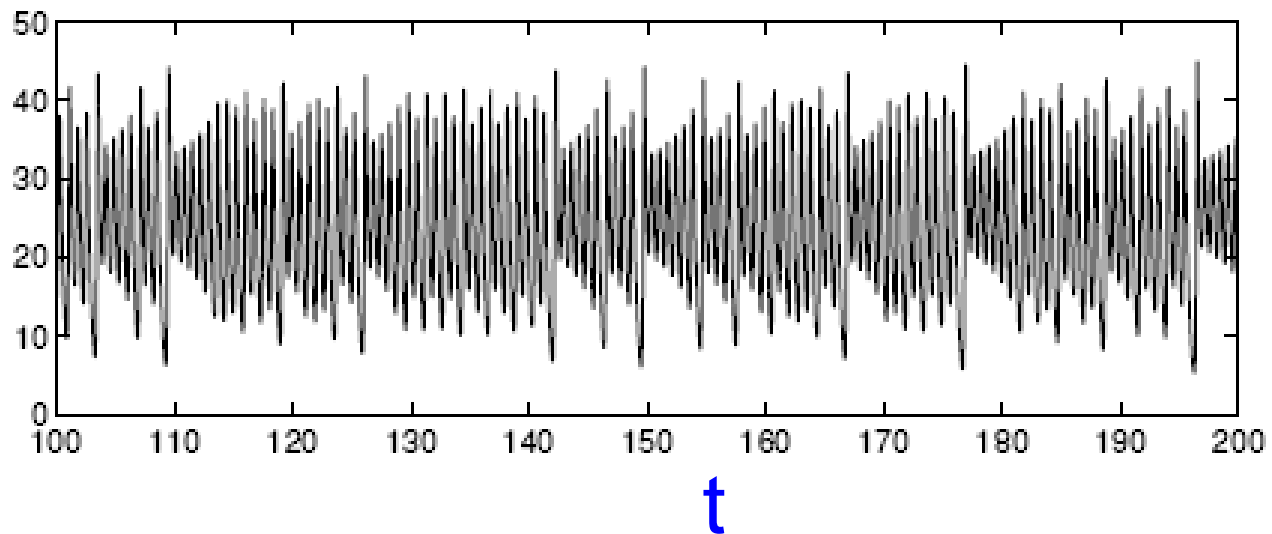


Here $R=28$, $b=8/3$, $P=10$
This is chaotic!

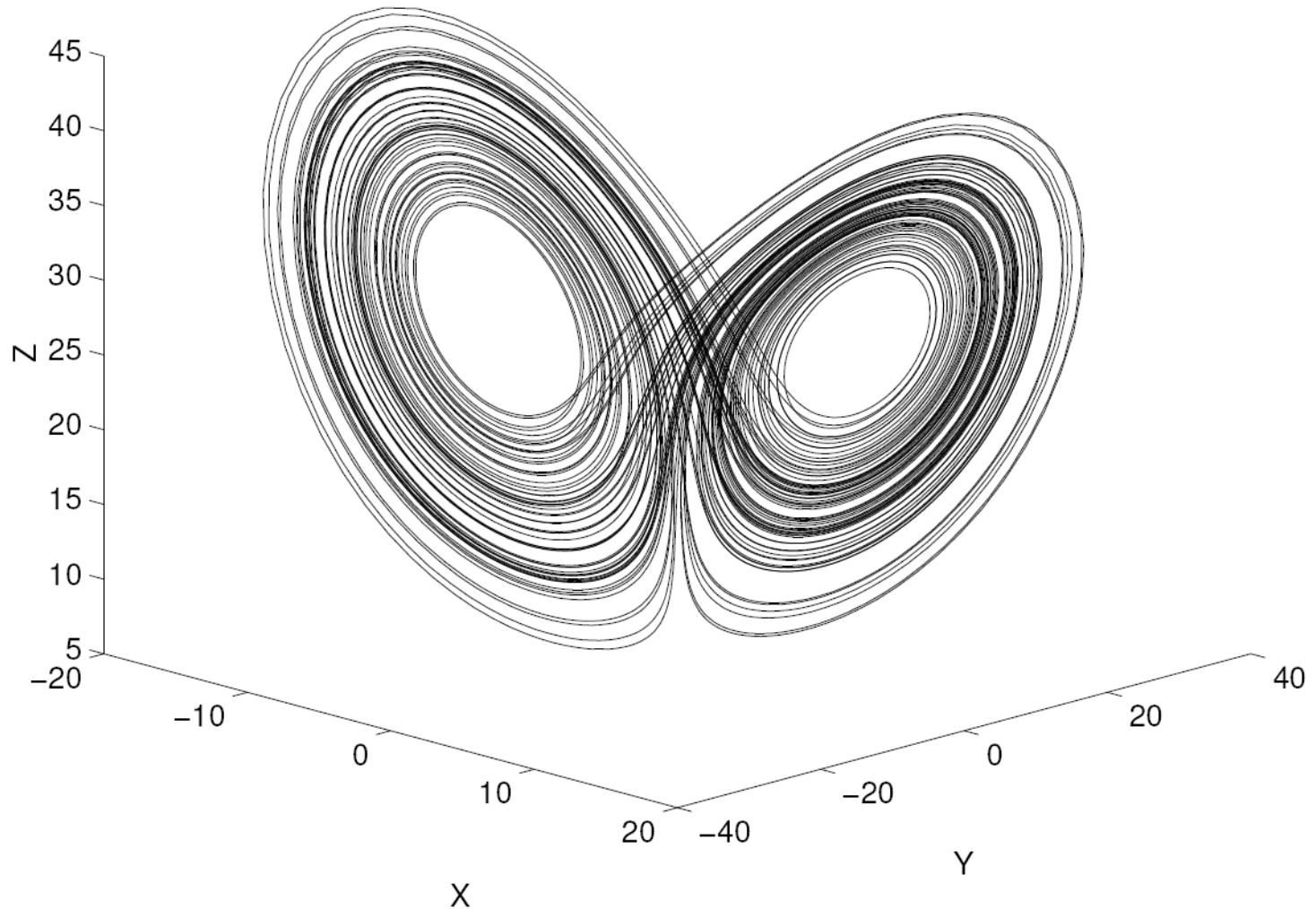
X



Y

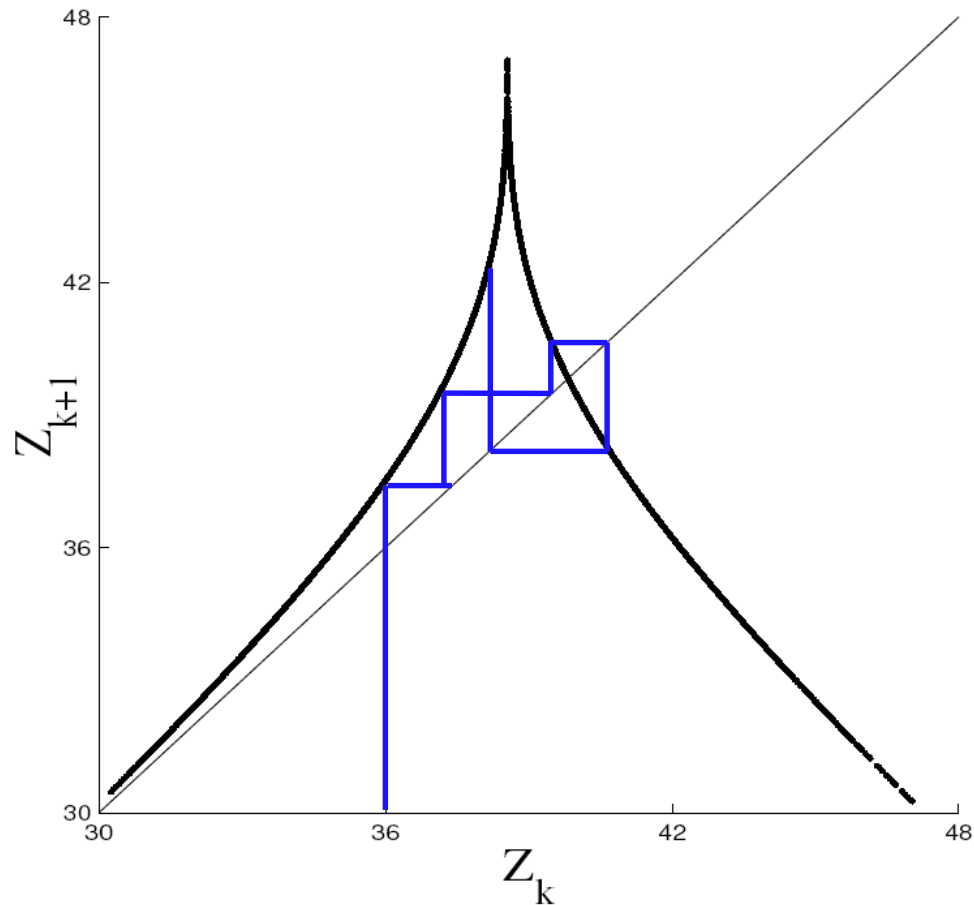


Here $R=28$, $b=8/3$, $P=10$
This is chaotic!



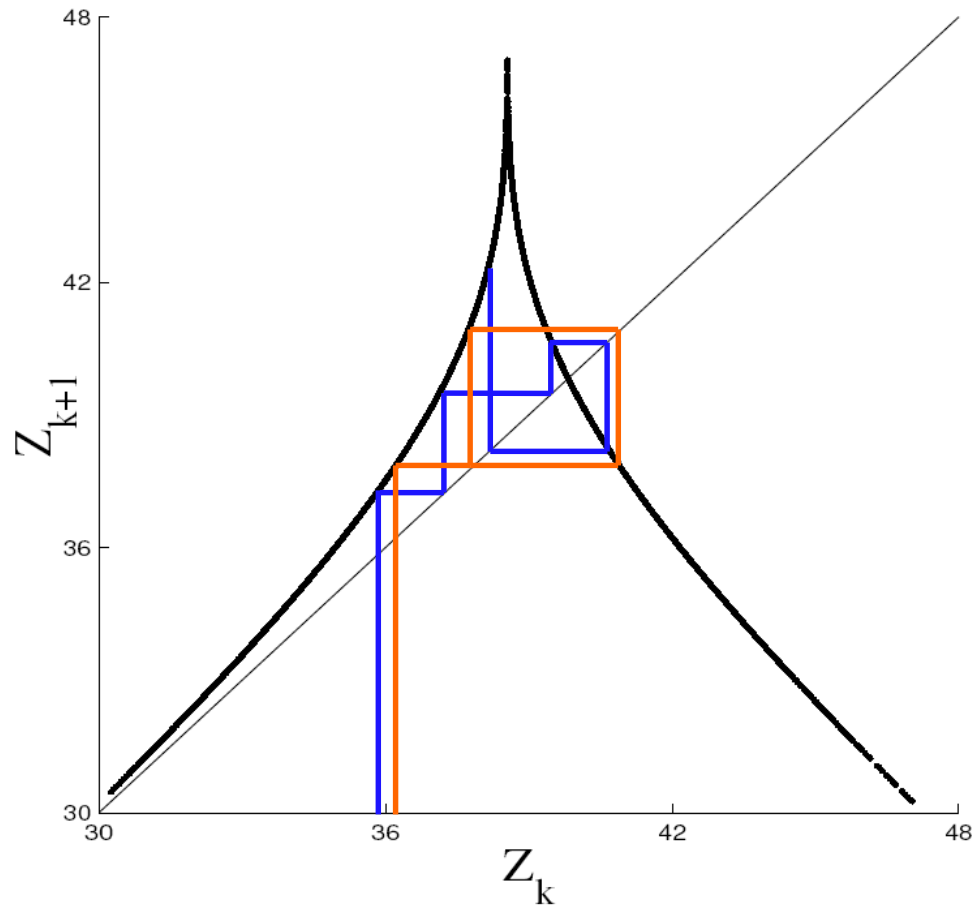
What happens?

Lorentz plot the successive maxima of Z , Z_k
In fact he plots Z_{k+1} as a function of Z_k



What happens?

Lorentz plot the successive maxima of Z , Z_k
In fact he plots Z_{k+1} as a function of Z_k



Discrete time iterations

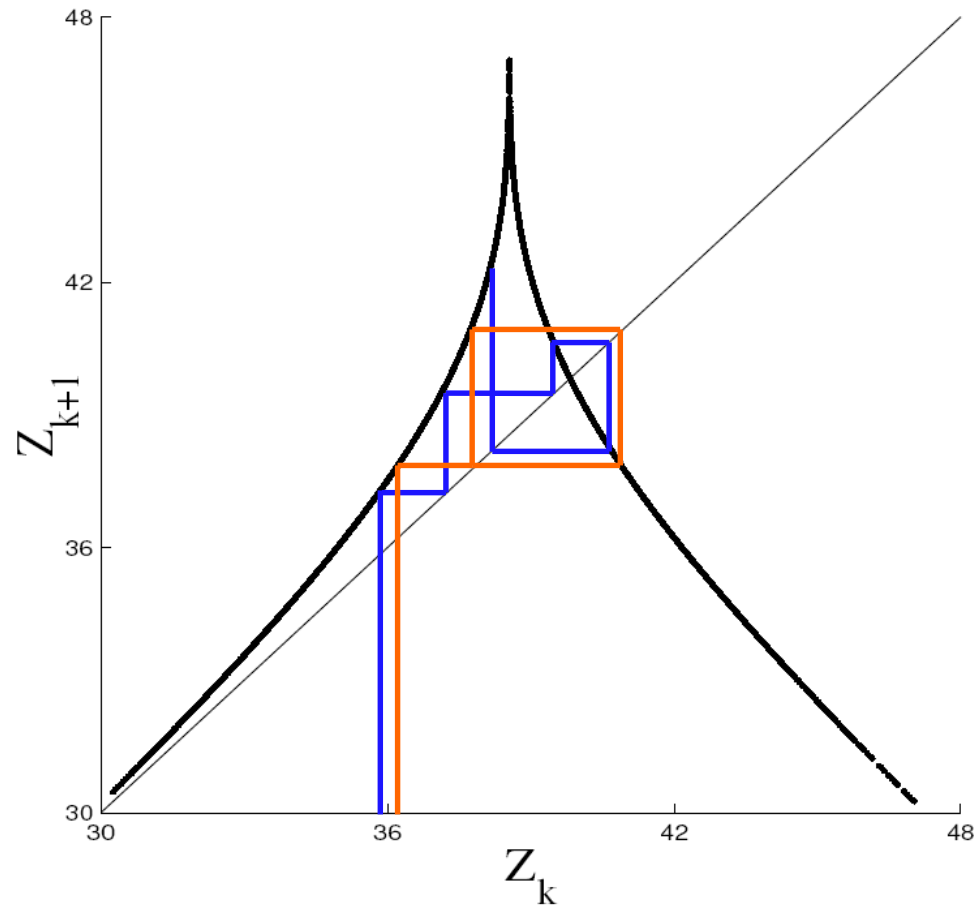
$$Z_{k+1} = f(Z_k)$$

Consider $Z_k' = Z_k + \delta Z$

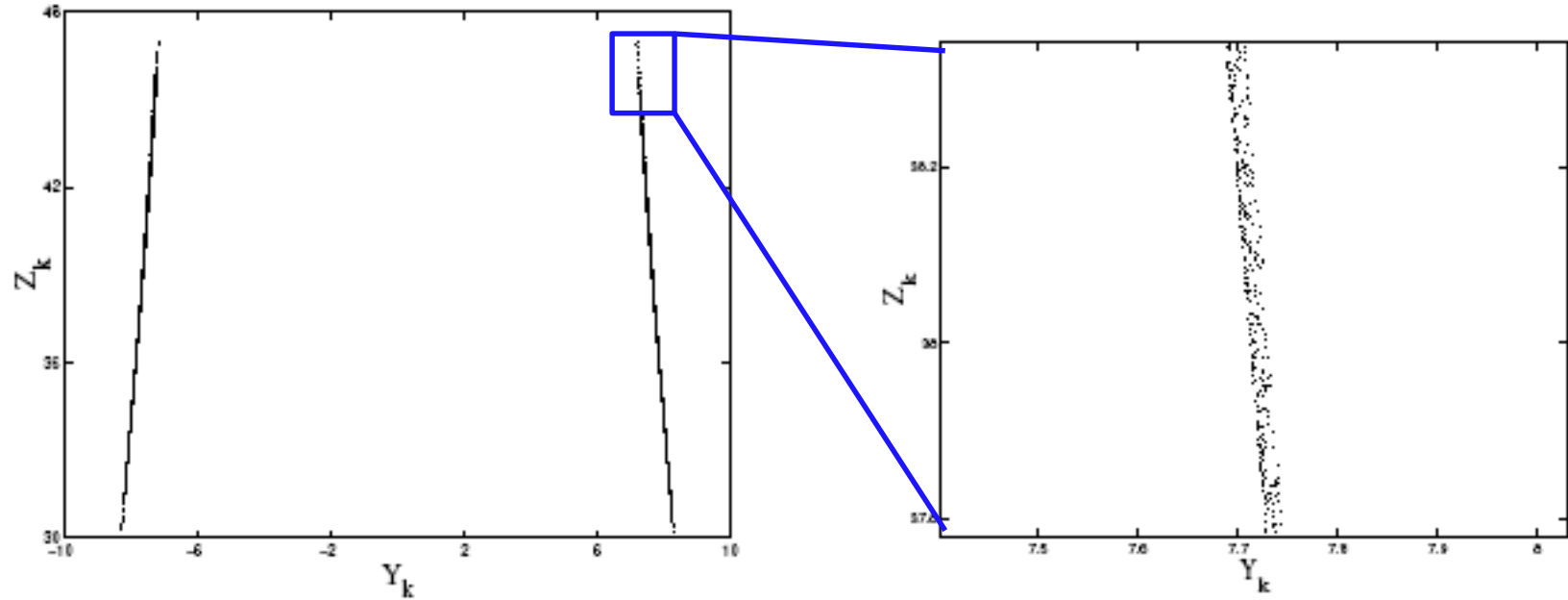
$$Z_{k+1}' = Z_{k+1} + df/dZ(Z_k) \delta Z$$

The divergence depends on the the sign of $|df/dZ| - 1$

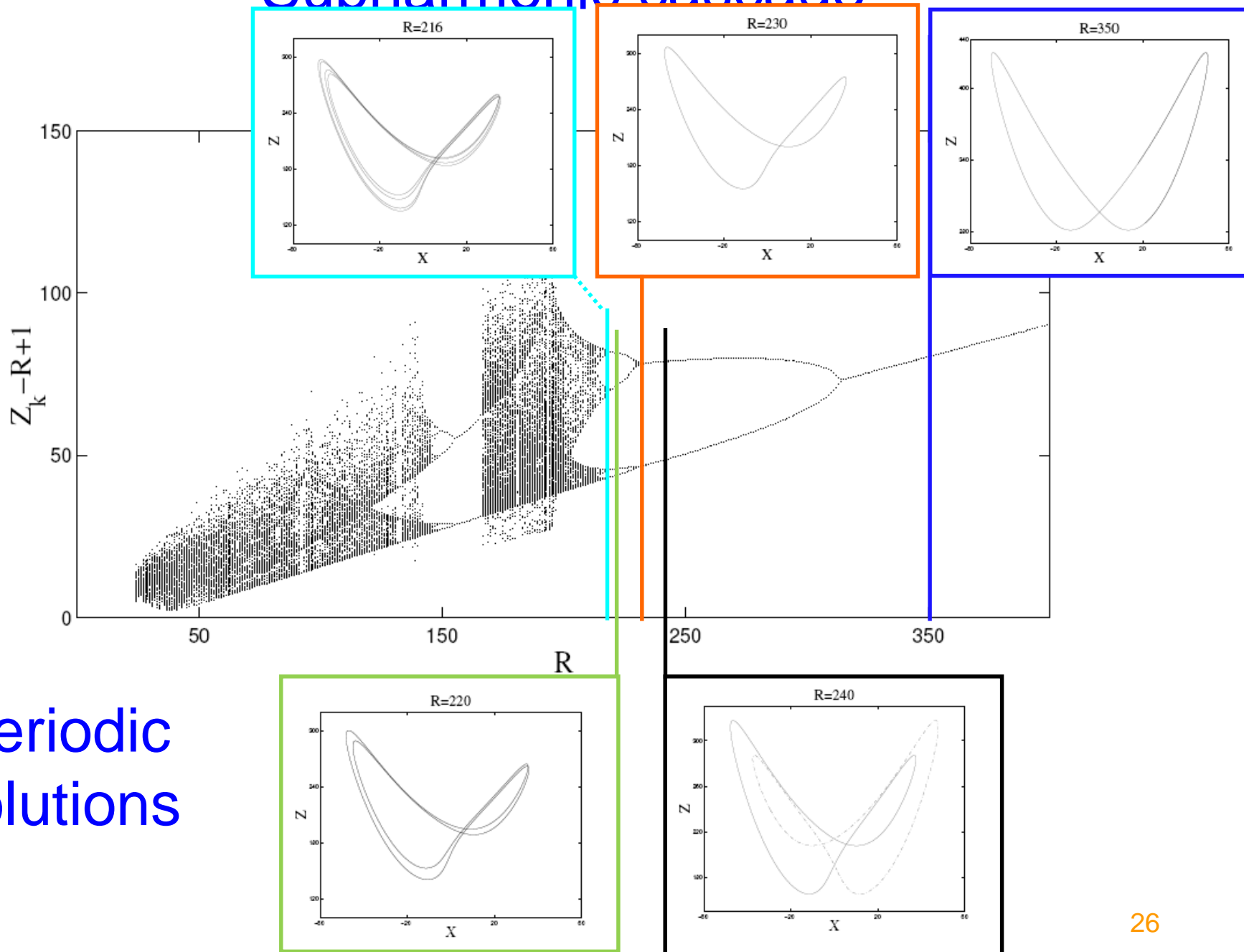
Here $|df/dZ| > 1$ everywhere!
Small initial differences get exponentially amplified
Loss of memory of initial condition
Butterfly effect



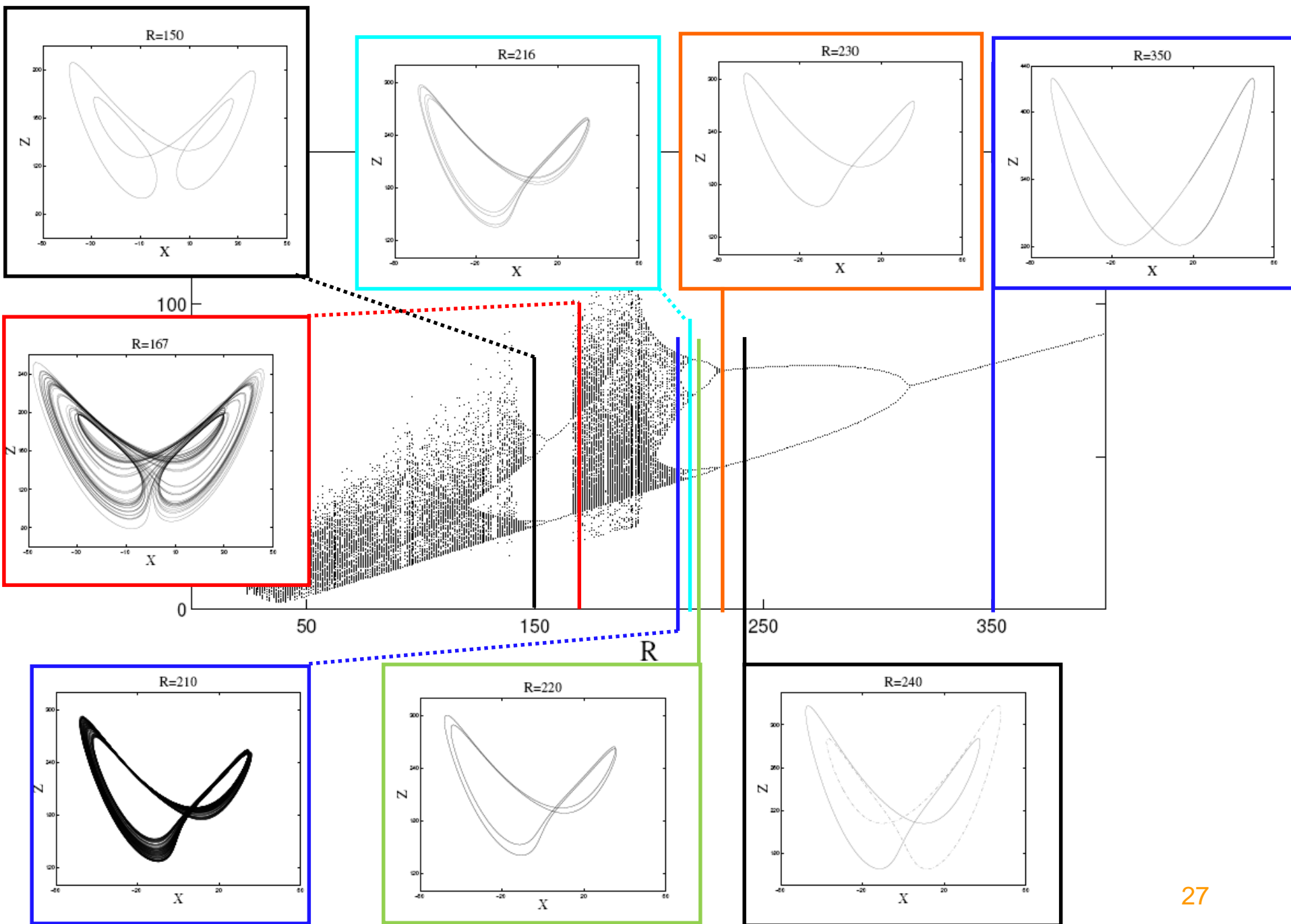
Beware, this is not a function



Subharmonic cascade



Periodic solutions



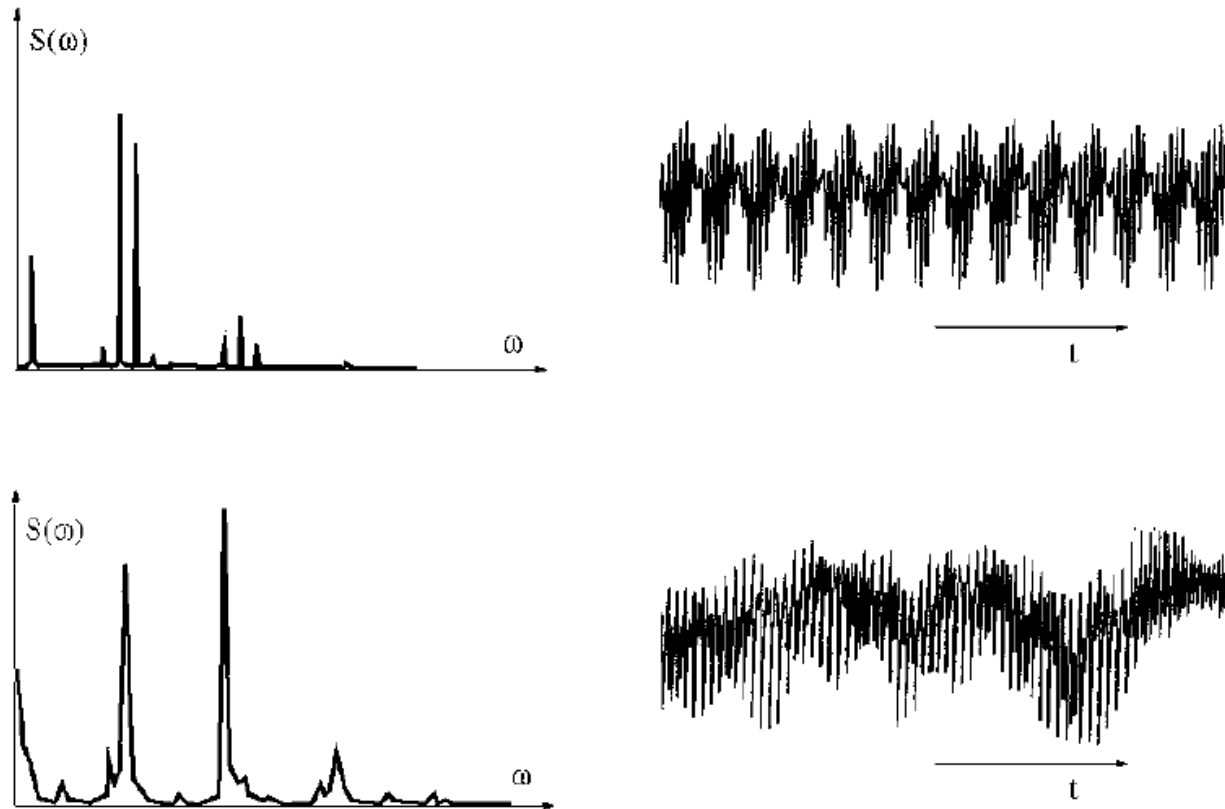
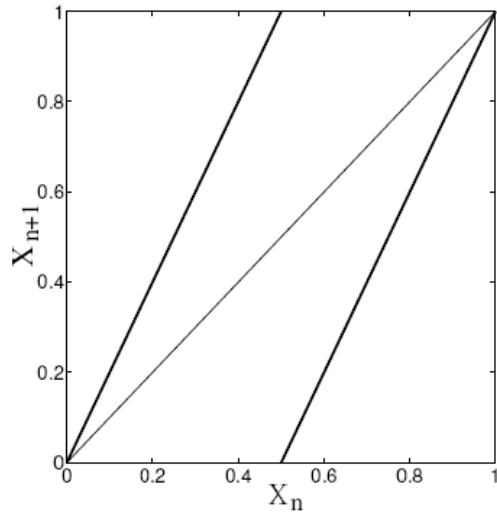


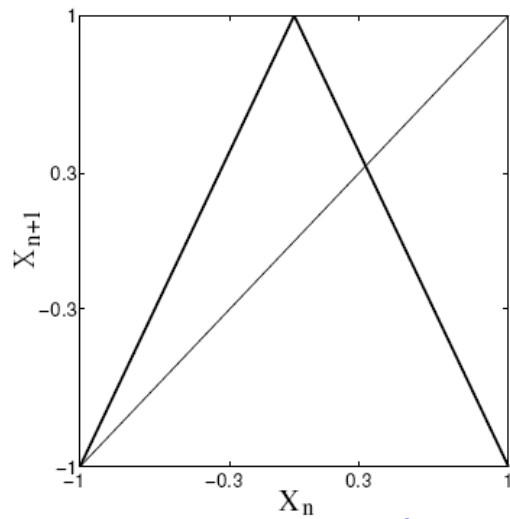
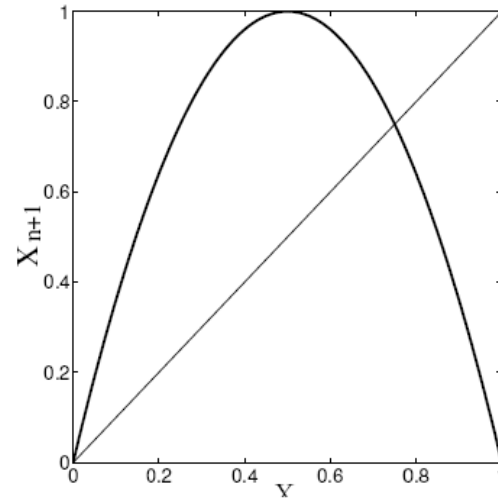
Figure 6.14: Spectres du signal de vitesse mesuré en un point de la cellule. En régime bipériodique (en haut), le spectre est composé de deux raies principales et de leurs combinaisons simples. Le régime chaotique est caractérisé par des raies au pied élargi et par la présence de bruit de basse fréquence ($\omega \approx 0$) dans le spectre (d'après Bergé et Dubois, 1981).

Different maps

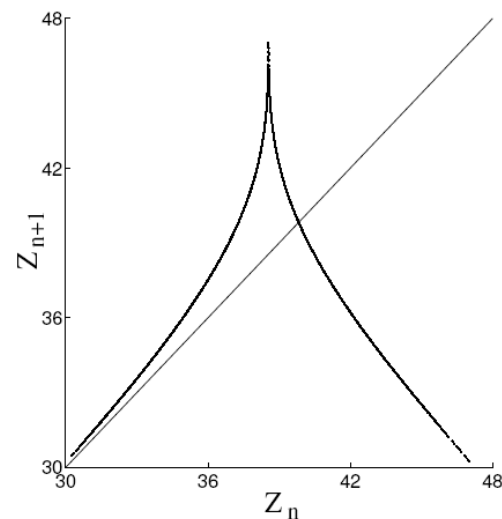
dyadic



logistic



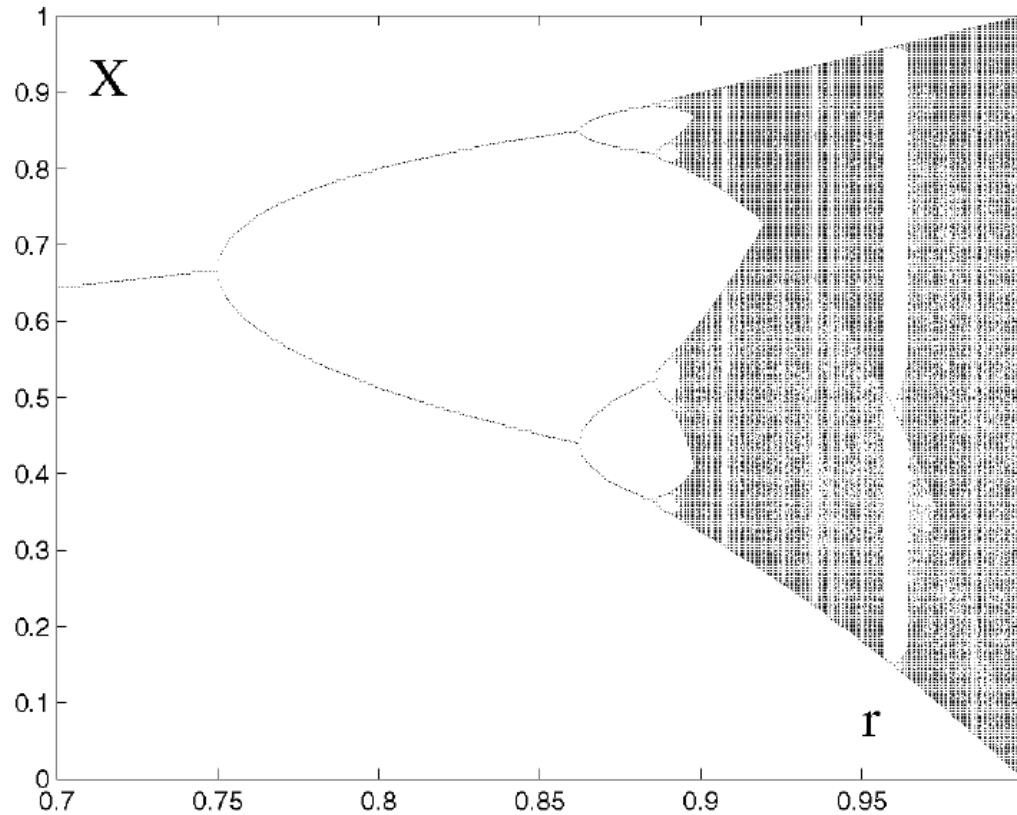
accent circonflexe



Lorentz R=28

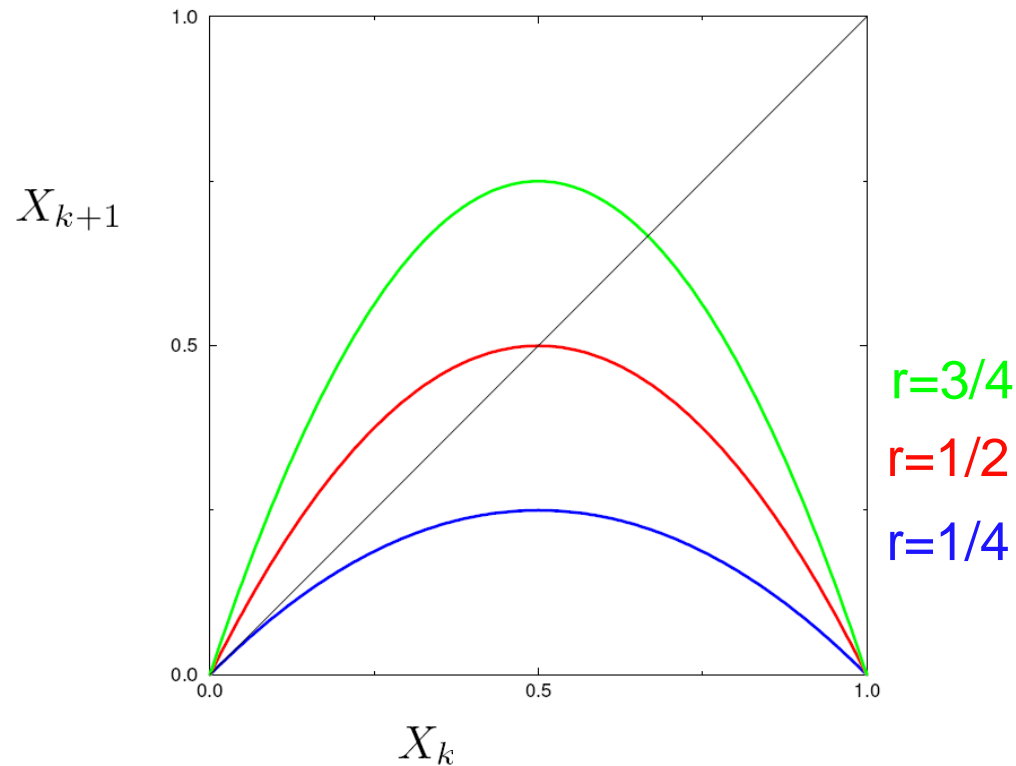
Logistic map

$$X_{k+1} = F_r(X_k) = 4r X_k(1 - X_k)$$



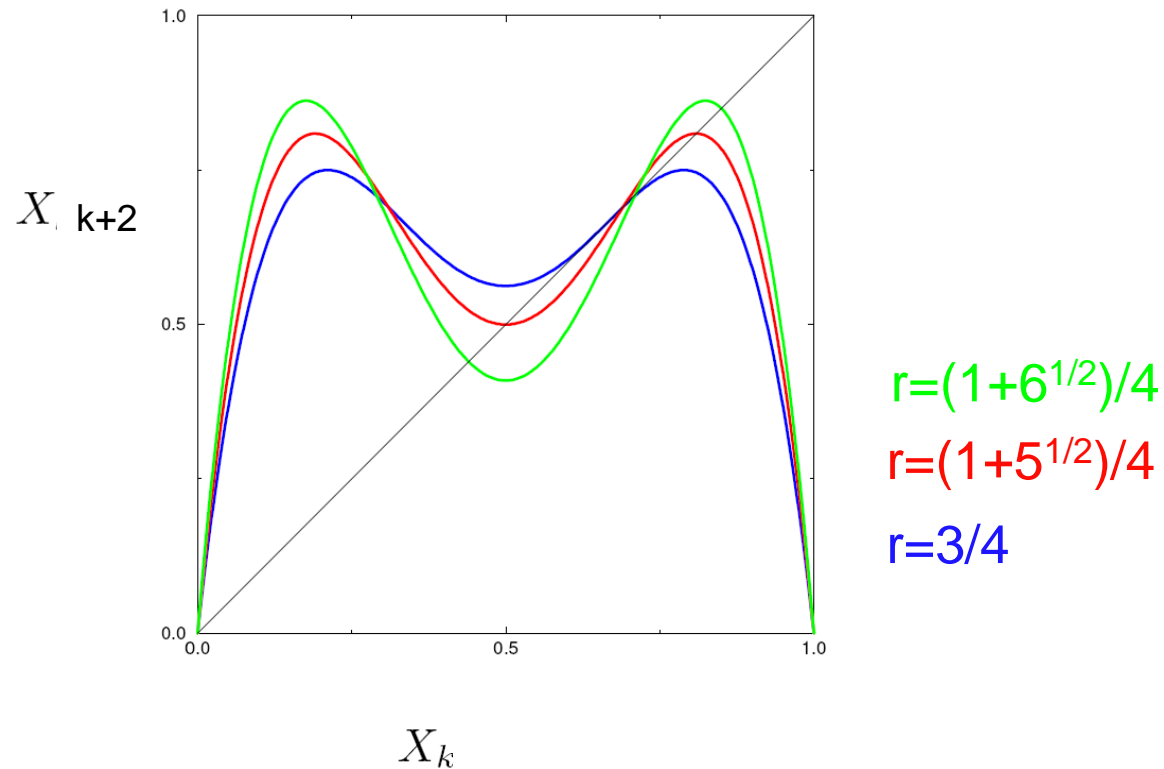
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Logistic map

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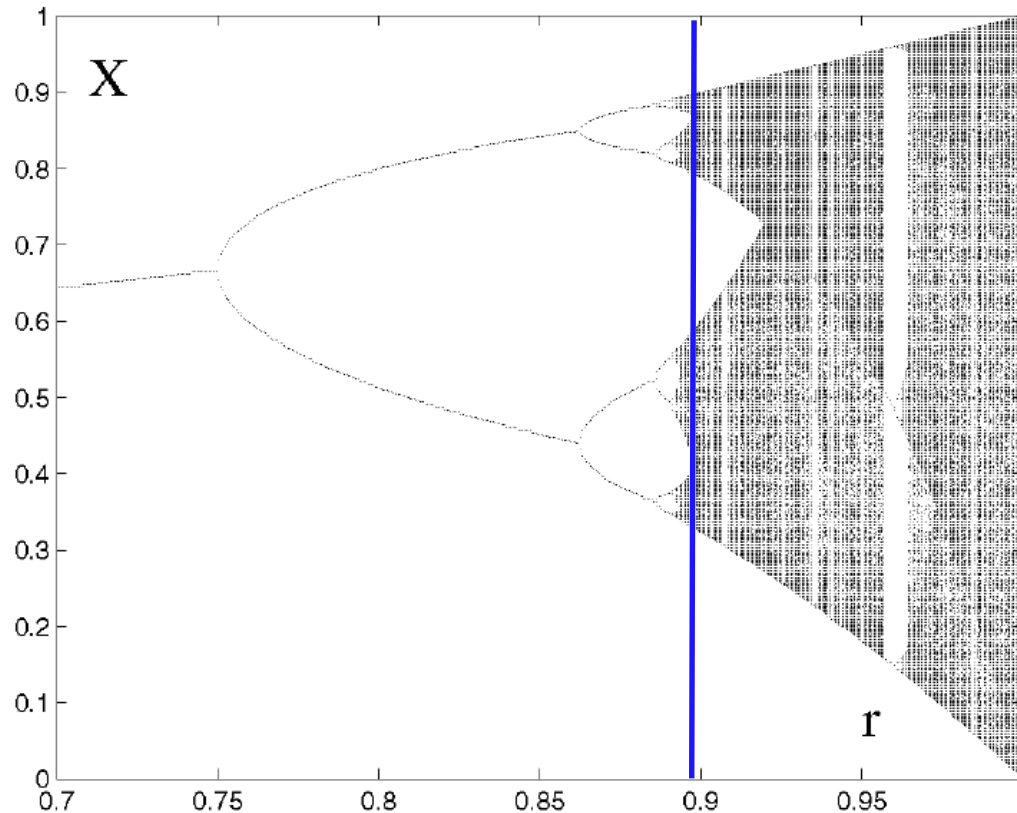
Logistic map

critical bifurcation parameters for period doubling

n	$T_{(n)}$	$r_{(n)}$	$r_{(n)}^{ss}$
0	1	1/4	1/2
1	2	3/4	0,80901...
2	4	0,86237...	0,87464...
3	8	0,88602...	0,88866...
4	16	0,89218...	...
\vdots	\vdots	\vdots	\vdots
∞	∞	0,89248...	0,89248...

Periodic window

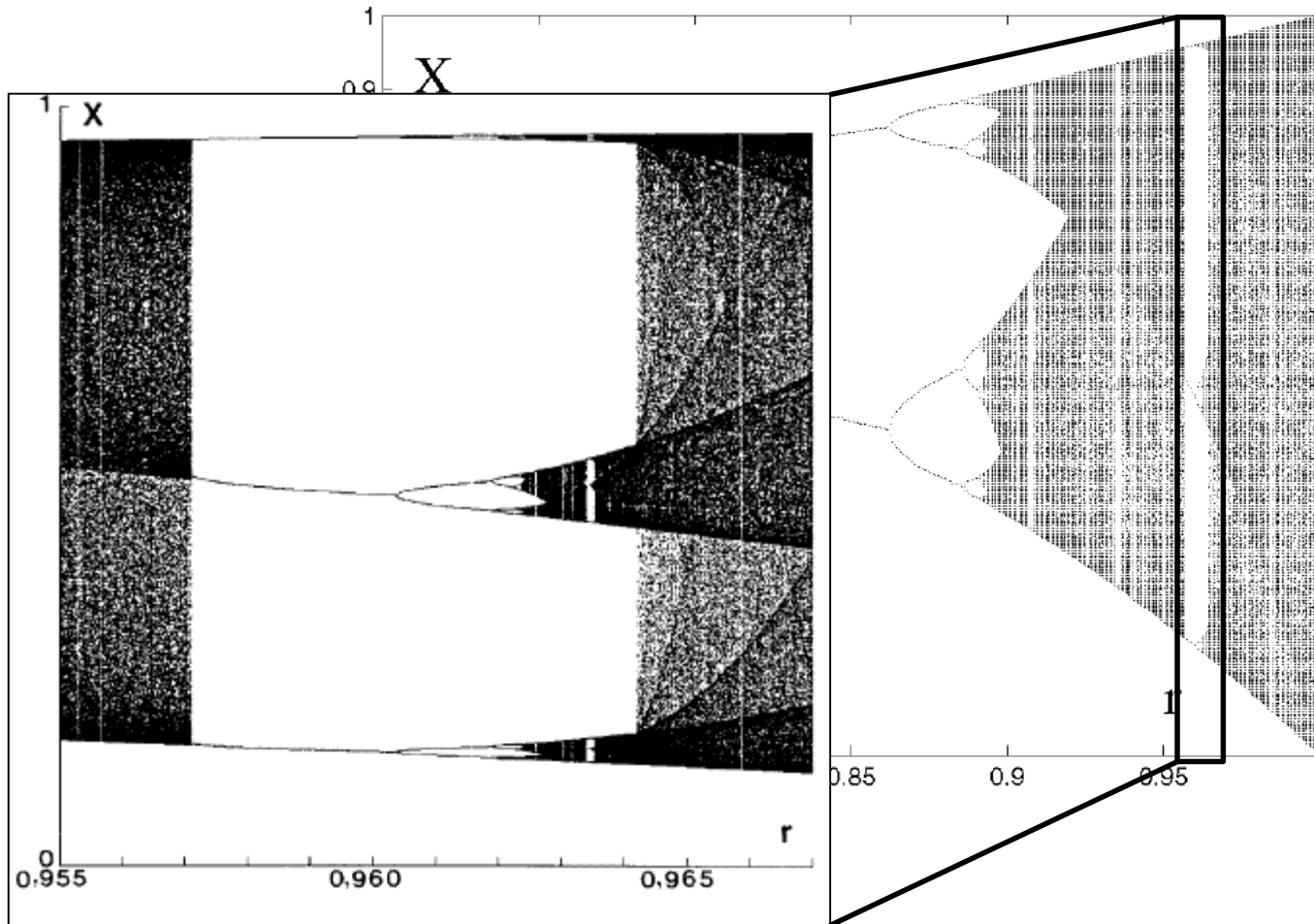
$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



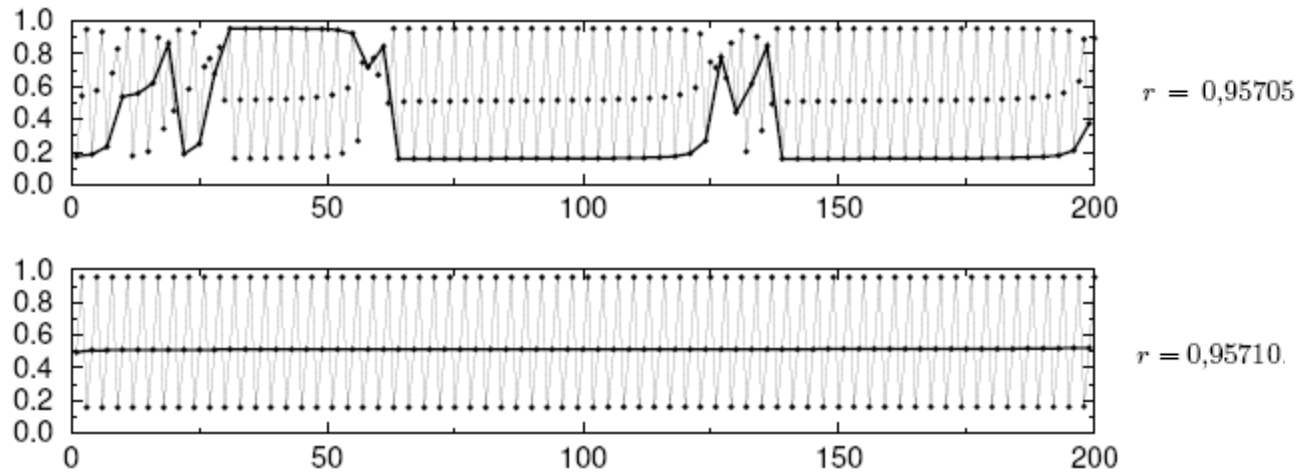
Logistic map

Periodic window

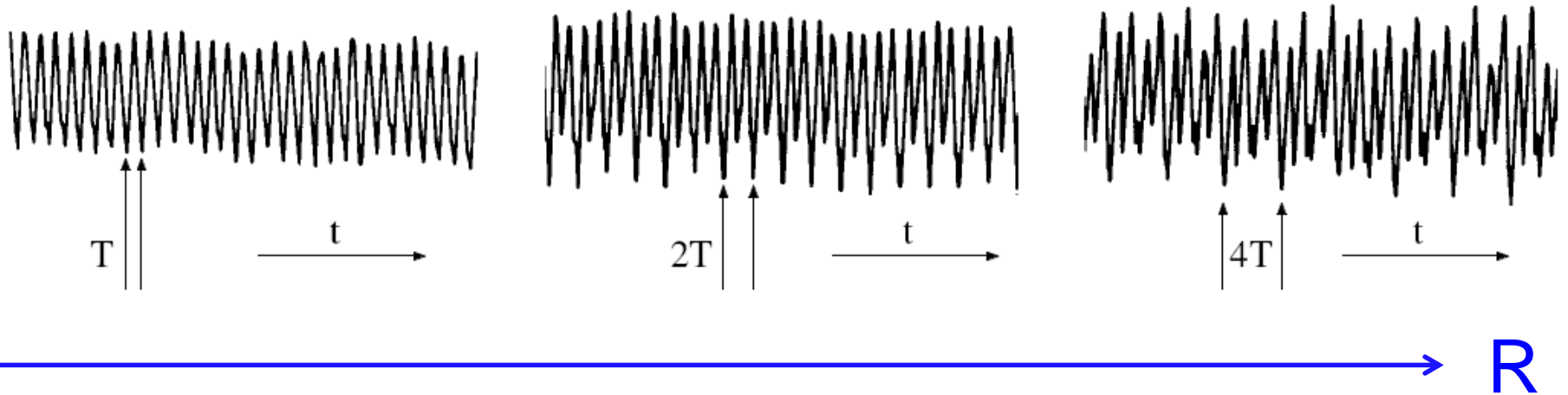
$$X_{k+1} = F_r(X_k) = 4rX_k(1 - X_k)$$



Very very sensitive system

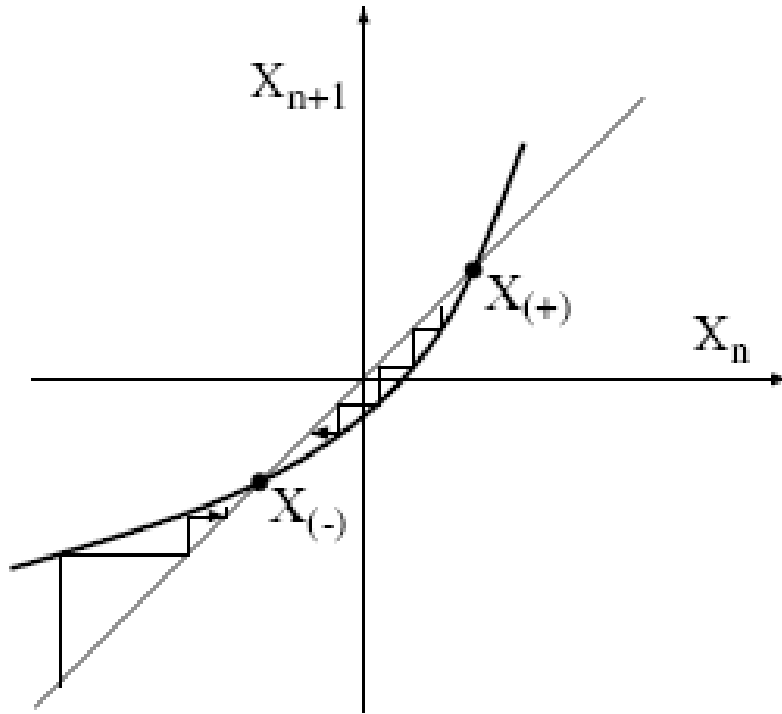


Subharmonic cascade in convection

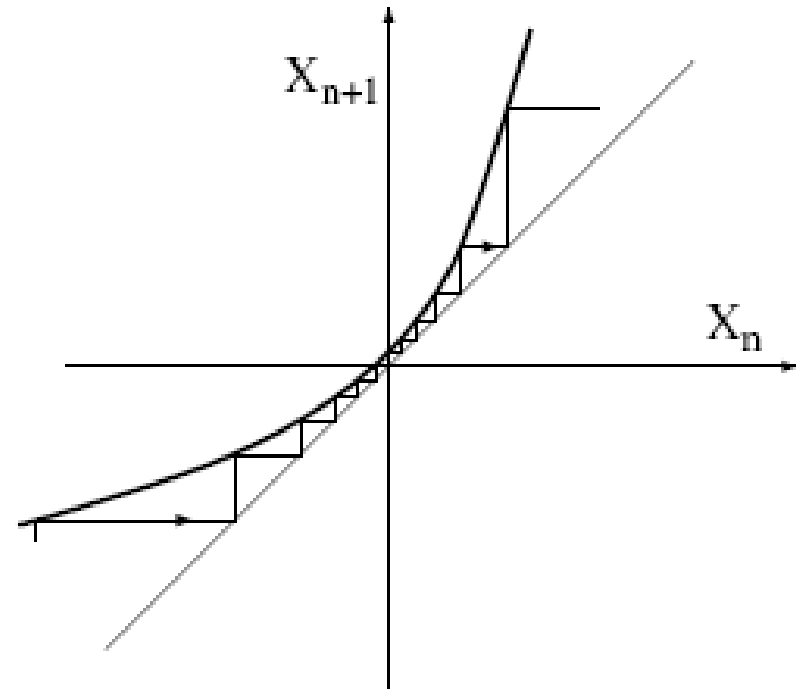


Libchaber and Maurer '80
Liquid Helium

Intermittency

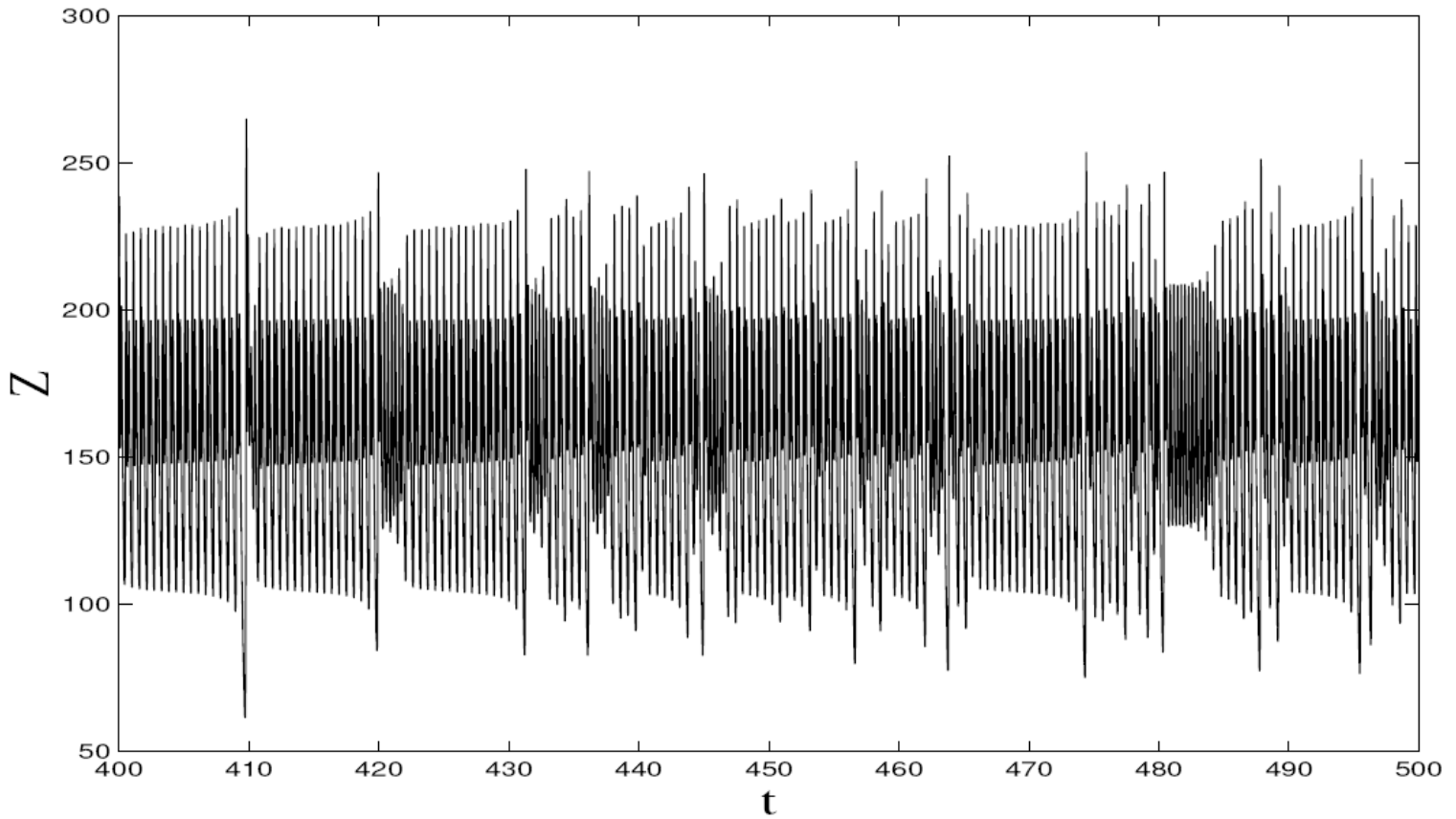


Stable



No fixed point-
Long attracting periods
Intermittent regime

Intermittency



Lorentz at $R=166$