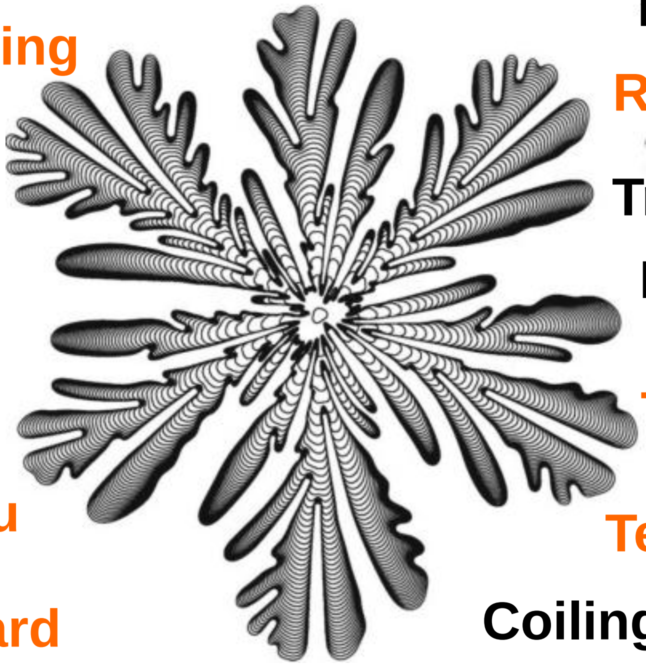
Most flows are unstable...



Vortex shedding

Saffman-Taylor

Flow separation

Tollmien-Schlichting

Rayleigh-Taylor

Lift-up and Streaks

Traffic waves

Meandering instability

Gravito-capillary waves

Taylor-Couette

Rayleigh-Plateau

Tearing instability

Coiling instability

Rayleigh-Benard

Benard-Marangoni

Kelvin-Helmholtz

Nonlinearities: bifurcations and amplitude equations

Agenda

1. The gravitational pendulum : a poor watch
   First use of Multiple scale weakly nonlinear approach
2. A simple example of bifurcation
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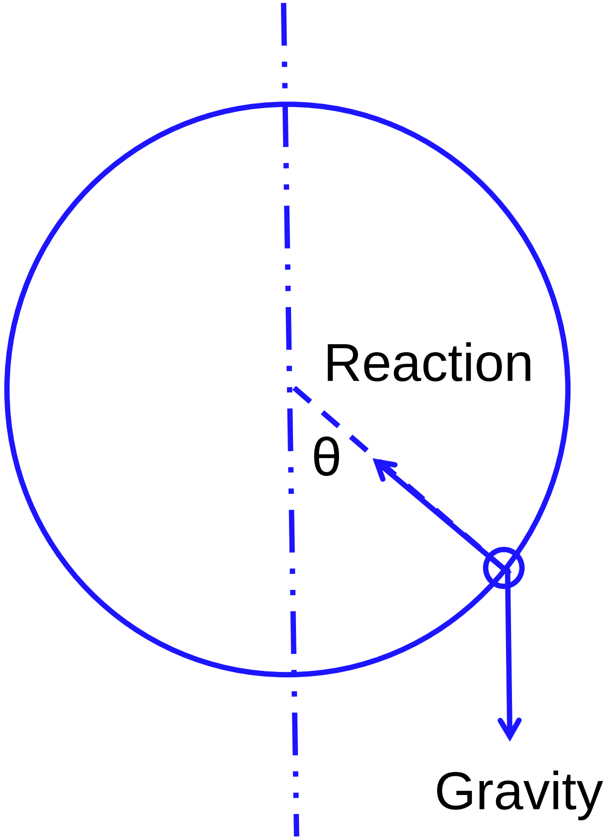
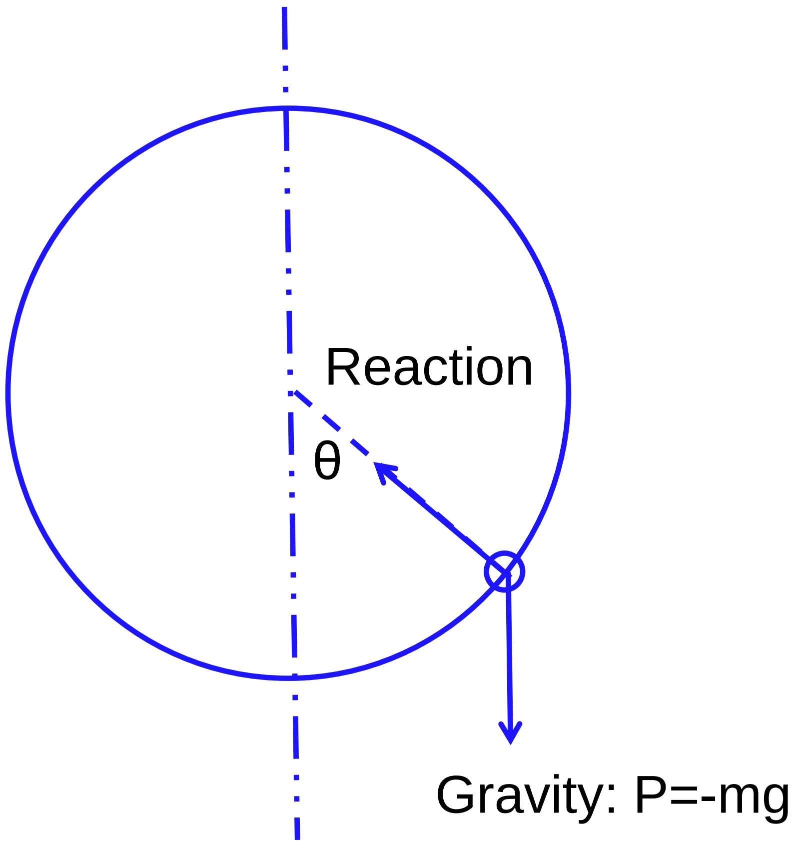
3. Classical bifurcations
4. Hopf bifurcation and Stuart-Landau equation

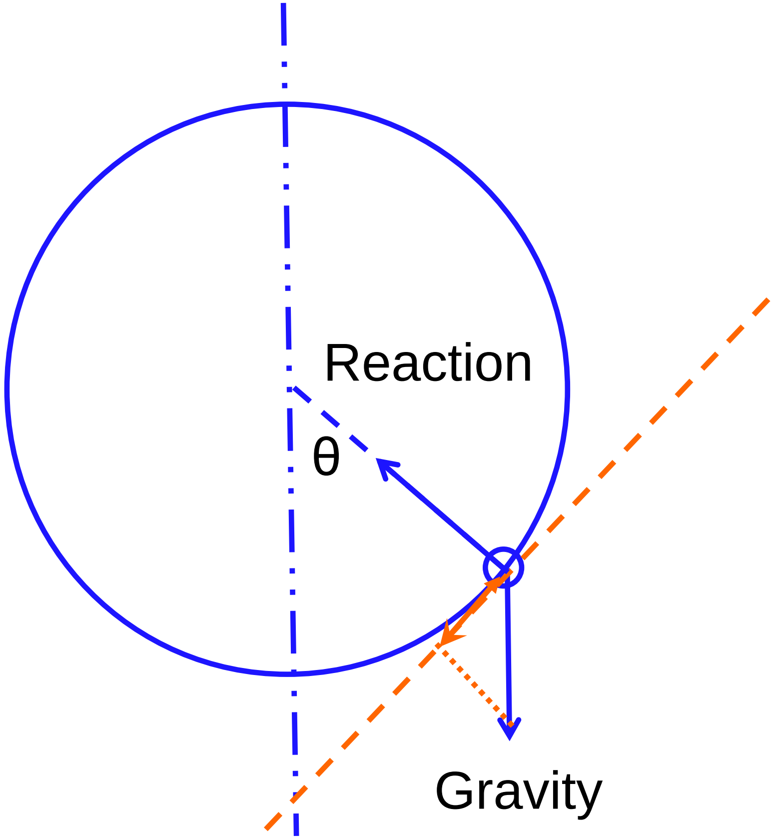
Period of a gravitational pendulum



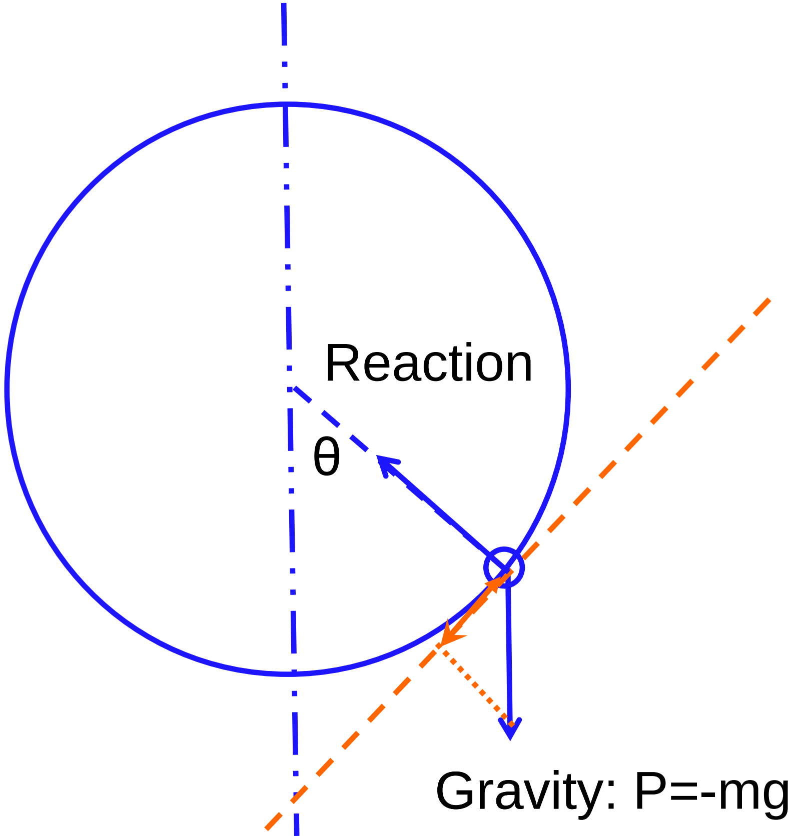
Period of a gravitational pendulum



Bifurcations and amplitude equations



Bifurcations and amplitude equations



$$\boxed{mR\ddot{\theta}=-mg\sin(\theta)}$$

Governing equations

$$\boxed{\ddot{\theta}=-\omega_0^2\sin(\theta)}$$

$\omega_0^2=g/R$

Pendulum frequency

Small perturbations

$$\theta=\varepsilon\,\theta'$$

Linearized equations

$$\boxed{\ddot{\theta}'=-\omega_0^2\theta'}$$

Governing equations

$$\boxed{\ddot{\theta}=-\omega_0^2\sin(\theta)}$$

$\omega_0^2=g/R$

Pendulum frequency

Small perturbations

$$\theta=\varepsilon\,\theta'$$

Linearized equations

$$\boxed{\ddot{\theta}'=-\omega_0^2\theta'}$$

Period seems independant of amplitude.

Can this be true? Effect of nonlinearity on period?

Multiple time scale expansion
Weakly nonlinear approach

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \ldots$$

$$T = \epsilon^2 t \qquad \frac{\partial}{\partial t} = \cdot \qquad \frac{\partial}{\partial T} = `$$

$\theta_i(t,T)$

$$\boxed{\ddot{\theta} = -\omega_0^2 \sin(\theta)}$$

Order ε^0

$$\ddot{\theta}_0 = 0 \Rightarrow \theta_0 = 0$$

Multiple time scale expansion
Weakly nonlinear approach

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \ldots$$

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$\theta_i(t,T)$

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Order ε^0 $\qquad \ddot{\theta}_0 = 0 \Rightarrow \theta_0 = 0$

Order ε^1 $\qquad \ddot{\theta}_1 = -\omega_0^2\theta_1 \Rightarrow \theta_1 = A(T)\cos(\omega_0 t + \varphi(T))$

Multiple time scale expansion
Weakly nonlinear approach

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \ldots$$

$$T = \epsilon^2 t \qquad \frac{\partial}{\partial t} = \cdot \quad \frac{\partial}{\partial T} = `$$

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Order ε^0 $\ddot{\theta}_0 = 0$ ⇨ $\theta_0 = 0$

Order ε^1 $\ddot{\theta}_1 = -\omega_0^2\theta_1$ ⇨ $\theta_1 = A(T)\cos(\omega_0 t + \varphi(T))$

Order ε^2 $\ddot{\theta}_2 = -\omega_0^2\theta_2$ ⇨ $\theta_2 = B(T)\cos(\omega_0 t + \psi(T))$

Multiple time scale expansion
Weakly nonlinear approach

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \ldots$$

$$T = \epsilon^2 t \qquad \frac{\partial}{\partial t} = \dot{} \qquad \frac{\partial}{\partial T} = {}'$$

$\theta_i(t,T)$

$$\boxed{\ddot{\theta} = -\omega_0^2 \sin(\theta)}$$

Order ε^0	$\ddot{\theta}_0 = 0 \Rightarrow \theta_0 = 0$
Order ε^1	$\ddot{\theta}_1 = -\omega_0^2\theta_1 \Rightarrow \theta_1 = A(T)\cos(\omega_0 t + \varphi(T))$
Order ε^2	$\ddot{\theta}_2 = -\omega_0^2\theta_2 \Rightarrow \theta_2 = B(T)\cos(\omega_0 t + \psi(T))$
Order ε^3	$\ddot{\theta}_3 = -\omega_0^2\theta_3 - 2\dot{\theta}'_1 + \theta_1^3/6$

Non-resonance condition

Order ε^3

$$\ddot{\theta}_3=-\omega_0^2\theta_3-2\dot{\theta}_1'+\omega_0^2\,\theta_1^3/6$$

$$\dot{\theta}_1'=-A'(T)\omega_0\sin(\omega_0 t+\varphi(T))+\omega_0 A(T)\varphi'(T)\cos(\omega_0 t+\varphi(T))$$

$$\cos^3(\omega_0 t+\varphi(T))=3/4\cos(\omega_0 t+\varphi(T))+1/4\cos(3\,\omega_0 t+\varphi(T))$$

$$\theta_1^3=A^3(T)\cos^3(\omega_0 t+\varphi(T))$$

Non-resonance condition

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$$\theta_1^3 = A^3(T)\cos^3(\omega_0 t+\varphi(T))$$

Attention! What happens if you force a linear system at its natural frequency?

Example: $\ddot{\theta}_3 + \omega_0^2\theta_3 = \cos(\omega_0 t)$

The particular solution is $\theta_{3f} = t\sin(\omega_0 t) / 2\omega_0$
It grows linearly in time and diverges! This should be avoided

Non-resonance condition

Order ε^3

$$\ddot{\theta}_3=-\omega_0^2\theta_3-2\dot{\theta}_1'+\omega_0^2\,\theta_1^3/6$$

$$\dot{\theta}_1'=-A'(T)\omega_0\sin(\omega_0t+\varphi(T))-\omega_0A(T)\varphi'(T)\cos(\omega_0t+\varphi(T))$$

$$\cos^3(\omega_0t+\varphi(T))=3/4\cos(\omega_0t+\varphi(T))+1/4\cos(3\,\omega_0t+\varphi(T))$$

$$\theta_1^3=A^3(T)\cos^3(\omega_0t+\varphi(T))$$

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The particular solution is $\theta_{3f}=t\sin(\omega_0t)\,/\,2\omega_0$
It grows linearly in time and diverges! This should be avoided

Therefore the resonant RHS should be zero
$$A'(T)\omega_0\sin(\omega_0t+\varphi(T))+\omega_0A(T)\varphi'(T)\cos(\omega_0t+\varphi(T))+\omega_0^2\,A^3(T)\cos(\omega_0t+\varphi(T))/16=0$$

Nonlinear frequency correction

Therefore the resonant RHS should be zero

$$A'(T)\omega_0\sin(\omega_0 t+\varphi(T))+\omega_0 A(T)\varphi'(T)\cos(\omega_0 t+\varphi(T))+\omega_0^2 A^3(T)\cos(\omega_0 t+\varphi(T))/16=0$$

Nonlinear frequency correction

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$A'(T)\omega_0\sin(\omega_0 t+\varphi(T))+\omega_0 A(T)\varphi'(T)\cos(\omega_0 t+\varphi(T))+\omega_0^2 A^3(T)\cos(\omega_0 t+\varphi(T))/16=0$

$+\omega_0 A(T)\varphi'(T)+\omega_0^2 A^3(T)/16=0$

$A'(T)\omega_0=0$

Nonlinea frequency correction

Therefore the resonant RHS should be zero

$A'(T)\omega_0\sin(\omega_0 t+\varphi(T))+\omega_0 A(T)\varphi'(T)\cos(\omega_0 t+\varphi(T))+\omega_0^2 A^3(T)\cos(\omega_0 t+\varphi(T))/16=0$

$+\omega_0 A(T)\varphi'(T)+\omega_0^2 A^3(T)/16=0$

$A'(T)\omega_0=0$

$\varphi(T)=\varphi_0-\omega_0 A_0^2 T/16$

$A(T)=A_0$

Nonlinea frequency correction

Therefore the resonant RHS should be zero

$A'(T)\omega_0\sin(\omega_0 t+\varphi(T))+\omega_0 A(T)\varphi'(T)\cos(\omega_0 t+\varphi(T))+\omega_0^2 A^3(T)\cos(\omega_0 t+\varphi(T))/16=0$

$+\omega_0 A(T)\varphi'(T) + \omega_0^2 A^3(T)/16=0$

$A'(T)\omega_0= 0$

$\varphi(T)= \varphi_0-\omega_0 A_0^2 T/16$

$A(T)=A_0$

The oscillation frequency depends on the amplitude

$\omega =\omega_0(1-A_0^2/16)$ Borda’s Formula

A gravitational pendulum is not a good oscillator for a watch!

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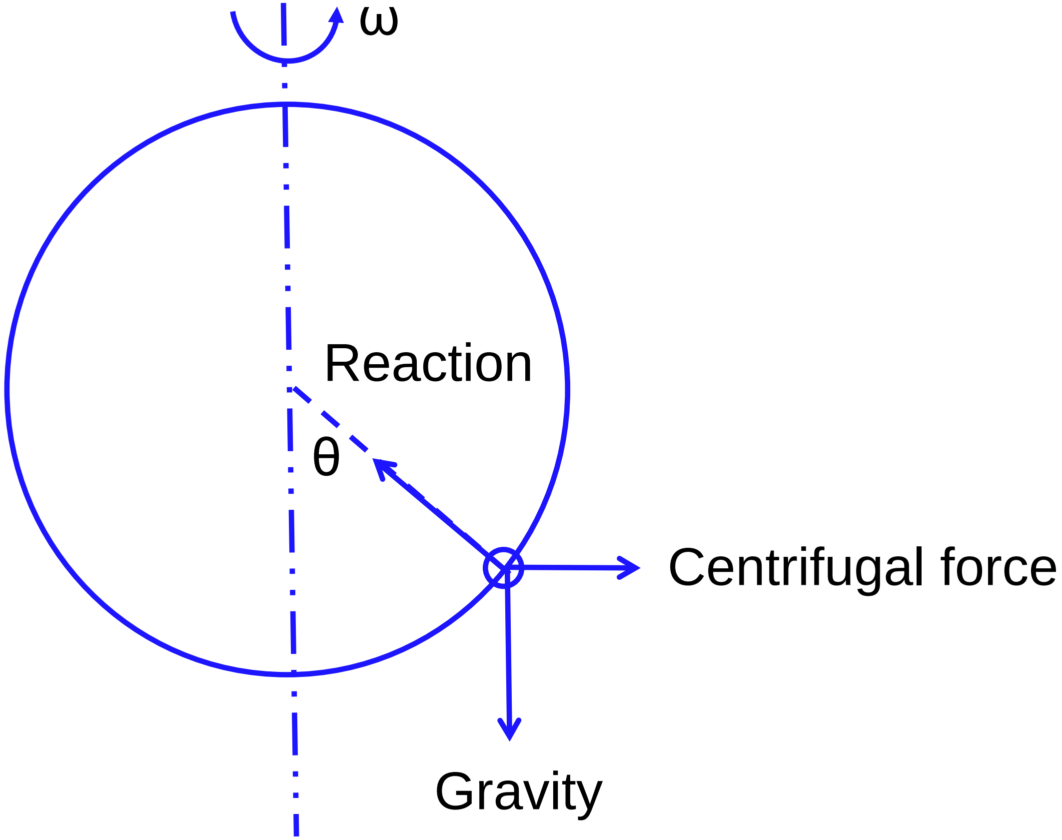
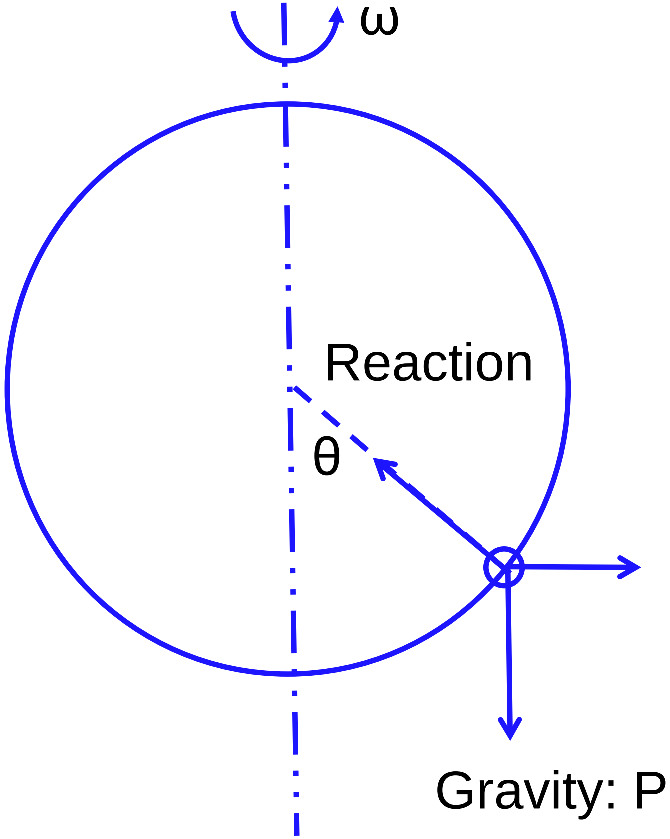
Bifurcations and amplitude equations



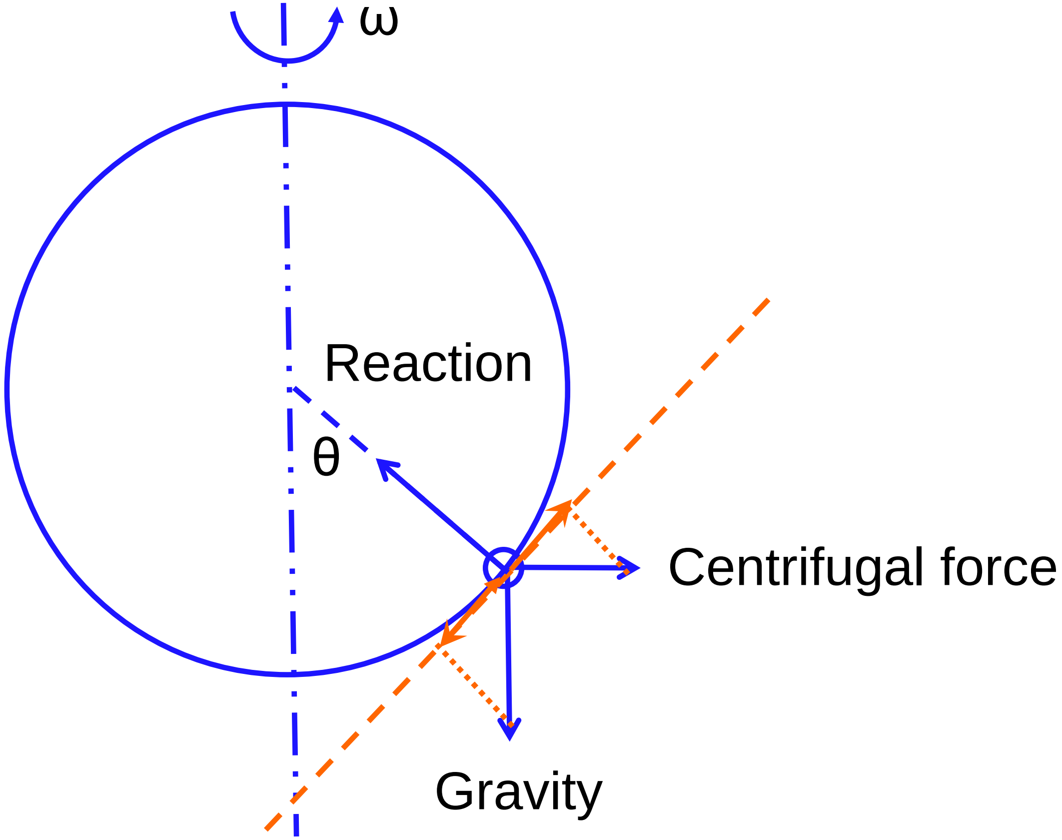
Bifurcations and amplitude equations



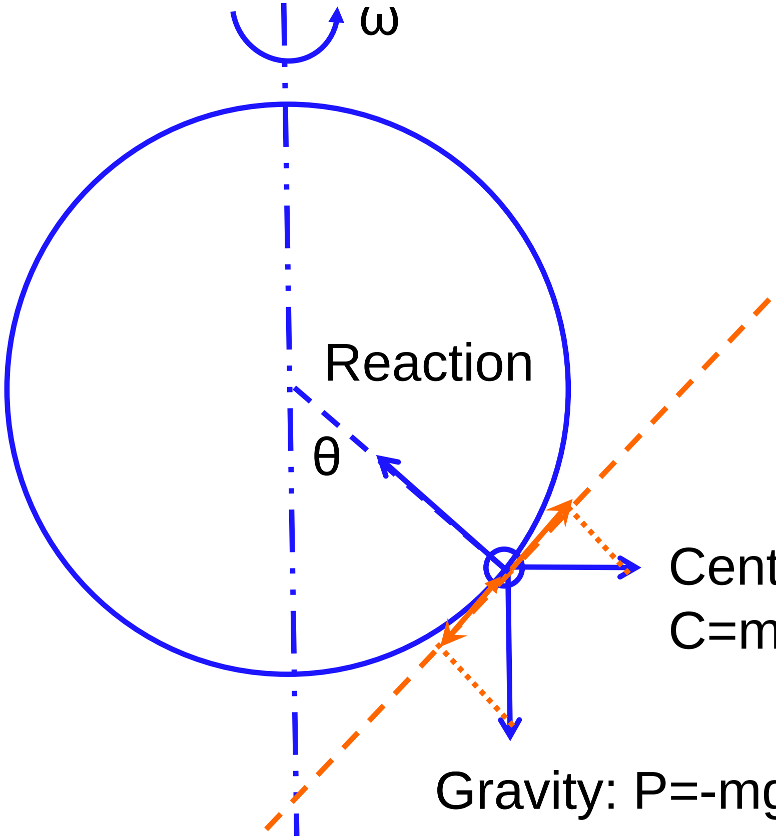
Centrifugal force:
$C=mR\sin(\theta)\omega^2$

Gravity: $P=-mg$

Bifurcations and amplitude equations



Bifurcations and amplitude equations



Centrifugal force
$C=mR\sin(\theta)\omega^2\cos(\theta)$

Gravity: $P=-mg\sin(\theta)$

$$\boxed{mR\ddot{\theta}=-mg\sin(\theta)+mR\sin(\theta)\omega^2\cos(\theta)}$$

Governing equations

$$\boxed{\ddot{\theta}=-\omega_0^2\sin(\theta)+\omega^2\sin(\theta)\cos(\theta)}$$

$\omega_0^2=g/R$

Pendulum frequency

Base flow

$\theta=0$

Small perturbations

$\theta=0+\varepsilon\,\theta'$

Linearized equations

$$\boxed{\ddot{\theta}'=-\omega_0^2\theta'+\omega^2\theta'}$$

Linearized equations

$$\boxed{\ddot{\theta}' = -\omega_0^2\theta' + \omega^2\theta'}$$

Normal mode

$$\theta' = A\exp(st)$$

Dispersion relation

$$\boxed{s^2 = \omega^2 - \omega_0^2}$$

$$\boxed{\omega^2 < \omega_0^2}$$

$$\boxed{\omega^2 > \omega_0^2}$$

$$\boxed{s = \pm i(\omega_0^2 - \omega^2)^{1/2}}$$

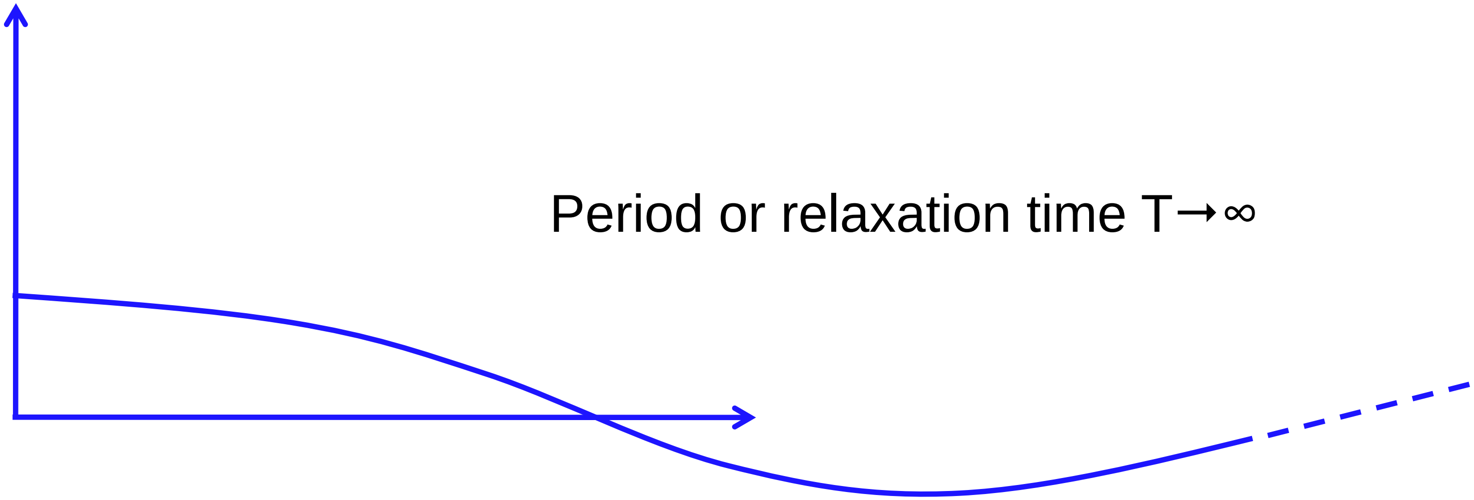
STABLE

$$\boxed{s = \pm(\omega^2 - \omega_0^2)^{1/2}}$$

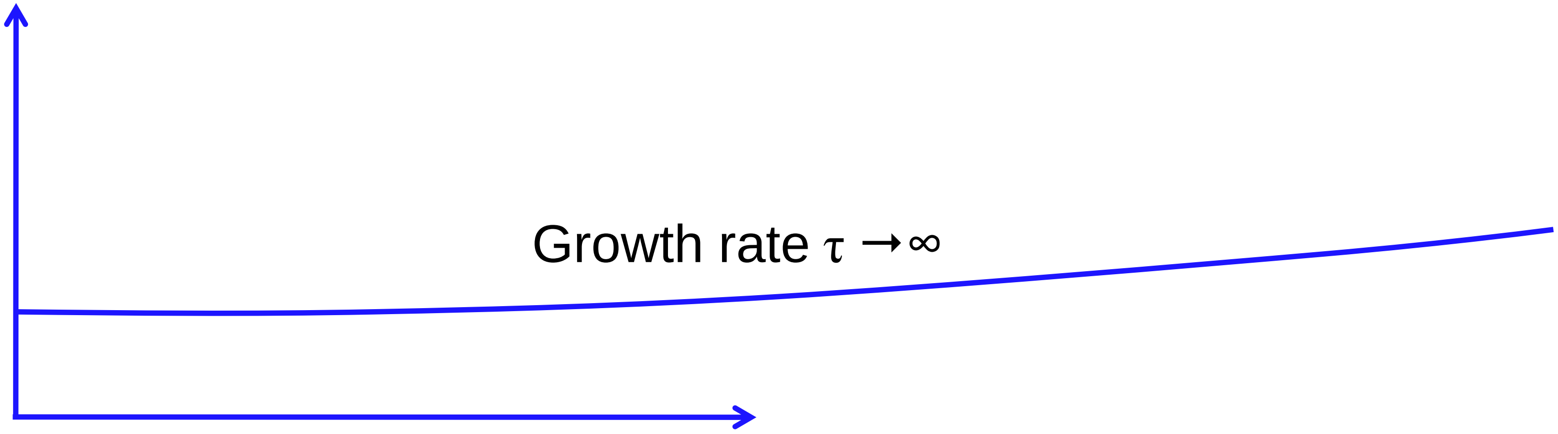
UNSTABLE

Important concept: critical slowing

When $\omega^2 \sim \omega_0^2$, the characteristic time $\tau=1/|s|$ diverges



Period or relaxation time T→∞



Growth rate $\tau \to \infty$

What about nonlinearities?

Weakly nonlinear theory : multiscale expansion

Slow time scale $T = \epsilon^2 t$

Close to threshold $\omega^2 = \omega_0^2 + \epsilon^2 \Delta$

Asymptotic expansion $\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \ldots$

$\theta_i(t,T)$

$$\boxed{\ddot{\theta}' = -\omega_0^2\theta' + \omega^2\theta'}$$

Order 0 $\theta_0 = 0$ Base state

Order 1 $\frac{\partial^2 \theta_1}{\partial t^2} = 0$

⇨ $\theta_1 = A_1(T)$ Constant 1st order perturbation

What about nonlinearities?

$$\boxed{\ddot{\theta}' = -\omega_0^2\theta' + \omega^2\theta'}$$

Order 0 $$\theta_0 = 0$$ Base state

Order 1 $$\frac{\partial^2\theta_1}{\partial t^2} = 0$$

⇨ $$\theta_1 = A_1(T)$$ Constant (in t!) 1st order perturbation

Order 2 $$\frac{\partial^2\theta_2}{\partial t^2} = 0$$

⇨ $$\theta_2 = A_2(T)$$ Constant (in t!) 2nd order perturbation

Order 3 $$\frac{\partial^2\theta_3}{\partial t^2} = \boxed{-\left(\frac{1}{2}\omega_0^2 A_1^3 - \Delta A_1\right)}$$

⇨ $= 0$

Secularity condition
= Non-resonance condition
= Compatilibity condition

If not, θ_3 would grow like t^2 and ruin the ordering in the expansion

What about nonlinearities?

$$\boxed{\ddot{\theta}'=-\omega_0^2\theta'+\omega^2\theta'}$$

Order 0 $$\theta_0 = 0$$

Order 1 $$\frac{\partial^2\theta_1}{\partial t^2} = 0$$

⇨ $$\theta_1 = A_1(T)$$

Order 2 $$\frac{\partial^2\theta_2}{\partial t^2} = 0$$

⇨ $$\theta_2 = A_2(T)$$

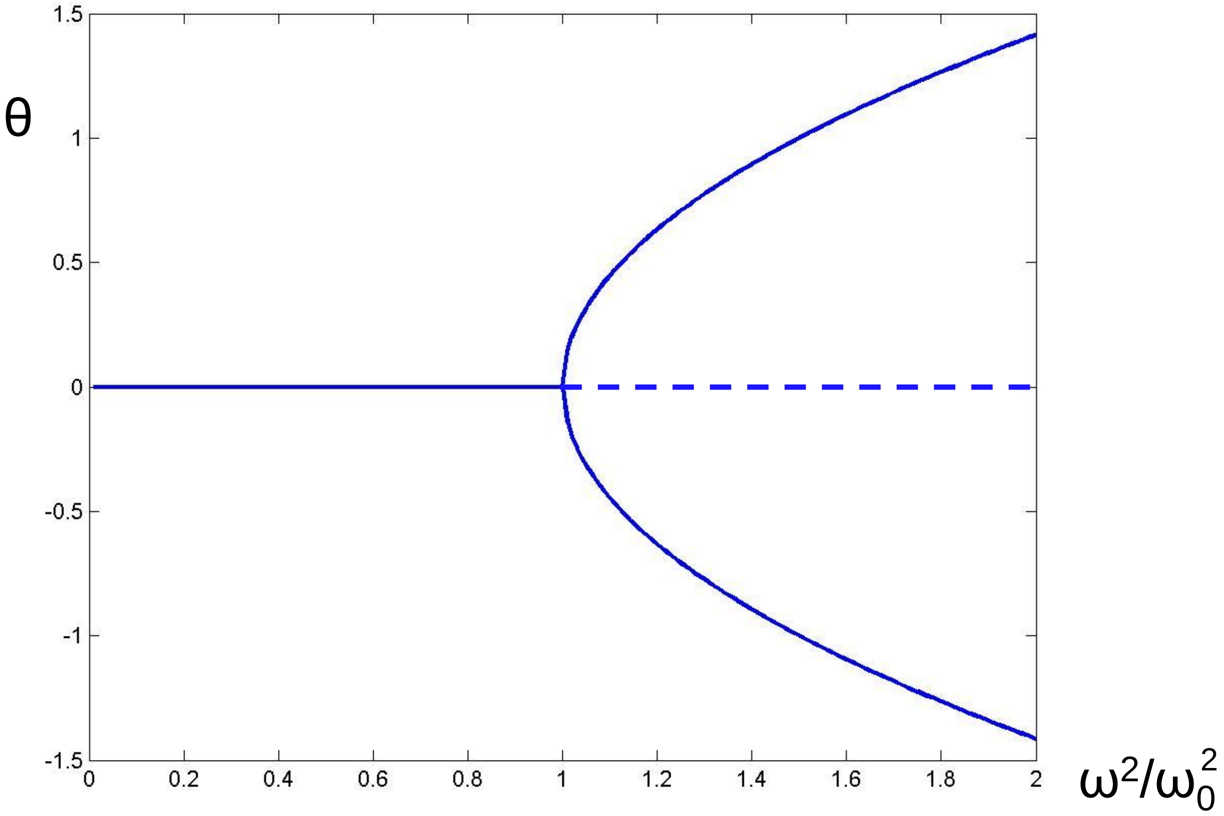
Order 3 $$\frac{\partial^2\theta_3}{\partial t^2} = -\left(\frac{1}{2}\omega_0^2 A_1^3 - \Delta A_1\right)$$

⇨ $$A_1 = \sqrt{2\frac{\Delta}{\omega_0^2}}$$

What about nonlinearities?

$$A_1 = \sqrt{2\frac{\Delta}{\omega_0^2}}$$

$$\epsilon A_1 = \sqrt{2\left(\frac{\omega^2}{\omega_0^2} - 1\right)}$$



What about nonlinearities?

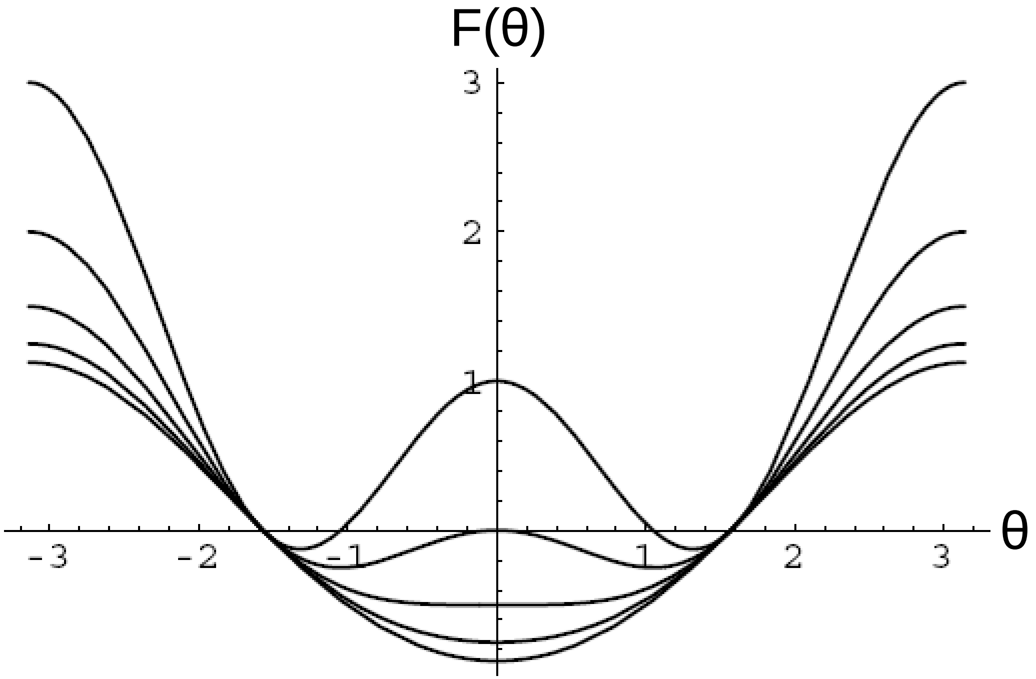
But recall the full nonlinear equation

$$\boxed{\ddot{\theta}=-\omega_0^2\sin(\theta)+\omega^2\sin(\theta)\cos(\theta)}$$

It has another 2 steady solutions

Physical interpretation (Potential)

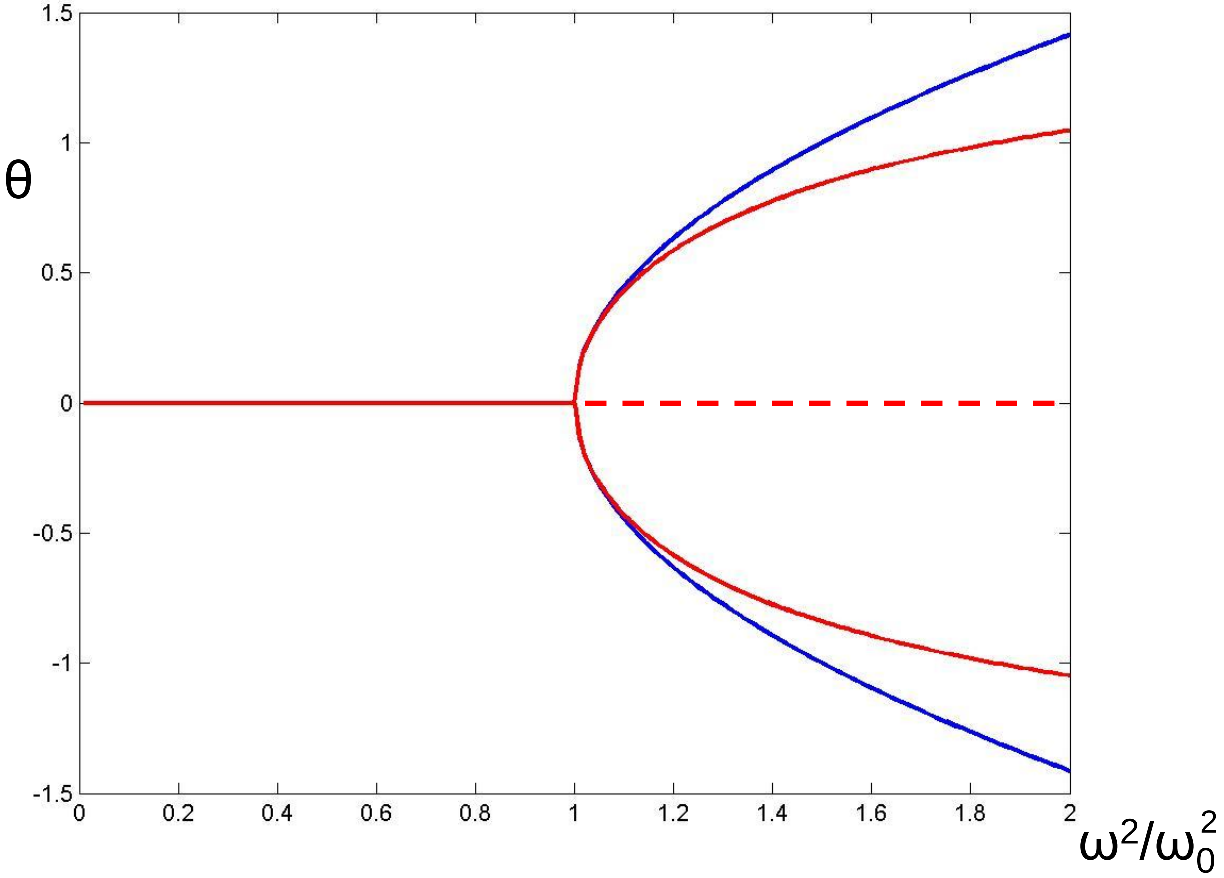
$$\boxed{m\ddot{\theta}=F'(\theta)}$$



What about nonlinearities?

It has another 2 steady solutions

$$\theta_s = \arccos((\omega_0/\omega)^2)$$



Stability of these bifurcated branches?

$$\boxed{\ddot{\theta}=-\omega_0^2\sin(\theta)+\omega^2\sin(\theta)\cos(\theta)}$$

$$\omega_0^2=\omega^2\cos(\theta_s)$$

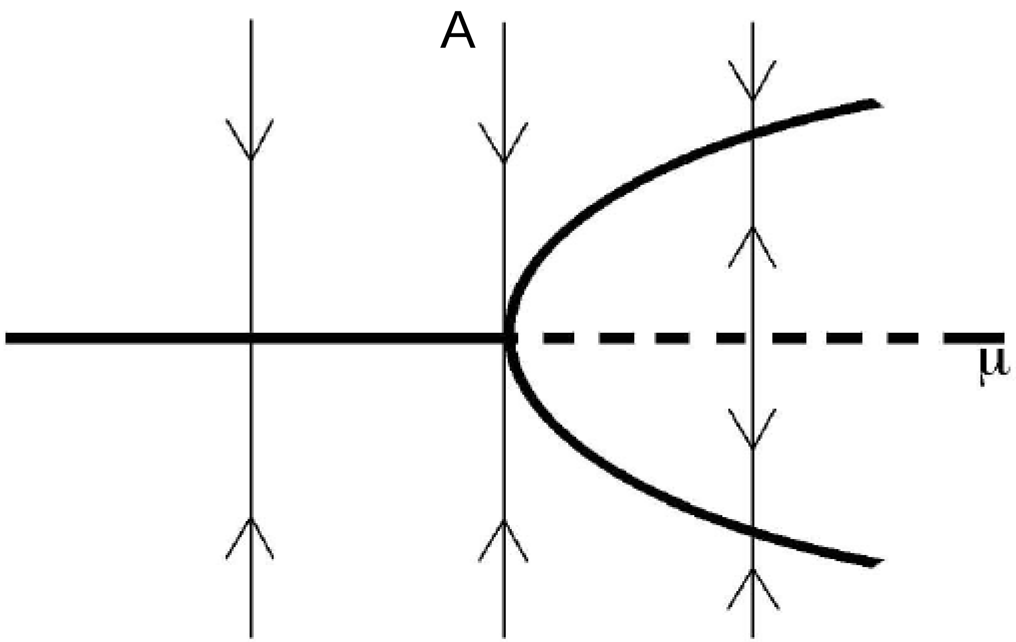
$$\theta=\theta_s+\theta'$$

$$\ddot{\theta}'=-\omega_0^2\cos(\theta_s)\,\theta'-\omega^2\sin^2(\theta_s)\,\theta'+\omega^2\cos^2(\theta_s)\,\theta'$$

$$\ddot{\theta}'=\cancel{-\omega^2\cos^2(\theta_s)\,\theta'}-\omega^2\sin^2(\theta_0)\,\theta'\cancel{+\omega^2\cos^2(\theta_0)\,\theta'}$$

$$\ddot{\theta}'=-\omega^2\sin^2(\theta_0)\,\theta'$$

STABLE!



Supercritical fork bifurcation

Agenda

1. The gravitational pendulum : a poor watch

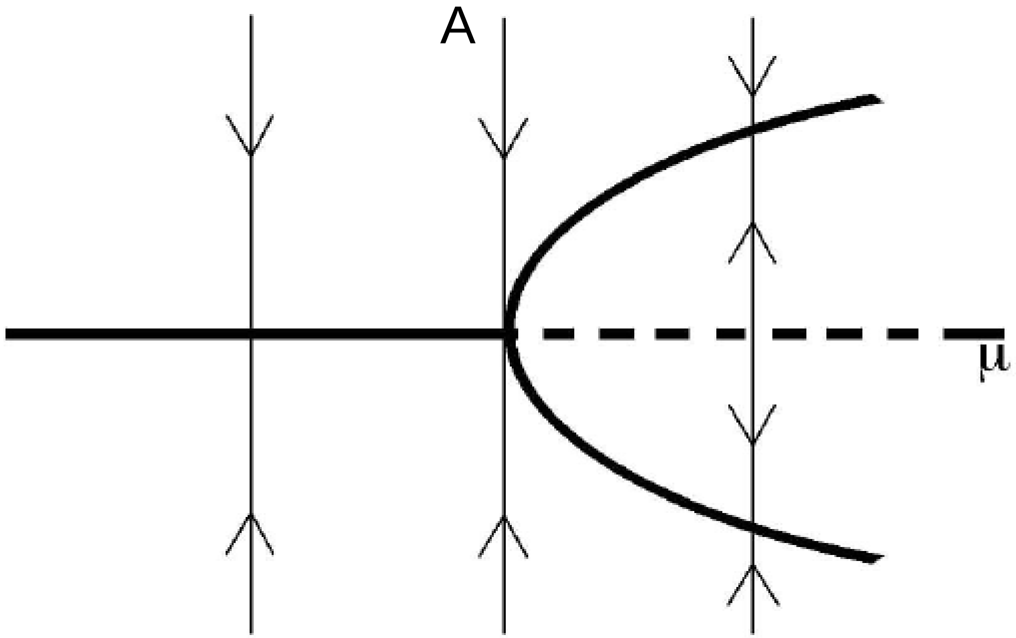
   First use of Multiple scale weakly nonlinear approach

2. A simple example of bifurcation

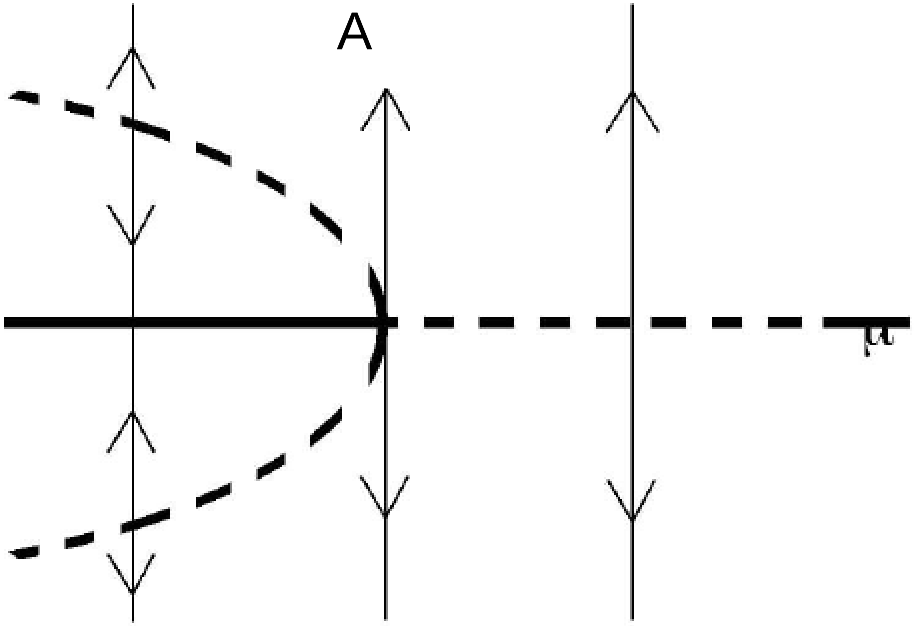
   First use of Multiple scale weakly nonlinear approach

3. Classical bifurcations

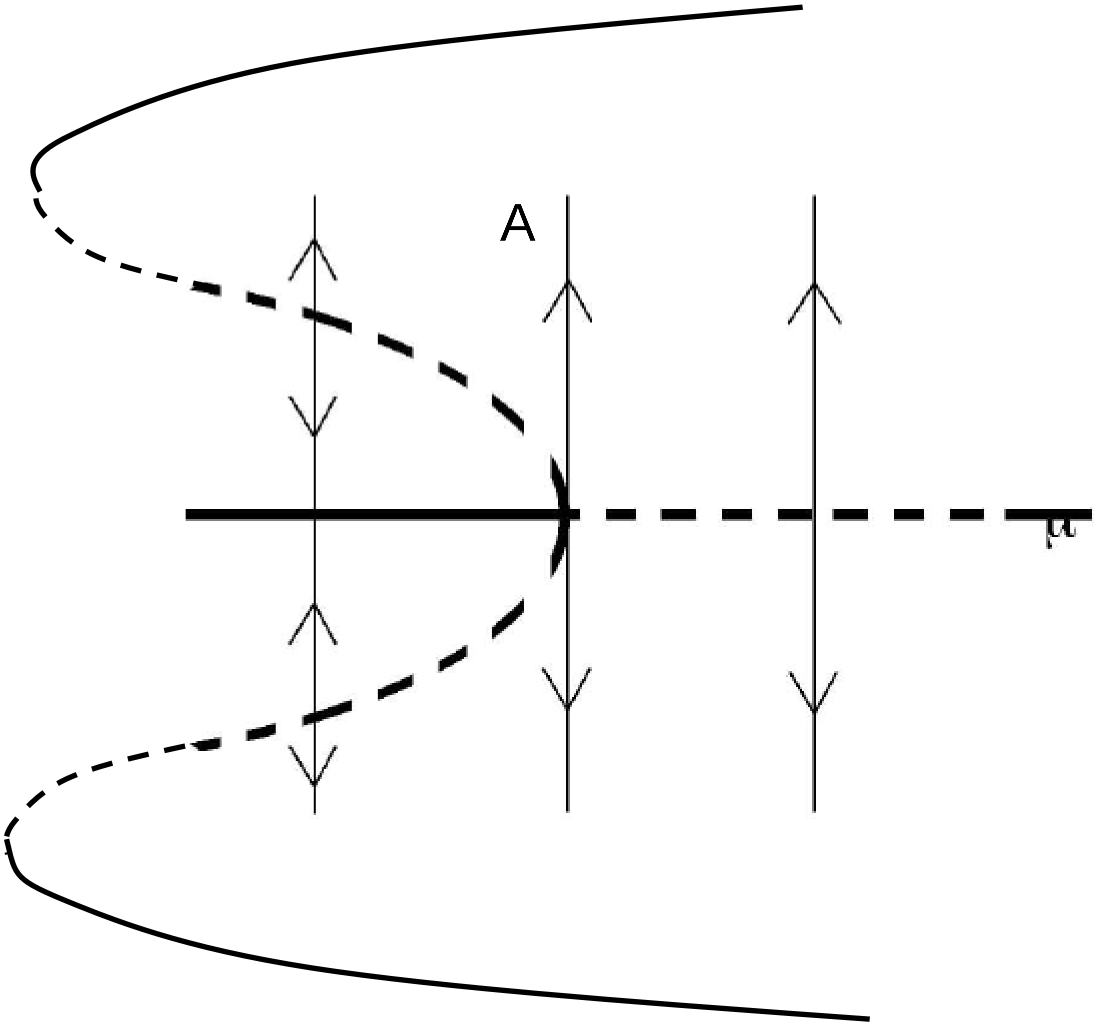
4. Hopf bifurcation and Stuart-Landau equation



Supercritical fork bifurcation

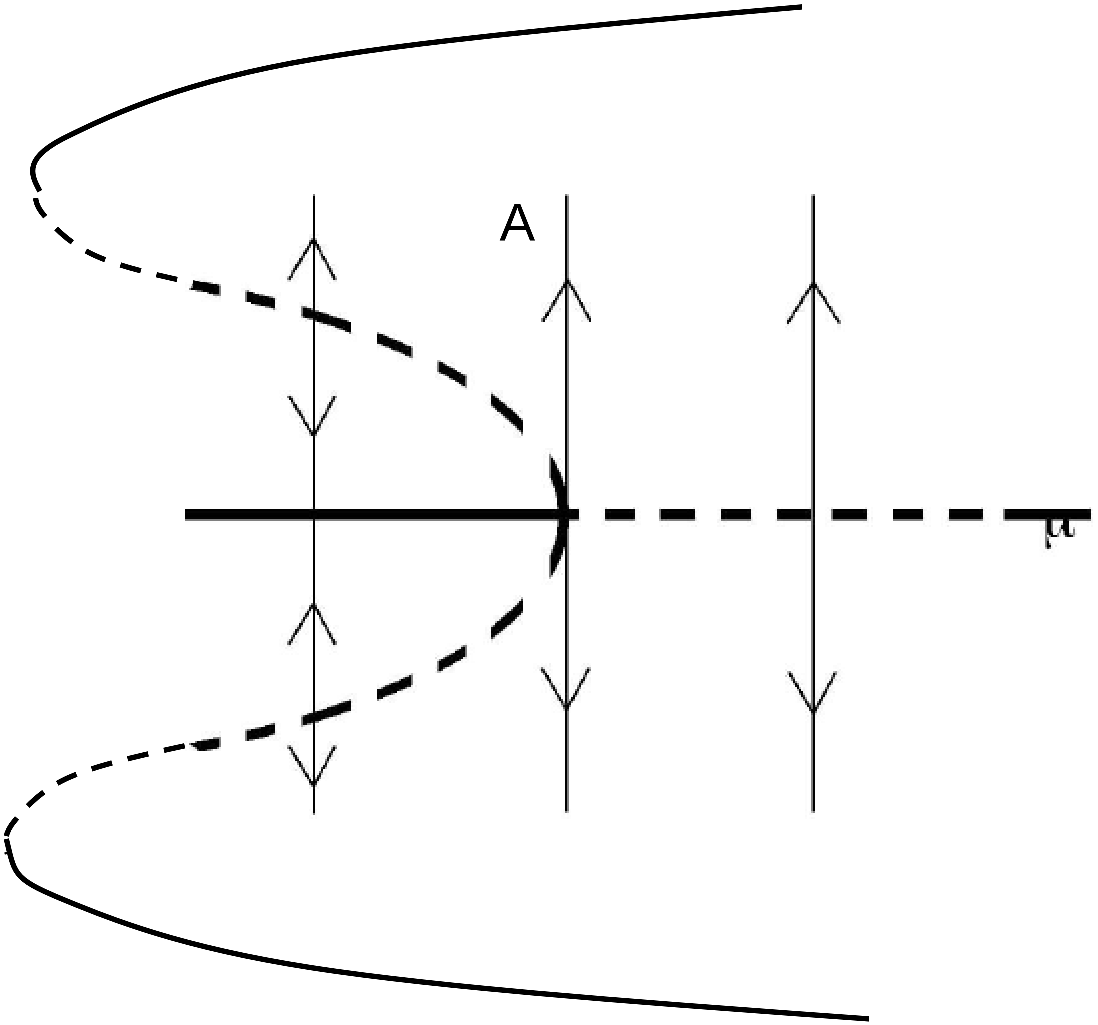


Subcritical fork bifurcation

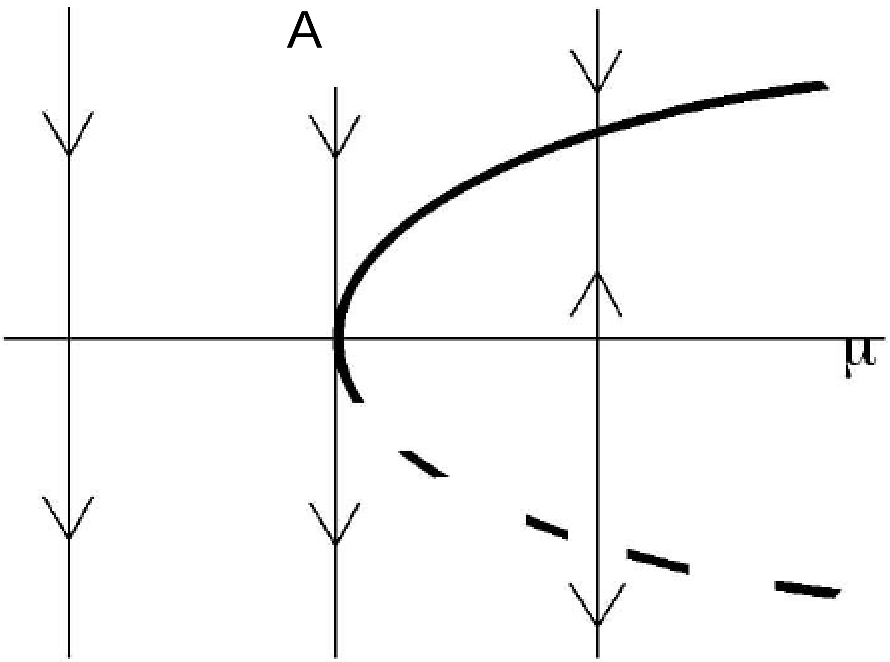


Subcritical bifurcation

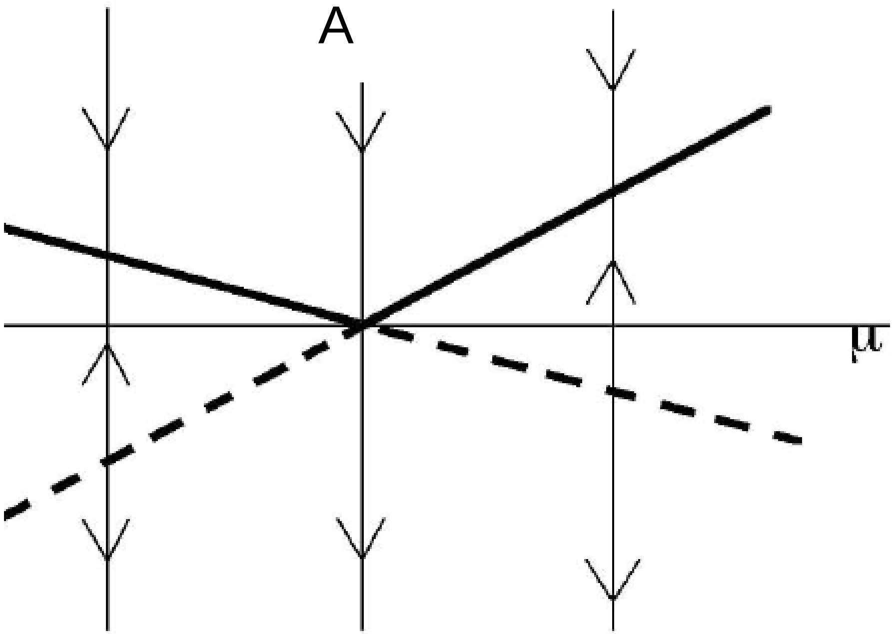
What about nonlinearities?



Hysteresis cycle!

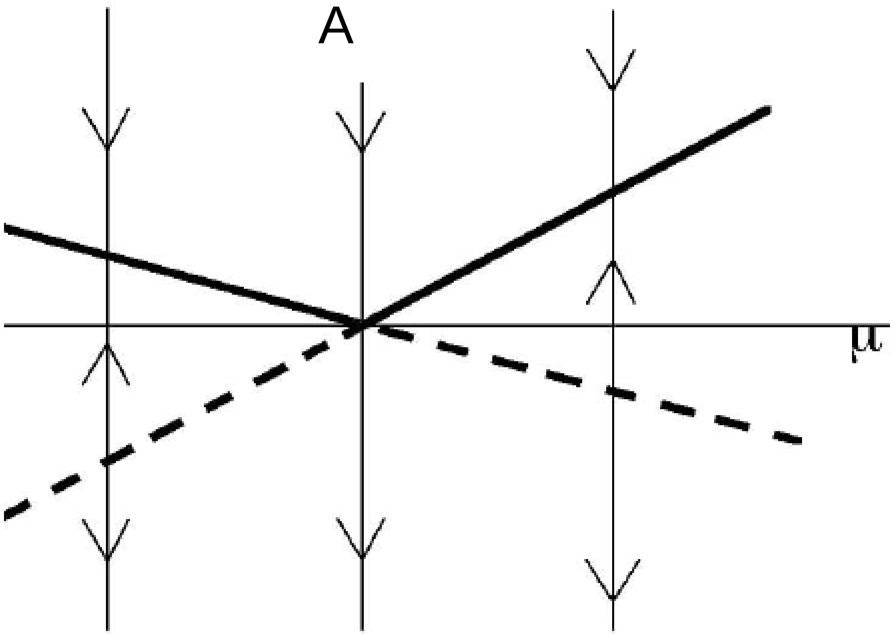


Saddle Node bifurcation

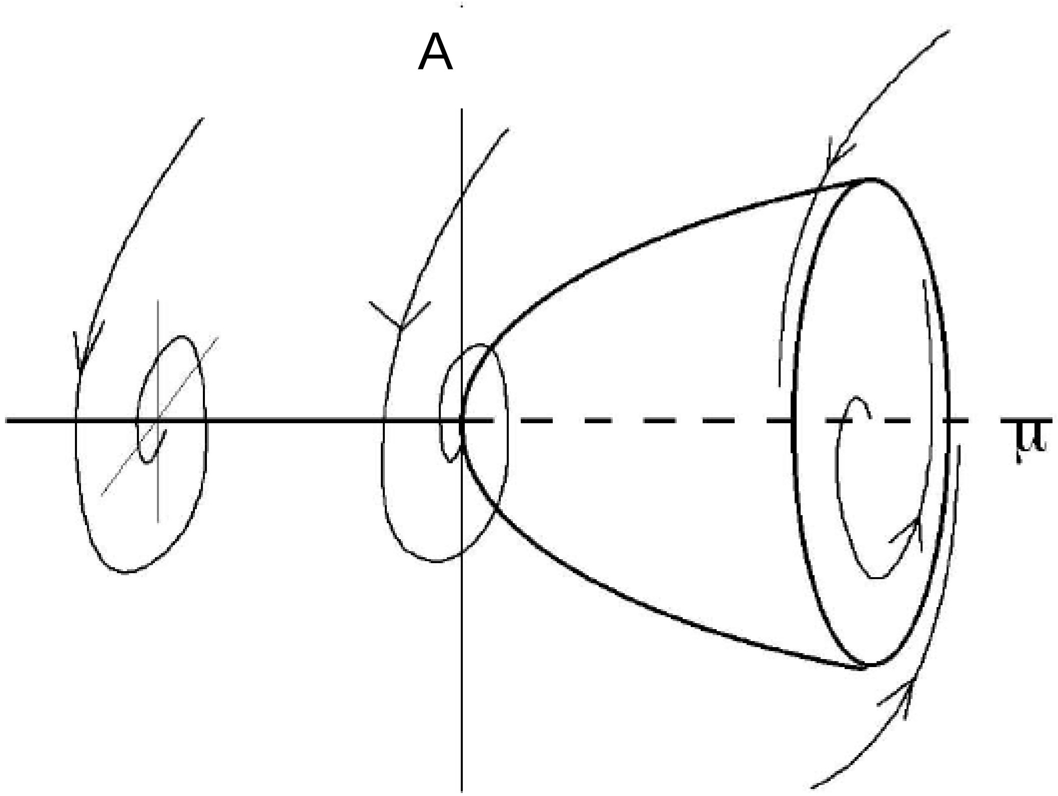


Transcritical bifurcation

What about nonlinearities?



Transcritical bifurcation



Hopf bifurcation

Agenda

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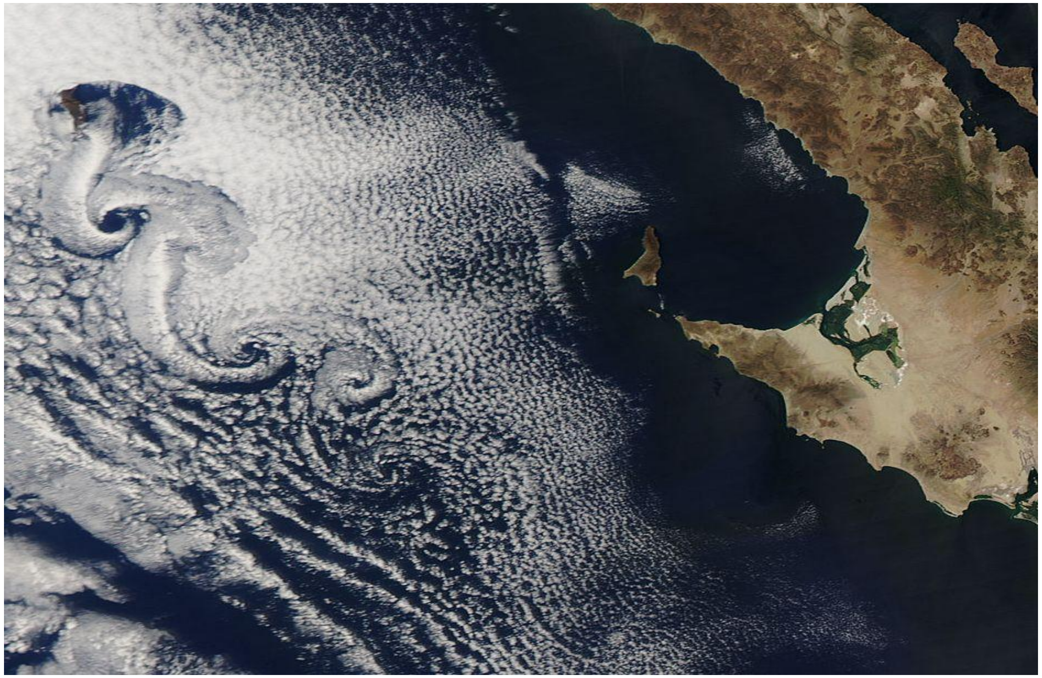
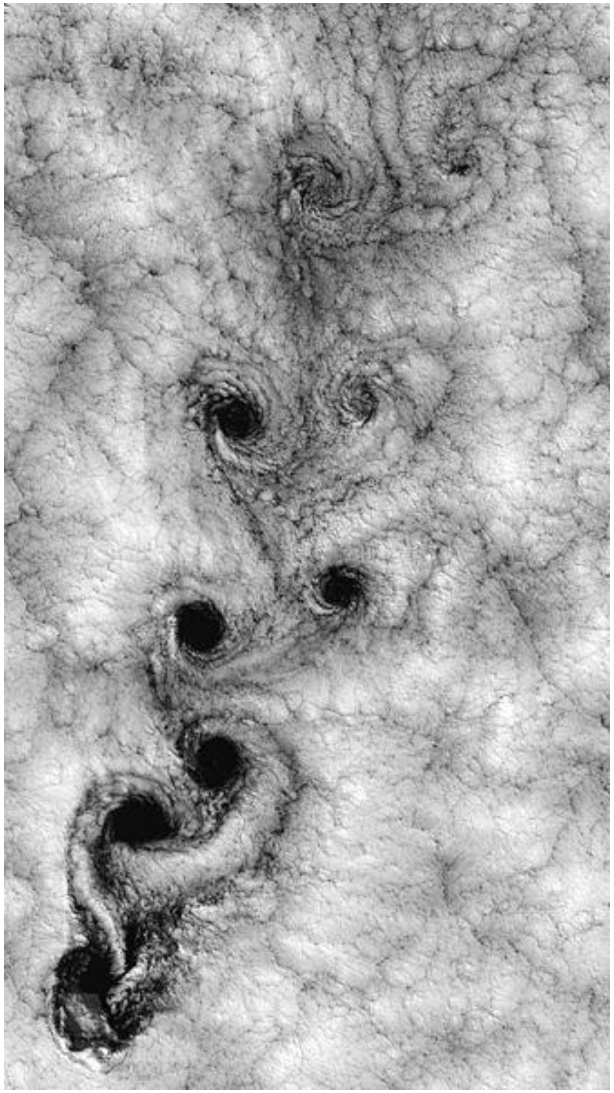
Natural oscillators



http://envsci.rutgers.edu/~lintner/teaching.html



en.wikipedia.org/wiki/File:Vortex-street-1.jpg

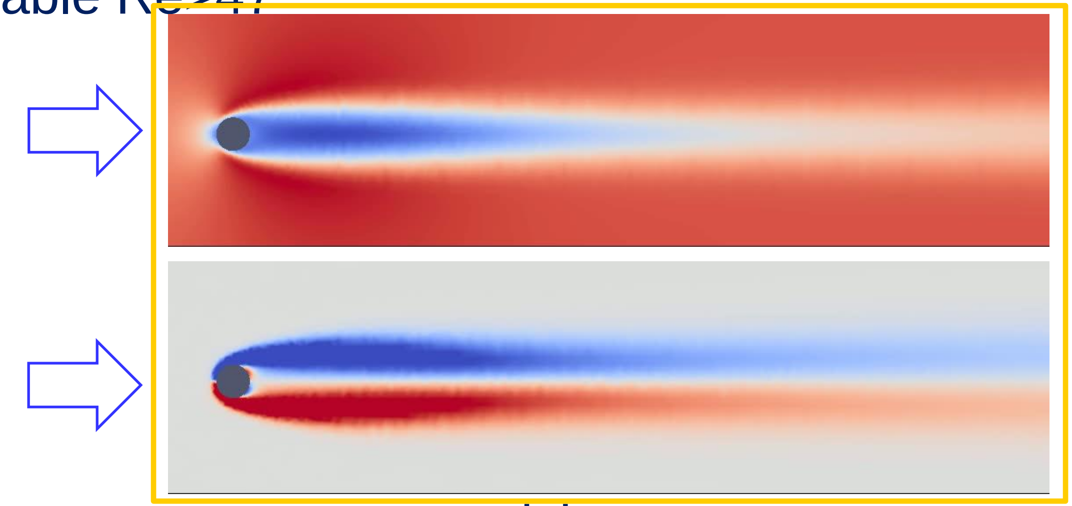
Cylinder wake

Cylinder wake

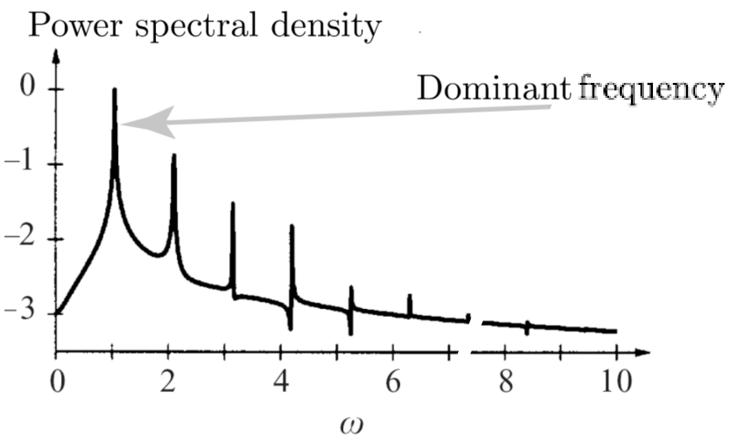
Oscillator, intrinsic dynamics, absolutely unstable (Triantafyllou 86, Monkewitz 88)

Globally unstable Re>47

axial velocity

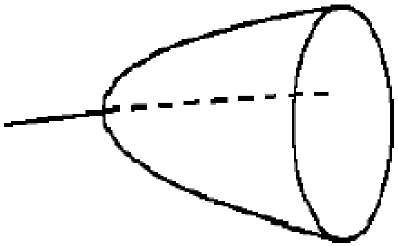


vorticity

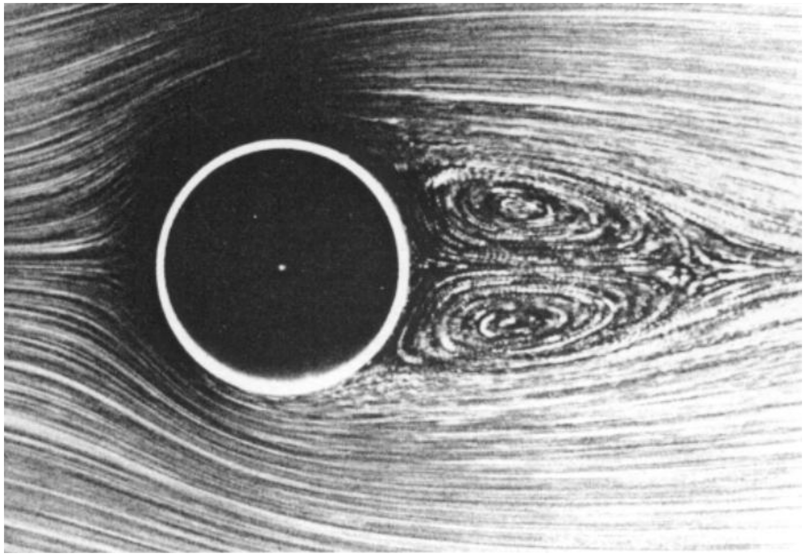


Canonical example of Hopf bifurcation
Bénard-von Karman street

Supercritical Hopf Bifurcation



$Re = 26 < Re_c$

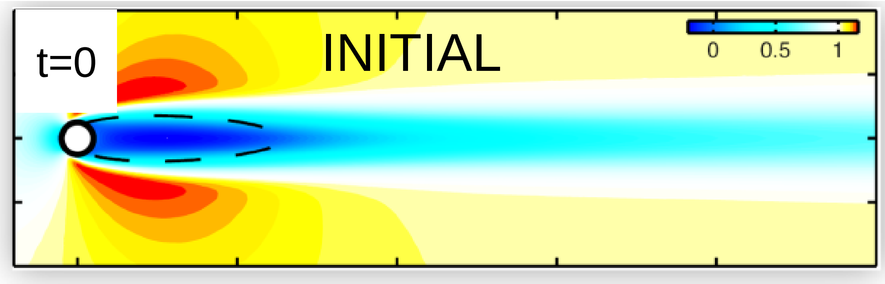


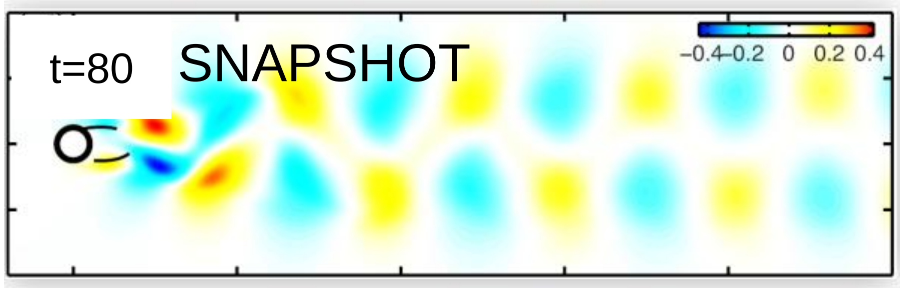
$Re = 140 > Re_c$

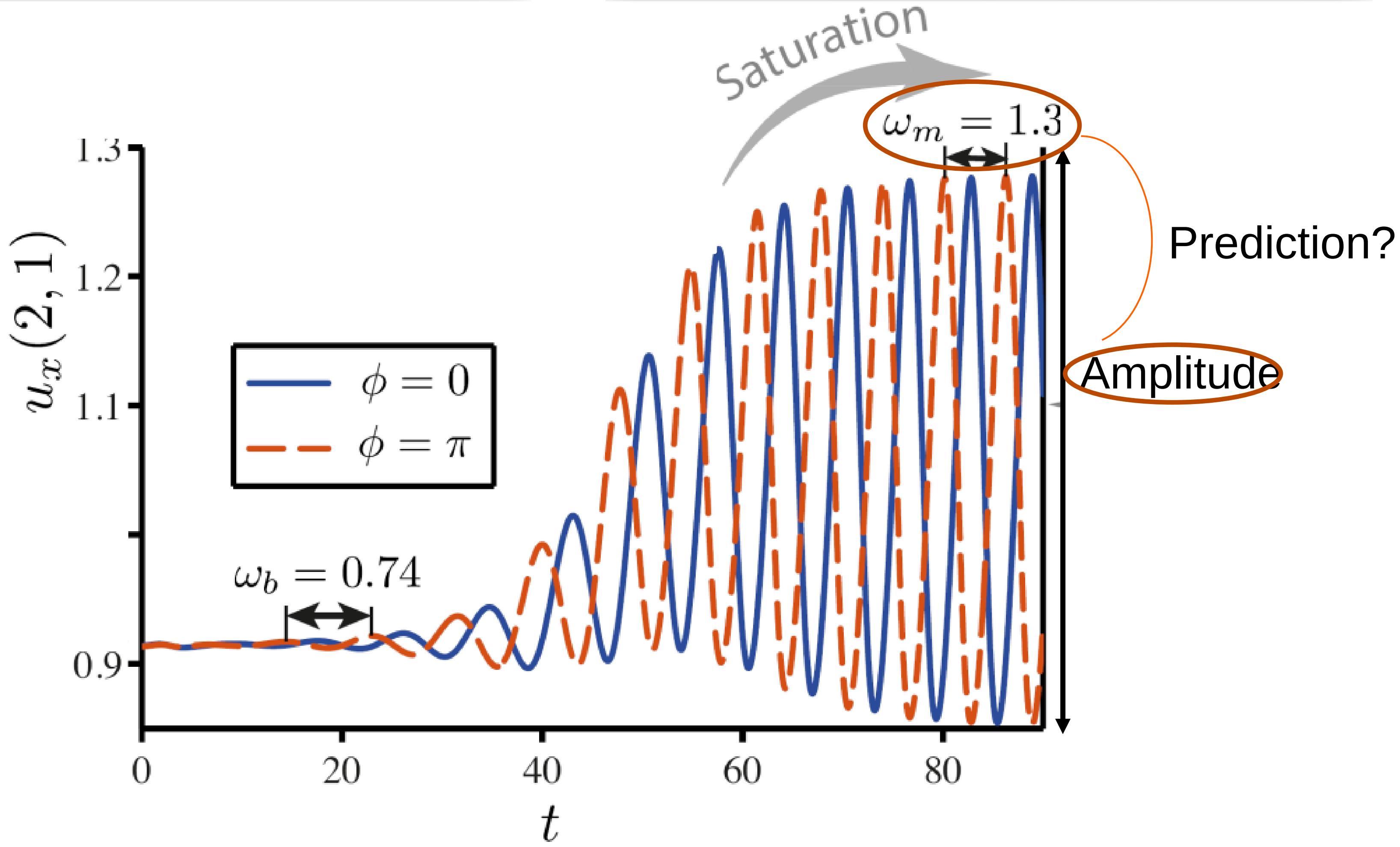
$Re_c \approx 47$

Threshold

Nonlinear Saturation





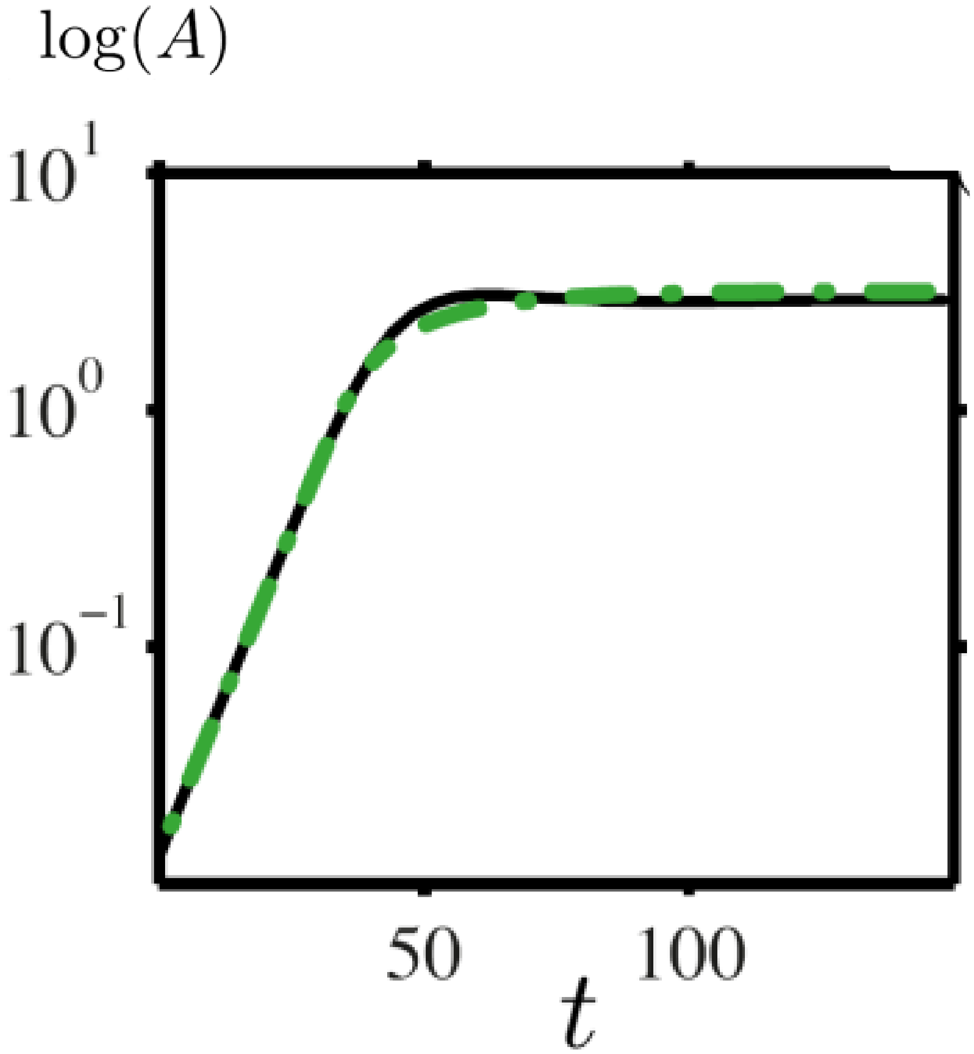


Saturation

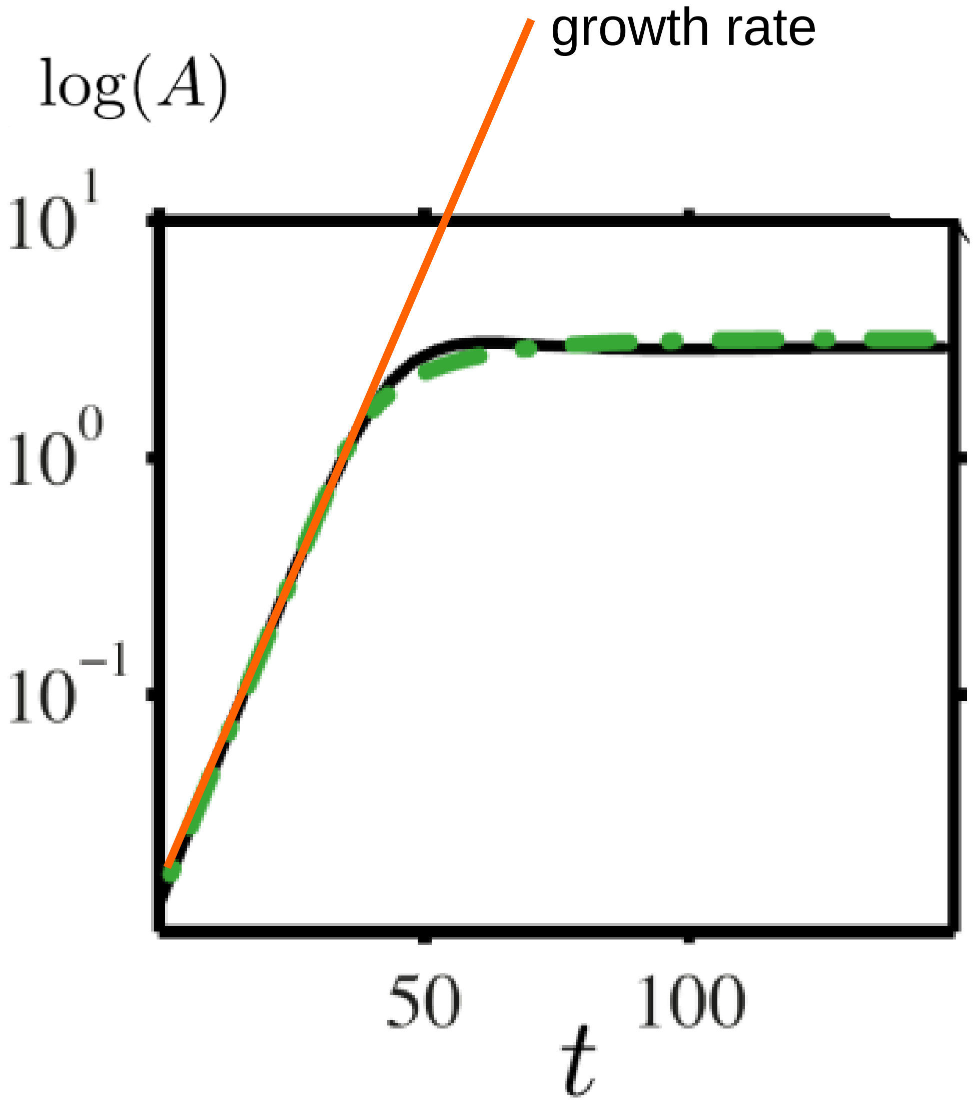
Prediction?

Amplitude

Saturation



Saturation…preceded by exponential growth



Linear stability analysis

Perturbation expansion

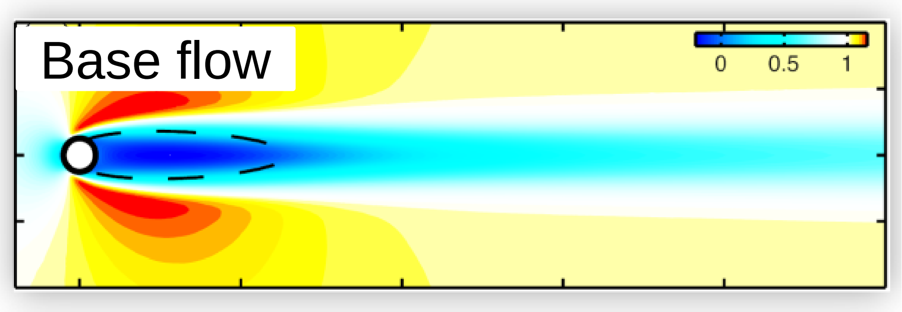
$$(\boldsymbol{u}, p) = \boxed{(\boldsymbol{U}, P)} + \boxed{(\boldsymbol{u}', p')}$$

Stationary base flow Perturbations

Base flow equations

$$\nabla \boldsymbol{U} \cdot \boldsymbol{U} = -\nabla P + Re^{-1} \nabla^2 \boldsymbol{U},$$

$$\nabla \cdot \boldsymbol{U} = 0$$



Linearized perturbation equations

$$\partial_t \boldsymbol{u}' + \nabla \boldsymbol{U} \cdot \boldsymbol{u}' + \nabla \boldsymbol{u}' \cdot \boldsymbol{U} + \xcancel{\nabla \boldsymbol{u}' \cdot \boldsymbol{u}'} = -\nabla p' + Re^{-1} \nabla^2 \boldsymbol{u}',$$

$$\nabla \cdot \boldsymbol{u}' = 0$$

Global stability analysis

$$(\boldsymbol{u}^{'}, p^{'})(x,y,t) = (\hat{\boldsymbol{u}}, \hat{p})(x,y)\exp[\sigma t]$$

Global mode

$$\sigma = \lambda + i\omega \qquad St = \frac{\omega}{2\pi}$$

Growth-rate

frequency

Singular generalized eigenvalue problem

$$\sigma\hat{\boldsymbol{u}} + \boldsymbol{\nabla}\hat{\boldsymbol{u}}\cdot\boldsymbol{U} + \boldsymbol{\nabla}\boldsymbol{U}\cdot\hat{\boldsymbol{u}} = -\boldsymbol{\nabla}\hat{p} + Re^{-1}\boldsymbol{\nabla}^{2}\hat{\boldsymbol{u}},$$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}} = 0,$$

Zebib, Jackson (1987)

Global stability analysis solvers

For a given value of Re, numerically solve

- Non linear equations,

$$\boldsymbol{\nabla} \boldsymbol{U} \cdot \boldsymbol{U} = -\boldsymbol{\nabla} P + Re^{-1} \boldsymbol{\nabla}^2 \boldsymbol{U},$$

(Newton method)

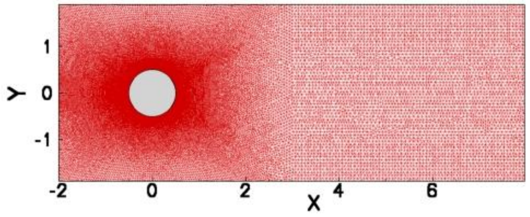
- Eigenvalue problem

$$\sigma \hat{\boldsymbol{u}} + \boldsymbol{\nabla} \hat{\boldsymbol{u}} \cdot \boldsymbol{U} + \boldsymbol{\nabla} \boldsymbol{U} \cdot \hat{\boldsymbol{u}} = -\boldsymbol{\nabla} \hat{p} + Re^{-1} \boldsymbol{\nabla}^2 \hat{\boldsymbol{u}},$$

(Krylov-Arnoldi method)

Spatial discretization = finite element methods

(FreeFem++ freeware)



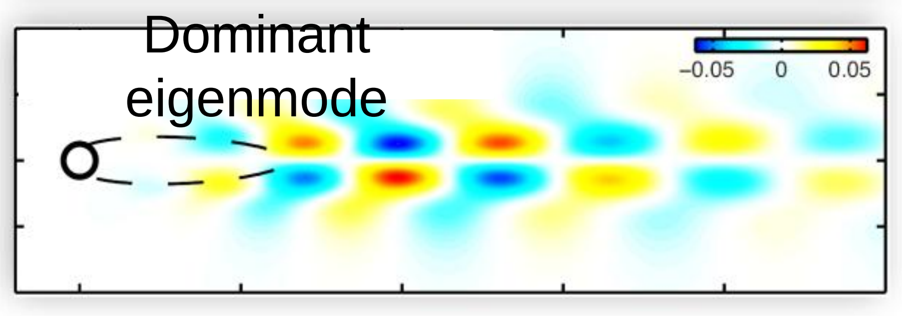
Taylor-Hood finite elements (P2,P2,P1)

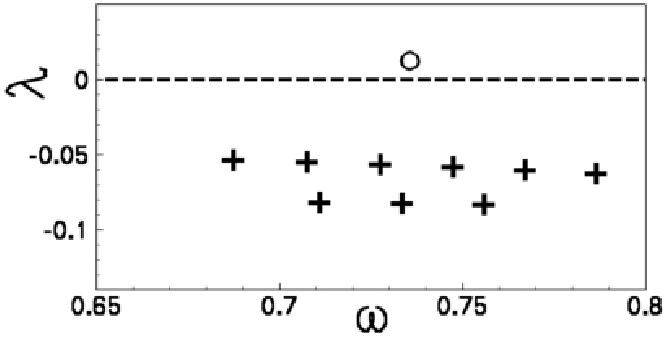
→ number of degrees of freedom~ O(10^6)

Dominant eigenvalue

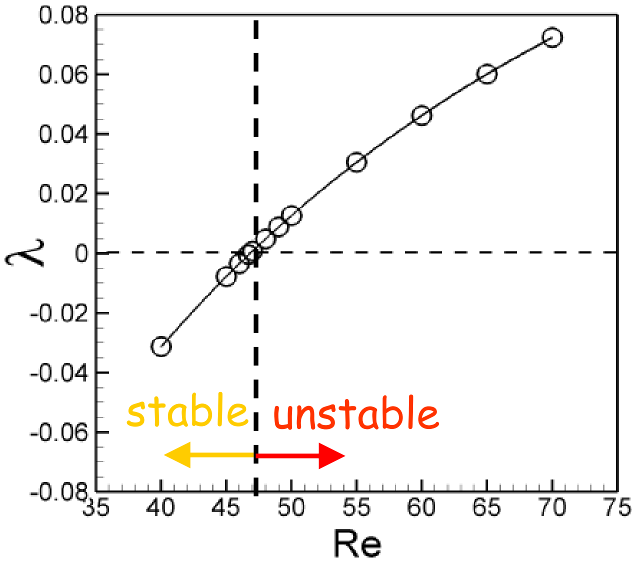
$(\hat{\boldsymbol{u}}, \hat{p}) \exp[\sigma t] \qquad \sigma = \lambda + i\omega$

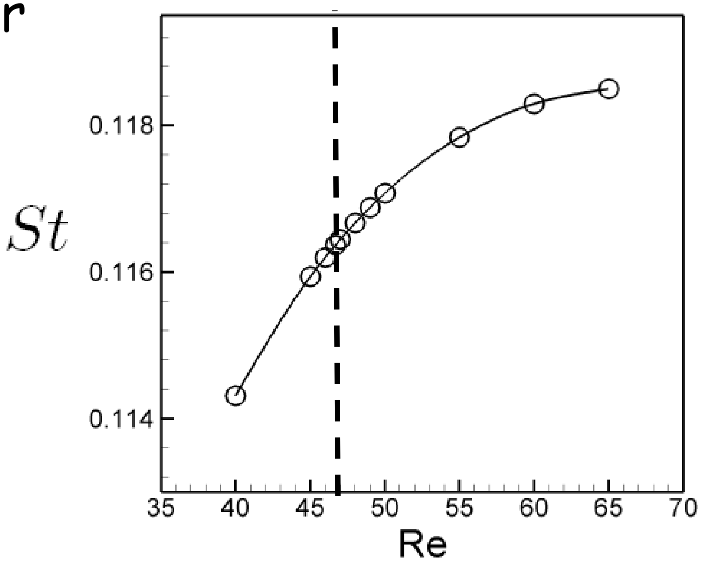
Spectrum at Re = 50


Dominant eigenmode



Evolution as a function of the Reynolds number



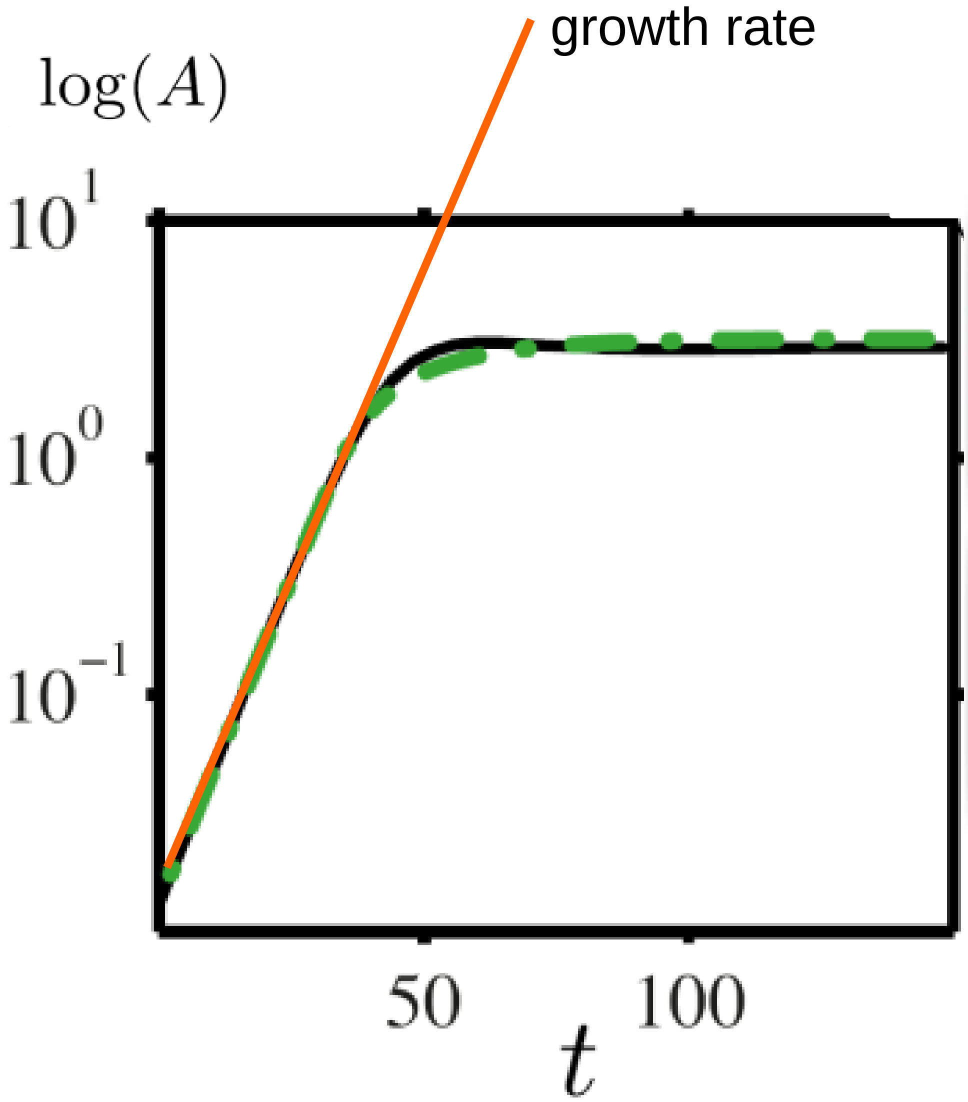


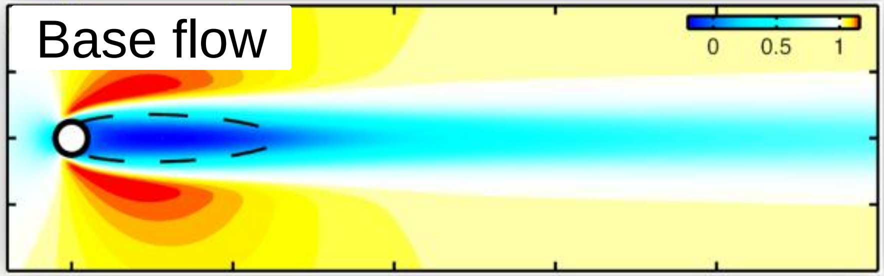
$Re_c \sim 47$

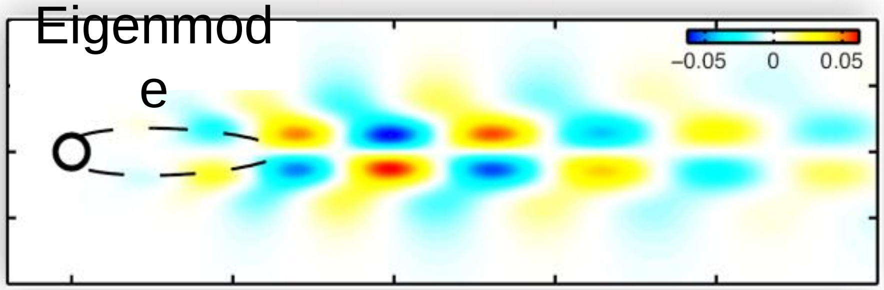
$St_c \sim 0.11$

Jackson (1987), Zebib (1987), Ding & Kawahara (1999), Barkley (2006), Giannetti & Luchini (2003, 2007), Sipp & Lebedev (2007), Marquet, Sipp & Jacquin (2009)…

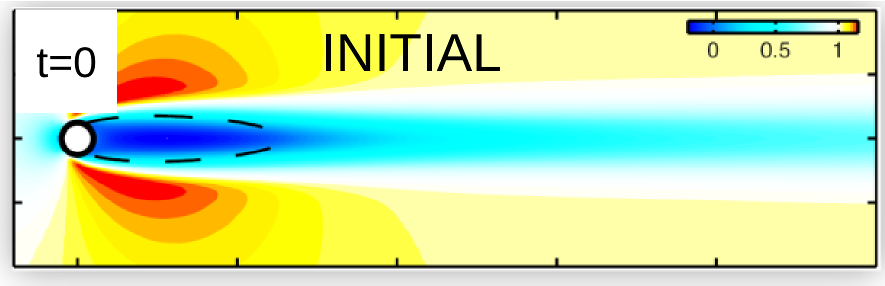
Saturation…preceded by exponential growth

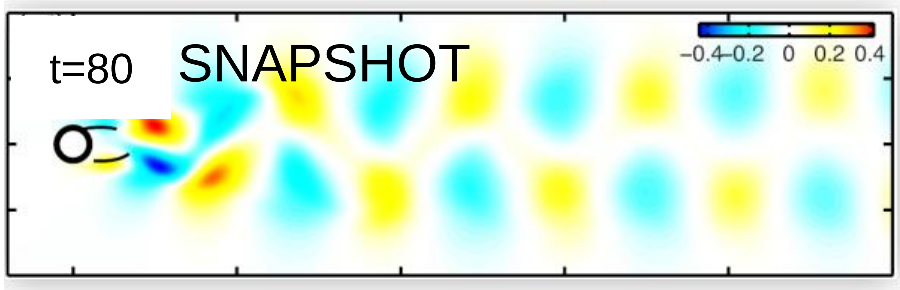


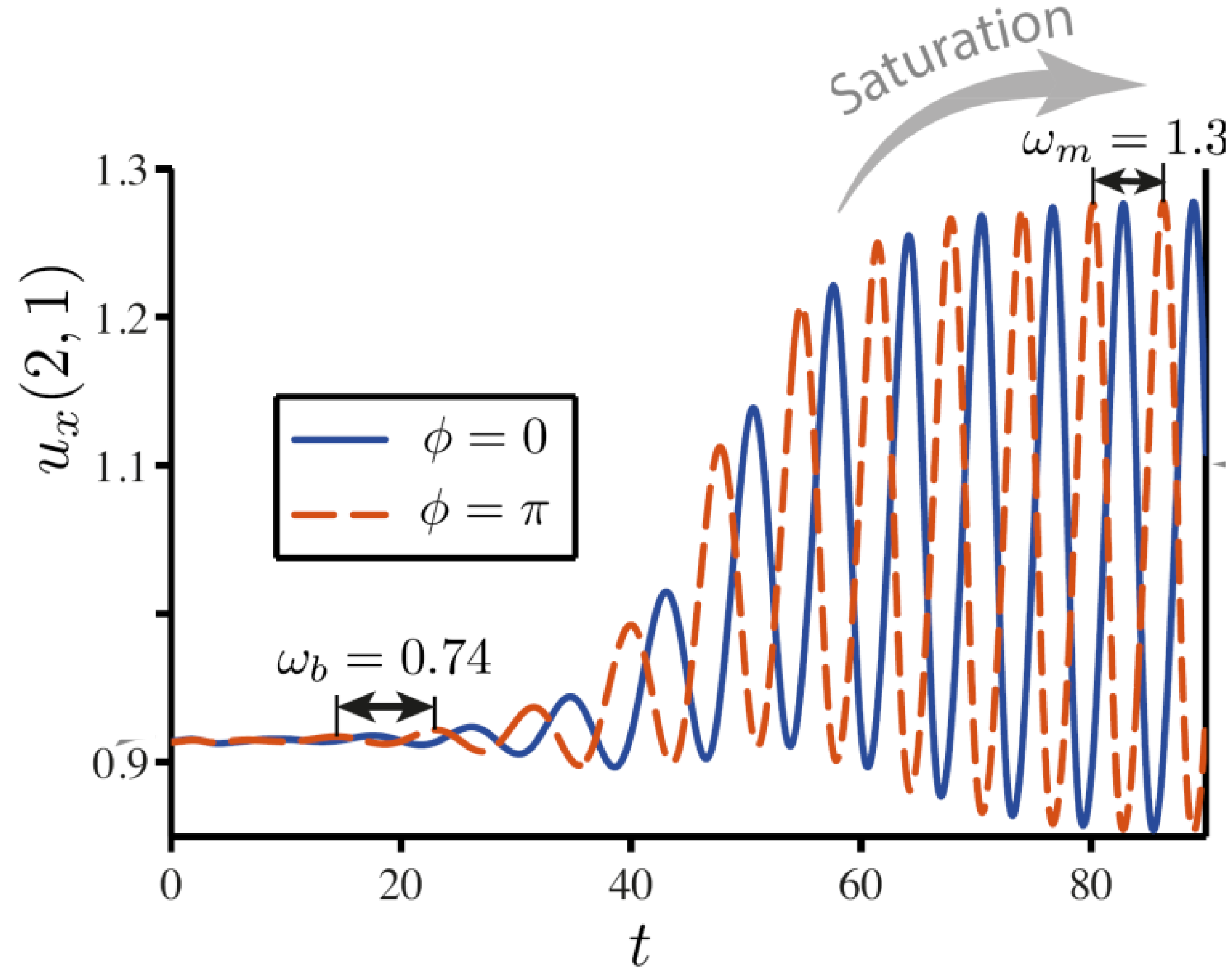




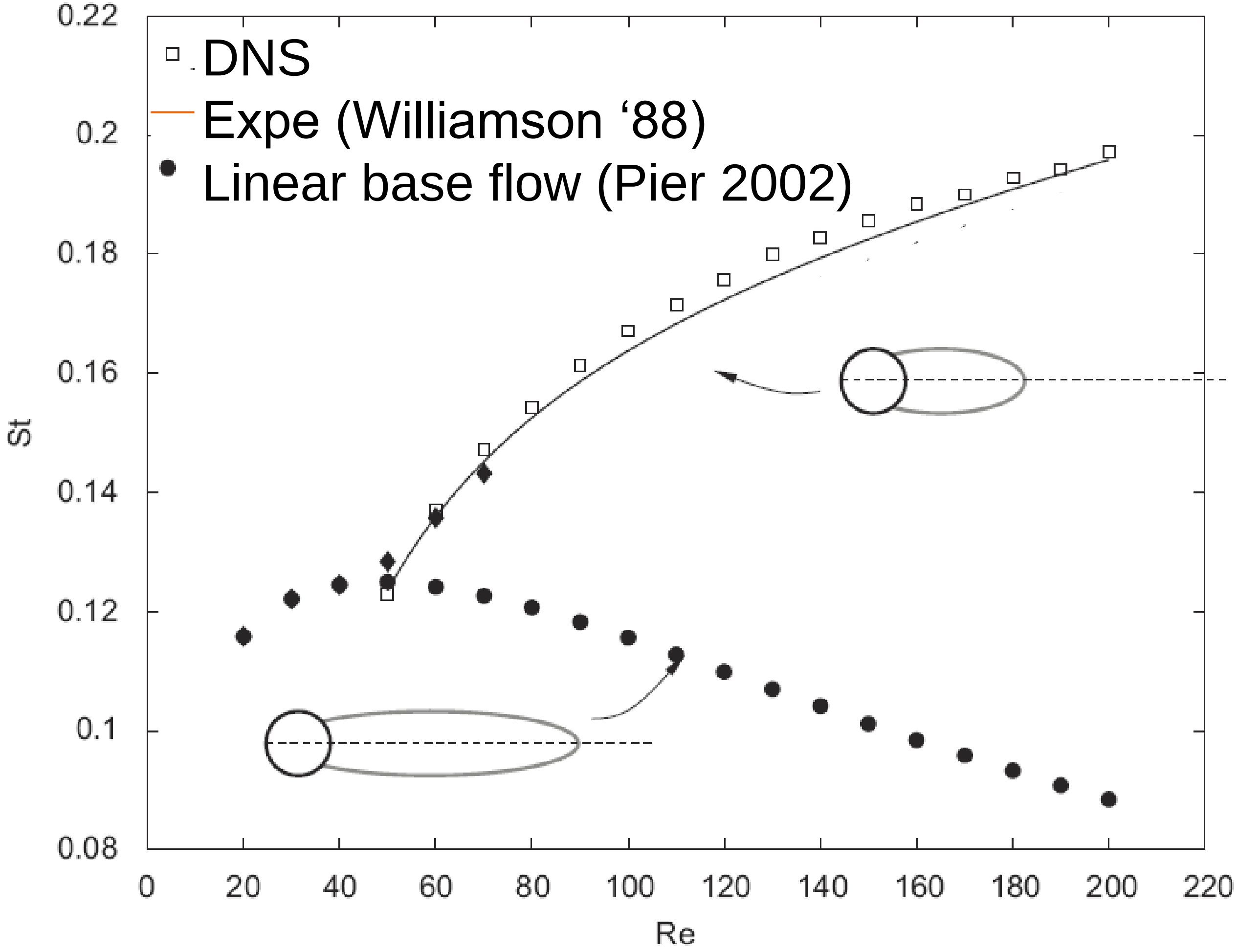
Frequency correction



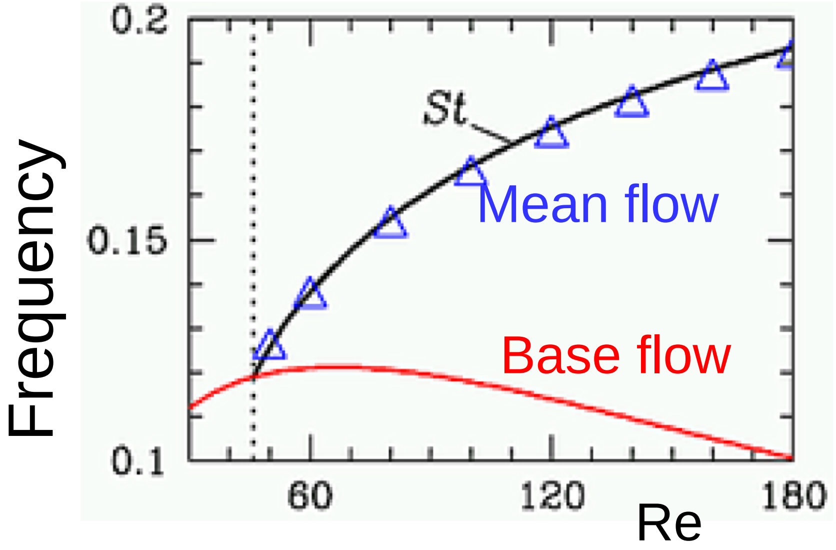


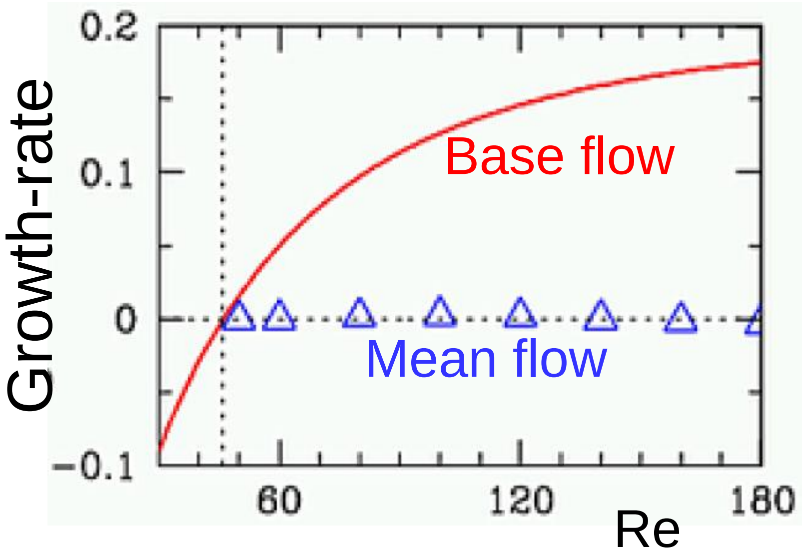


Frequency correction



The mean flow is neutrally (marginally) stable





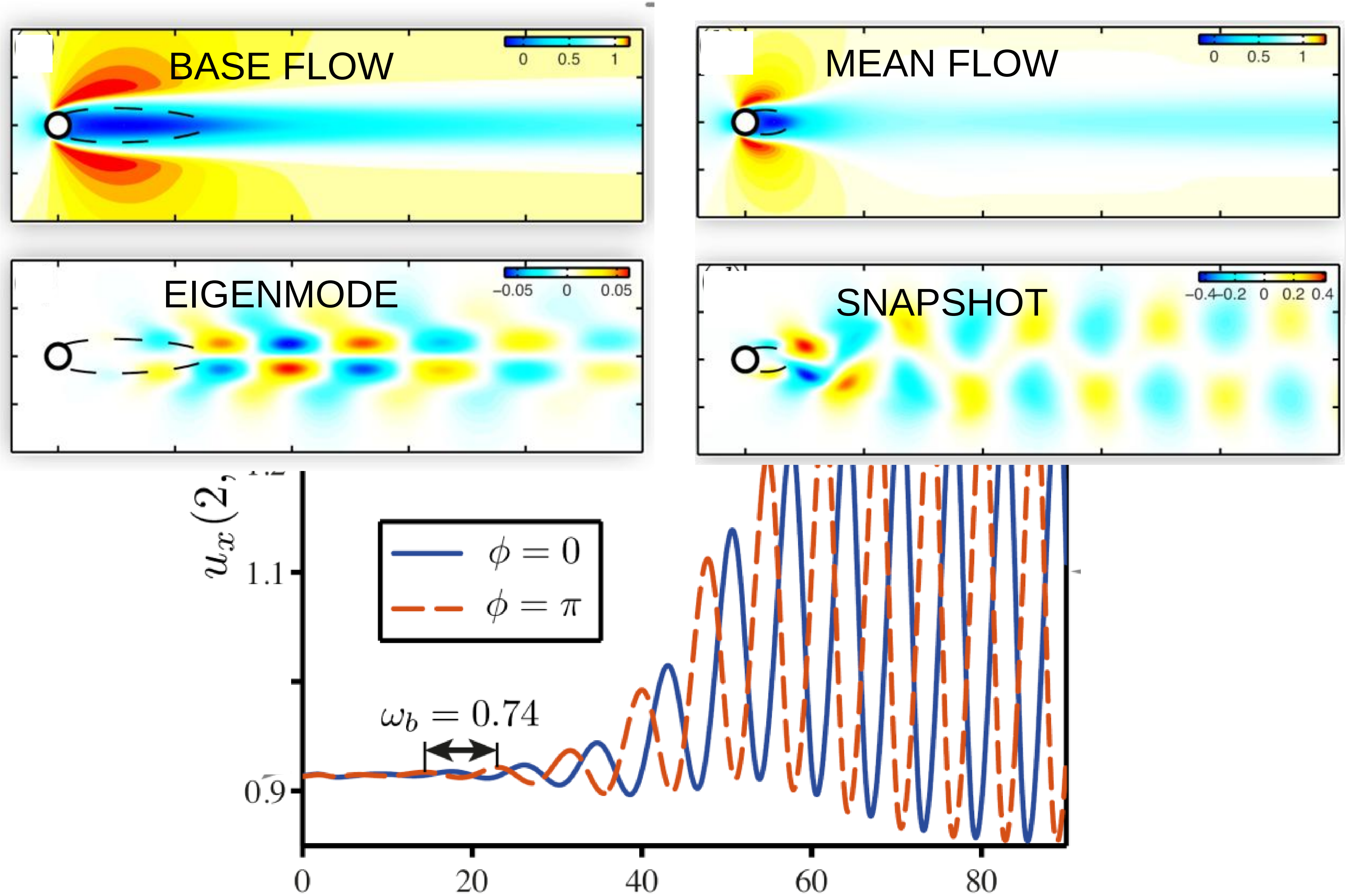
Barkley (2006), Malkus (1956)

Two limitations:

A posteriori prediction: need mean flow

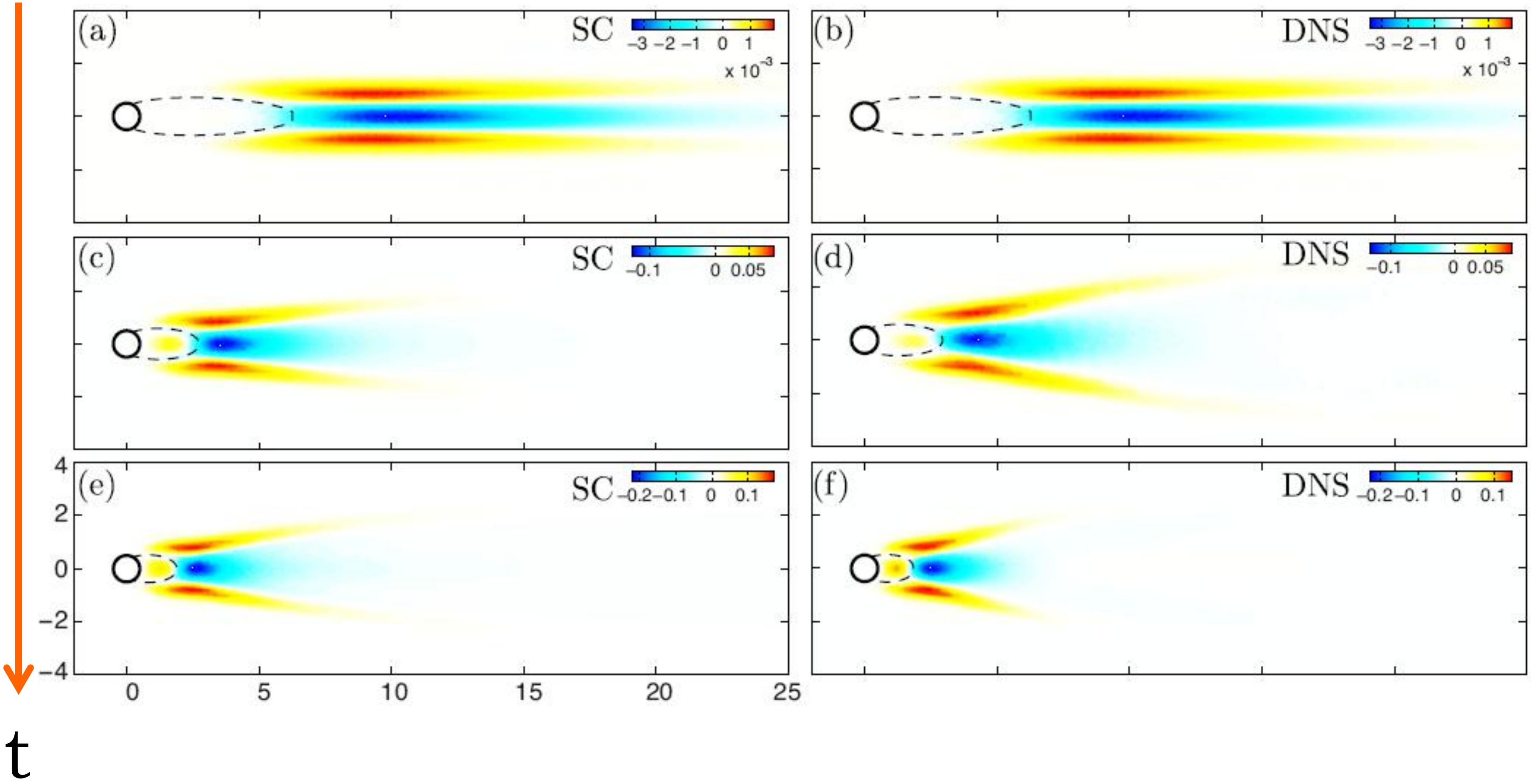
No information on amplitude

Mean flow distortion



Stuart 1958, Maurel, Pagneux and Wesfreid 1995 , Barkley 2006

Transient mean flow correction



Stuart-Landau amplitude equation

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}T} = \lambda\delta A - \mu A|A|^2\,,}$$

δ : distance from threshold

Experimental

Sreenivasan, Strykowsky & Olinger (1986)

Provansal, Mathis & Boyer (1987)

Numeric

Dusek, Le Gal & Fraunié (1994)

Analytic

Sipp & Lebedev (2007)

Bifurcation theory

Stuart (1960) , Sipp & Lebedev (2007)

Departure from threshold:

$$\frac{1}{Re} - \frac{1}{Re_*} = O(\epsilon^2) \equiv \epsilon^2 \delta$$

$$slow \text{ time scale } T = \epsilon^2 t$$

Expansion:

$$\boldsymbol{q} = \boldsymbol{q}_0 + \epsilon \boldsymbol{q}_1 + \epsilon^2 \boldsymbol{q}_2 + \epsilon^3 \boldsymbol{q}_3 + \ldots .$$

q_0 base flow,

q_1 leading order perturbation $\boldsymbol{q}_1 = A(T)\hat{\boldsymbol{q}}_{1A} e^{\mathrm{i}\omega_* t} + \text{ c.c.}.$

A(T) unknown

q_2 second order perturbation, no secular terms with frequency ω_*

$$\boldsymbol{q}_2 = \delta \hat{\boldsymbol{q}}_{2\delta} + |A|^2 \hat{\boldsymbol{q}}_{2|A|^2} + \left(A^2 \hat{\boldsymbol{q}}_{2A^2} e^{2\mathrm{i}\omega_* t} + \text{c.c.}\right)$$

BF diffusion

Base flow modifications

harmonics

Bifurcation theory

q_2 second order perturbation, no secular terms with frequency ω_*

$$\boldsymbol{q}_2 = \delta \hat{\boldsymbol{q}}_{2\delta} + |A|^2 \hat{\boldsymbol{q}}_{2|A|^2} + \left(A^2 \hat{\boldsymbol{q}}_{2A^2} e^{2\mathrm{i}\omega_* t} + \mathrm{c.c.}\right)$$

BF diffusion

Base flow modifications

harmonics

$$(\partial_t \mathscr{L} + \mathscr{M})\, \boldsymbol{q}_2 = \boldsymbol{F}_2^1 + |A|^2 \boldsymbol{F}_2^{|A|^2} + (A^2 e^{2\mathrm{i}\omega_0 t} \boldsymbol{F}_2^{A^2} + \mathrm{c.c.})$$

$$\boldsymbol{F}_2^1 = \begin{pmatrix} -\Delta \boldsymbol{u}_0 \\ 0 \end{pmatrix},$$

$$\boldsymbol{F}_2^{|A|^2} = \begin{pmatrix} -\nabla \boldsymbol{u}_1^A \cdot \overline{\boldsymbol{u}_1^A} - \nabla \overline{\boldsymbol{u}_1^A} \cdot \boldsymbol{u}_1^A \\ 0 \end{pmatrix},$$

$$\boldsymbol{F}_2^{A^2} = \begin{pmatrix} -\nabla \boldsymbol{u}_1^A \cdot \nabla \boldsymbol{u}_1^A \\ 0 \end{pmatrix}.$$

Resonance at third order

q_3 Third order **secular** **(resonant)** **forcing terms**

$$\mathcal{B}\partial_t \boldsymbol{q}_3 + \mathrm{L}_* \boldsymbol{q}_3 = (\hat{\boldsymbol{F}}_{3r} \boxed{e^{\mathrm{i}\omega_* t}} + \boldsymbol{F}_{3nr} + \text{c.c.}, 0)^T .$$

$$\hat{\boldsymbol{F}}_{3r} = -\frac{\mathrm{d}A}{\mathrm{d}T}\hat{\boldsymbol{u}}_{1A} + \delta A \hat{\boldsymbol{F}}_{3A} + A|A|^2 \hat{\boldsymbol{F}}_{3A|A|^2} ,$$

$$\boldsymbol{F}_3^A = \begin{pmatrix} -\nabla \boldsymbol{u}_1^A \cdot \nabla \boldsymbol{u}_2^1 - \boldsymbol{u}_2^1 \cdot \nabla \boldsymbol{u}_1^A - \Delta \boldsymbol{u}_1^A \\ 0 \end{pmatrix} ,$$

$$\boldsymbol{F}_3^{A|A|^2} = \begin{pmatrix} -\nabla \boldsymbol{u}_1^A \cdot \nabla \boldsymbol{u}_2^{|A|^2} - \boldsymbol{u}_2^{|A|^2} \cdot \nabla \boldsymbol{u}_1^A \\ 0 \end{pmatrix} ,$$

$$\boldsymbol{F}_3^{\bar{A}A^2} = \begin{pmatrix} -\nabla \overline{\boldsymbol{u}_1^A} \cdot \nabla \boldsymbol{u}_2^{A^2} - \boldsymbol{u}_2^{A^2} \cdot \nabla \overline{\boldsymbol{u}_1^A} \\ 0 \end{pmatrix} ,$$

Resonance at third order

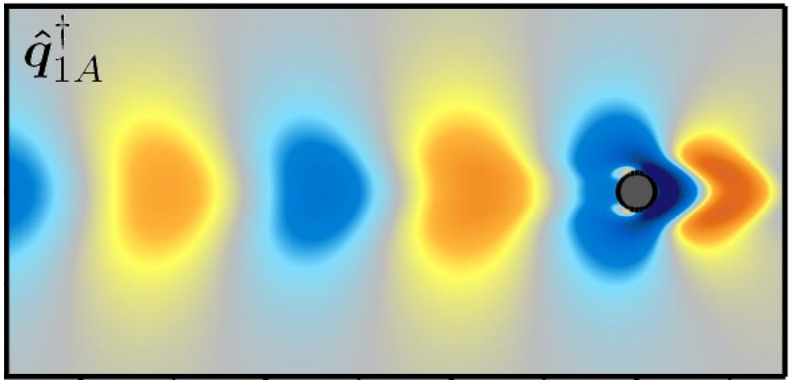
q_3 Third order **secular** **(resonant)** **forcing terms**

$$\mathcal{B}\partial_t \boldsymbol{q}_3 + \mathrm{L}_* \boldsymbol{q}_3 = (\hat{\boldsymbol{F}}_{3r} e^{\mathrm{i}\omega_* t} + \boldsymbol{F}_{3nr} + \mathrm{c.c.}, 0)^T .$$

$$\hat{\boldsymbol{F}}_{3r} = -\frac{\mathrm{d}A}{\mathrm{d}T}\hat{\boldsymbol{u}}_{1A} + \delta A \hat{\boldsymbol{F}}_{3A} + A|A|^2 \hat{\boldsymbol{F}}_{3A|A|^2} ,$$

⇨ The Fredholm alternative $\hat{\boldsymbol{F}}_{3r}$ orthogonal to the adjoint of $\hat{\boldsymbol{q}}_{1A}$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}^\dagger = 0 , \qquad \partial_t \hat{\boldsymbol{u}}^\dagger + \boldsymbol{\nabla}\boldsymbol{U}^{\mathrm{T}}\cdot\hat{\boldsymbol{u}}^\dagger - \boldsymbol{\nabla}\hat{\boldsymbol{u}}^\dagger\cdot\boldsymbol{U} + \boldsymbol{\nabla}\hat{p}^\dagger - \mathrm{Re}^{-1}\boldsymbol{\nabla}^2\hat{\boldsymbol{u}}^\dagger = \boldsymbol{0} ,$$



Giannetti & Luchini (2003), Sipp & Lebedev (2007), Marquet, Sipp & Jacquin (2009)…

Compatibility condition yields closure and the normal form

- $A\varepsilon$ leading order determined by resonant terms at ε^3

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}T} = \lambda\delta A - \mu A|A|^2 \,,}$$

$$\lambda = \int_\Sigma \hat{\boldsymbol{q}}_{1A}^\dagger \cdot \hat{\boldsymbol{F}}_{3A} \mathrm{d}x\mathrm{d}y \,,$$

$$\mu = \int_\Sigma \hat{\boldsymbol{q}}_{1A}^\dagger \cdot \hat{\boldsymbol{F}}_{3A|A|^2} \mathrm{d}x\mathrm{d}y \,.$$

$$\int_\Sigma \hat{\boldsymbol{q}}_{1A}^\dagger \cdot \hat{\boldsymbol{q}}_{1A} \mathrm{d}x\mathrm{d}y = 1$$

Sipp & Lebedev (2007)

Normal form

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}T} = \lambda\delta A - \mu A|A|^2\,,}$$

$\mu_r > 0$ ⇨ predicts saturation $|A|^2 = \dfrac{\lambda_r \delta}{\mu_r}$

⇨ nonlinear frequency correction $\delta\omega = \lambda_i \delta - \mu_i |A|^2$

Sipp & Lebedev (2007)