Exercise on the notion of global instability : a flowing cylindrical thread held between two nozzles

We are considering a viscous liquid thread (viscosity μ, surface tension γ) injected at velocity U_0 by a nozzle of inner radius R_0 and collected at a distance L by a similar nozzle of same radius. We introduce the capillary number $Ca = \mu U_0/\gamma$ (check the dimensions) as rescaled velocity and rescale the time with $\tau = \mu R_0/\gamma$. We will assume the dimensionless equation governing the evolution of the radius perturbation $R = R_0\,(1+\epsilon h(z,t))$ is governed by

$$\frac{\partial h}{\partial t} = -Ca\frac{\partial h}{\partial z} + \frac{h}{6} + \frac{1}{6}\frac{\partial^2 h}{\partial z^2} \tag{1}$$

with boundary conditions

$$h(0,t) = h(l,t) = 0 \tag{2}$$

where $l = L/R_0$.

1. Ignoring for now the boundary conditions, use a normal mode expansion $h(z,t) = H\exp(i(kz-\omega t))$ to obtain the following dispersion relation

   $$\omega = f(k) = Ca\,k + i\frac{1}{6}(1-k^2) \tag{3}$$

2. Remembering that the transition from absolute to convective instability is attained when the imaginary part of the absolute frequency is zero, $\omega_{0,i} = 0$, where $\omega_0 = f(k_0)$ with k_0 such that

   $$\left.\frac{\partial f_r}{\partial k_r}\right|_{k_0} = \left.\frac{\partial f_i}{\partial k_r}\right|_{k_0} = 0, \tag{4}$$

   show that the transition from absolute to convective instability takes place at $Ca^* = 1/3$.

3. Show that this normal mode assumption does not satisfy the boundary conditions.

4. In order to correctly account for the boundary conditions, a more general expansion should be used

   $$h(z,t) = \eta(z)\exp(\lambda t) \tag{5}$$

   where the λ is the so-called global eigenvalue and $\eta(z)$ the global eigenfunction. With this definition, global instability will happen if there exists an eigenvalue with positive real part.

   Show that with this new Ansatz, the PDE becomes an ODE.

   $$(\lambda - \frac{1}{6})\eta = -Ca\frac{d\eta}{dz} + \frac{1}{6}\frac{d^2\eta}{dz^2}, \tag{6}$$

   $$\eta(0) = \eta(l) = 0 \tag{7}$$

5. To solve this ODE, we look for solutions under the form $\eta(z) = \exp(\alpha z)$. Show that the roots of these fundamental solutions write

   $$\alpha_{\pm} = 3Ca \pm \frac{1}{2}\sqrt{36Ca^2 + 24\lambda - 4} \tag{8}$$

6. We look for a combination of these two fundamental solutions $\eta = \eta_+ \exp(\alpha_+ z) + \eta_- \exp(\alpha_-)$. By imposing the boundary conditions, show that

$$\begin{pmatrix} \exp(\alpha^+)l & 1 \\ \exp(\alpha^-)l & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

   Deduce that

$$\sqrt{36Ca^2 + 24\lambda - 4} = i\frac{2\pi m}{l} \tag{10}$$

   where m is an integer.

7. Show that the eigenvalues equal

$$\lambda = -\frac{3}{2}Ca^2 + \frac{1}{6} - \frac{\pi^2 m^2}{6l^2} \tag{11}$$

8. Observe that local absolute instability is a necessary condition for global instability. While this is not always true (it is also dependent on the boundary conditions), this condition is often accepted.

   NB: eq. (1) can actually be obtained from the following two conservation laws, expressed here with dimensions:

   conservation of mass $\partial R/\partial t=-U\partial R/\partial z- 1/2\ \partial U/\partial z\ R$- explain where it is coming from by drawing a sketch.

   force balance (static at Re=0) $3\mu\partial U/\partial z=\gamma(-1/R+ \partial^2 R/\partial z^2)$- explain the origin of the terms

   Linearize these two equations around U_0 R_0 to obtain (still with dimensions)
   $dR'/dt=-U_0\partial R'/\partial z- 1/2\partial U'/\partial z\ R_0$
   $3\mu\partial U'/\partial z=\gamma(R'/R_0^2+ \partial^2 R'/\partial z^2)$

   Combining these equations and making things dimensionless, obtain (1).