Viscous Taylor Couette instability

The purpose of this exercise is to determine the stability of the Taylor Couette flow with respect to perturbations that are governed by the linearized Navier-Stokes equations without assuming vanishing viscosity.

1. Recall the governing nonlinear incompressible Navier-Stokes equations in cylindrical coordinates, assuming axisymmetry only. The flow is not assumed unidirectional at this stage.

2. Recall the base flow solution $U_\theta(r)$ between two walls at R_1 and R_2 rotating respectively at Ω_1 and Ω_2. We further assume that that only the inner cylinder is rotating and that $\Omega_2 = 0$.

3. Linearize the equations to obtain

$$\frac{\partial u_r}{\partial t} - \frac{2U_\theta u_\theta}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\tilde{\Delta} - \frac{1}{r^2})u_r, \tag{1}$$

$$\frac{\partial u_\theta}{\partial t} + \frac{dU_\theta}{dr}u_r + \frac{U_\theta u_r}{r} = \nu(\tilde{\Delta} - \frac{1}{r^2})u_\theta, \tag{2}$$

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\tilde{\Delta}u_z, \tag{3}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0, \tag{4}$$

(5)

where

$$\tilde{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \tag{6}$$

4. Assuming now that the gap $d = R_2 - R_1$ is small with respect to the median radius $\bar{r} = (R_2 + R_1)/2$, it is natural to introduce following change of variables

$$r = \bar{r} + y. \tag{7}$$

Show that the base flow rewrites

$$U_\theta = \bar{r}\Omega_1(\frac{1}{2} - \frac{y}{d}) \tag{8}$$

The following scales are then chosen for non-dimensionalisation

$$\tilde{u}_r = u_r d/\nu, \tag{9}$$

$$\tilde{u}_\theta = u_\theta d/\nu, \tag{10}$$

$$\tilde{u}_z = u_z d/\nu, \tag{11}$$

$$\tilde{t} = t\nu/d^2, \tag{12}$$

$$\tilde{y} = y/d \tag{13}$$

$$\tilde{p} = pd^2/\rho\nu^2 \tag{14}$$

(15)

In the sequel, use the limit $d/\bar{r} \ll 1$ to simplify the equations which become (dropping the tildes)

$$\begin{aligned}
\frac{\partial u_r}{\partial t} - \frac{2d^2\Omega_1}{\nu}(1/2 - y)u_\theta &= -\frac{\partial p}{\partial y} + \Delta u_r, && (16)\\
\frac{\partial u_\theta}{\partial t} - \frac{\bar{r}d\Omega_1}{\nu}u_r &= \Delta u_\theta, && (17)\\
\frac{\partial u_z}{\partial t} &= -\frac{\partial p}{\partial z} + \Delta u_z, && (18)\\
\frac{\partial u_r}{\partial y} + \frac{\partial u_z}{\partial z} &= 0, && (19)\\
& && (20)
\end{aligned}$$

where

$$\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad (21)$$

5. With the classical normal mode ansatz

$$\begin{aligned}
u_r &= \hat{u}_r \exp(i(kz - \omega t)), && (22)\\
u_\theta &= \hat{u}_\theta \exp(i(kz - \omega t)), && (23)\\
u_z &= \hat{u}_z \exp(i(kz - \omega t)), && (24)\\
p &= \hat{p} \exp(i(kz - \omega t)), && (25)\\
& && (26)
\end{aligned}$$

and introducing

$$\check{u}_r = -\frac{\bar{r}d\Omega_1}{\nu}\hat{u}_r, \quad (27)$$

show that the system reduces to

$$\begin{aligned}
(\hat{D}^2 - k^2 + i\omega)\hat{u}_\theta &= \check{u}_r, && (28)\\
(\hat{D}^2 - k^2 + i\omega)(\hat{D}^2 - k^2)\check{u}_r &= -k^2 T(1/2 - y)\hat{u}_\theta, && (29)
\end{aligned}$$

where

$$\hat{D} = \frac{d}{dy}, \quad (30)$$

and T is the Taylor number

$$T = \frac{2\bar{r}d^3\Omega_1^2}{\nu^2}. \quad (31)$$

6. Solve the above linear eigenvalue problem? Observe and comment the link with the linearized equations governing the onset of Rayleigh Benard convection. What is the critical Taylor number?