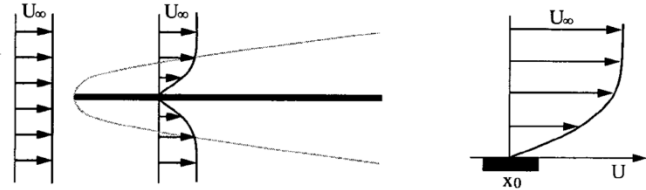


# Most flows are unstable...

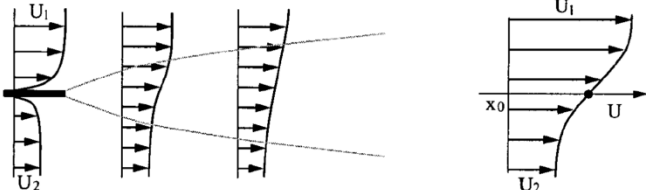
1. Intro-definitions
2. Rayleigh-Taylor
3. Rayleigh Plateau (destabilization through surface tension)
4. Rayleigh-Benard (convection)
5. Taylor Couette-Centrifugal instability
6. Kelvin-Helmholtz
7. Inflection point theorem Rayleigh ! Orr sommerfeld
8. Tollmien schlichting waves+ transient growth
9. Waves (phase velocity-group velocity)
10. Spatial growth

# SPATIALLY DEVELOPING SHEAR FLOWS

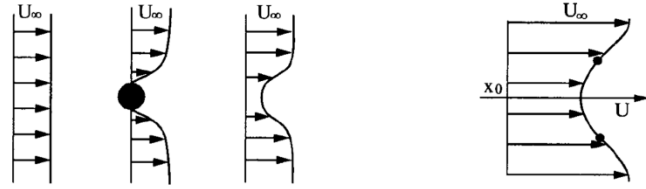
Flat plate boundary layer



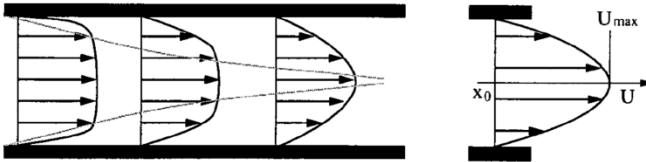
Mixing layer



Cylinder wake



Plane channel flow



2D jet



## 2D PARALLEL FLOW CONCEPTS

Dispersion relation

**2D vorticity equation**

$$\left( \frac{\partial}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \right) \nabla^2 \Psi = \frac{1}{Re} \nabla^4 \Psi$$

**Basic flow + perturbation**

$$\Psi(x, t) = \int U(y) dy + \psi(x, y, t)$$

**Linear vorticity equation**

$$\left( \frac{\partial}{\partial t} + U(y) \frac{\partial}{\partial x} \right) \nabla^2 \psi - U''(y) \frac{\partial \psi}{\partial x} = \frac{1}{Re} \nabla^4 \psi$$

## Dispersion relation

$$D(k, \omega) = 0$$

Temporal approach:

$k$  is real;  $\omega$  is complex

Perturbation grow and decay in time!

Spatial approach:

$\omega$  is real;  $k$  is complex

Perturbations grow and decay in space!

# Shear layer is inviscidly unstable!

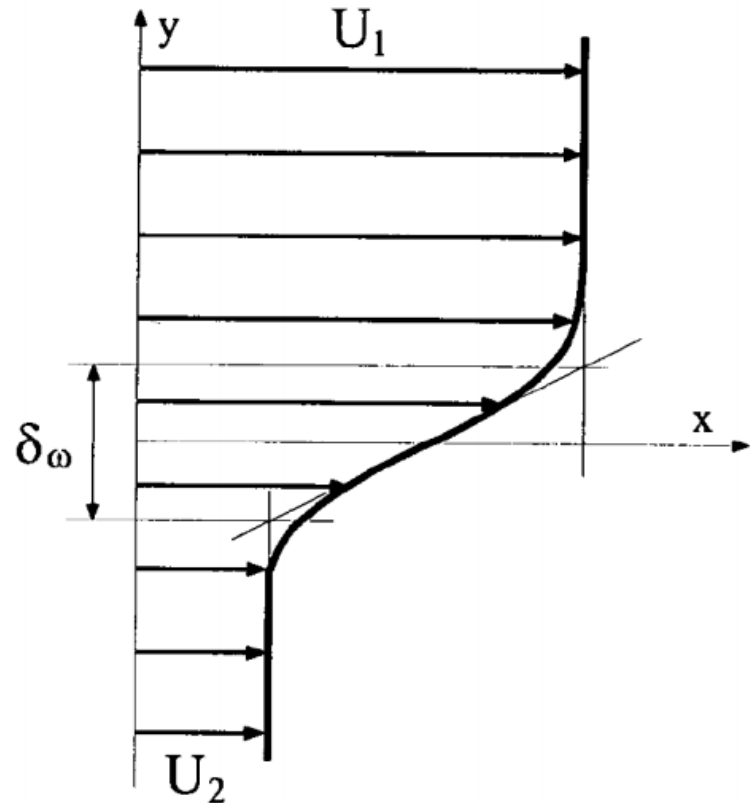
Hyperbolic tangent mixing layer

$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

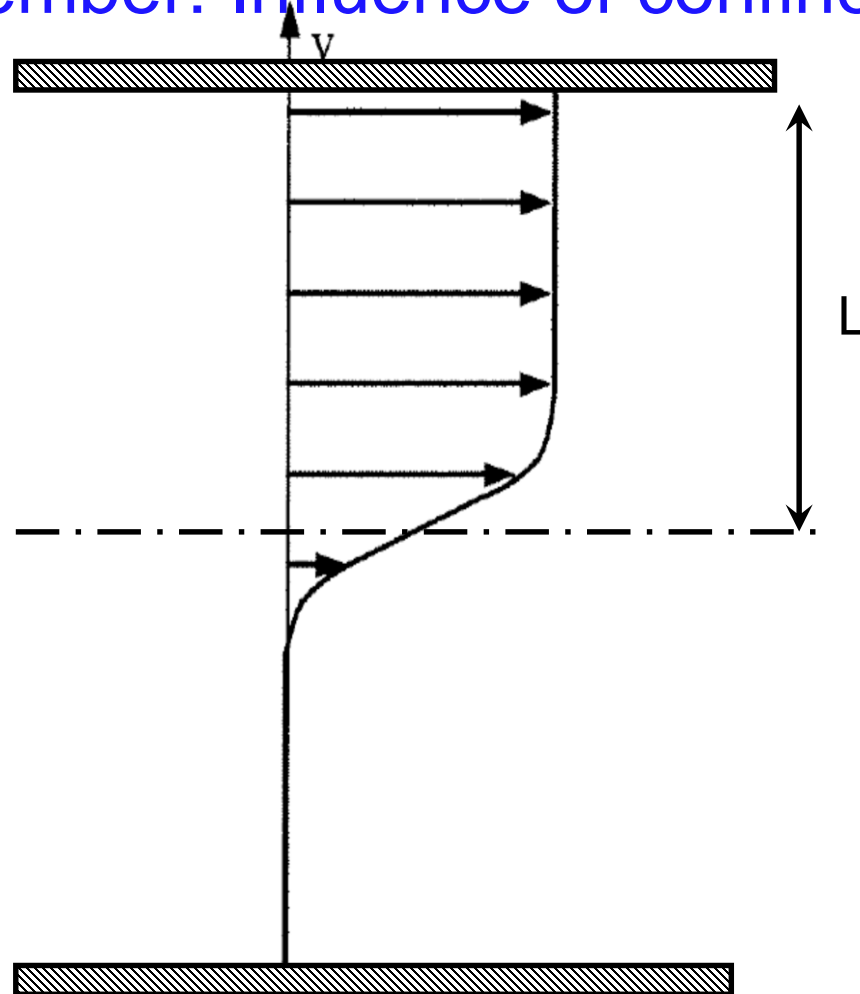
Velocity ratio

$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$



Why?

*Only a necessary condition for instability!*  
Remember: Influence of confinement



$$R = 1$$

## 2D PARALLEL FLOW CONCEPTS

Hyperbolic tangent mixing layer

$$U(y; R) = 1 + R \tanh y$$

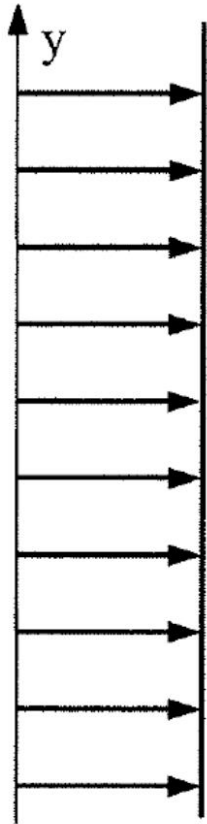
$$U_1(y) = \tanh y$$

Dispersion relation

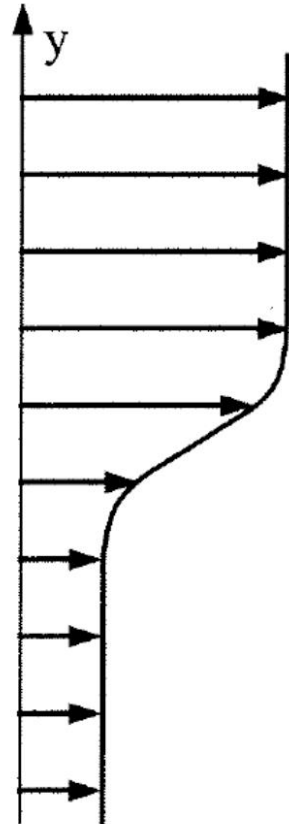
$$\omega(k; R) = k + R \omega_1(k)$$

# PARALLEL FLOW CONCEPTS

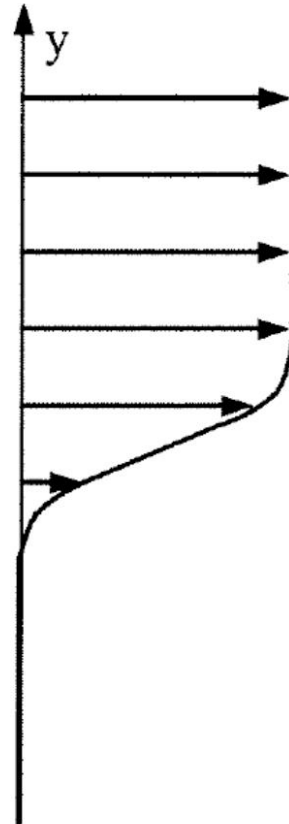
## Effect of velocity ratio



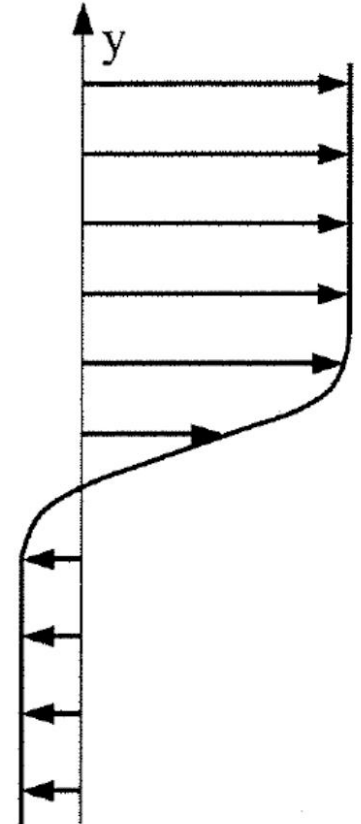
$R = 0$



$0 < R < 1$



$R = 1$



$R > 1$

## 2D PARALLEL FLOW CONCEPTS

### Hyperbolic tangent mixing layer

#### Temporal approach

$$\omega_1(k) = i \omega_{1,i}(k)$$

$$\omega_i(k; R) = R \omega_{1,i}(k)$$

$$c_r = \omega_r / k = 1$$

Temporal approach:  $k$  is real;  $\omega$  is complex

# 2D PARALLEL FLOW CONCEPTS

## Hyperbolic tangent mixing layer

**Spatial approach**

$$k + R \omega_1(k) = \omega$$

$$R \ll 1$$

$$-k_i(\omega, R) \sim R \omega_{1,i}(\omega)$$

Gaster transformation

## 2D PARALLEL FLOW CONCEPTS

### Broken-line profile mixing layer

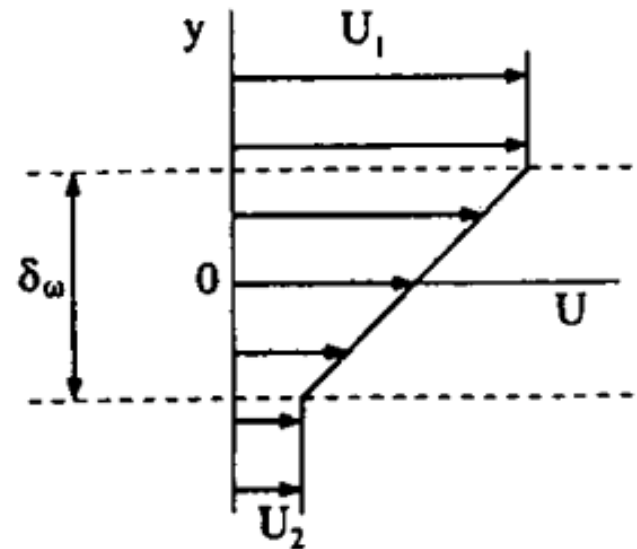
$$U(y) = \begin{cases} U_1, & y > \delta_\omega/2 \\ (U_1 + U_2)/2 + (U_1 - U_2)y/\delta_\omega, & |y| < \delta_\omega/2 \\ U_2, & y < -\delta_\omega/2 \end{cases}$$

$$\phi'' - k^2 \phi = 0$$

$$\phi_1(y) = A_1 e^{-ky}, \quad y > \delta_\omega/2,$$

$$\phi_2(y) = B_2 e^{ky}, \quad y < -\delta_\omega/2,$$

$$\phi_0(y) = A_0 e^{-ky} + B_0 e^{ky}, \quad |y| < \delta_\omega/2$$



## 2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$A_1 e^{-k\delta_\omega/2} = A_0 e^{-k\delta_\omega/2} + B_0 e^{k\delta_\omega/2},$$

$$B_2 e^{-k\delta_\omega/2} = A_0 e^{k\delta_\omega/2} + B_0 e^{-k\delta_\omega/2},$$

$$\begin{aligned} -k(U_1 - c)A_1 e^{-k\delta_\omega/2} &= k(U_1 - c)(-A_0 e^{-k\delta_\omega/2} + B_0 e^{k\delta_\omega/2}) \\ &\quad - \frac{\Delta U}{\delta_\omega}(A_0 e^{-k\delta_\omega/2} + B_0 e^{k\delta_\omega/2}), \end{aligned}$$

$$\begin{aligned} k(U_2 - c)B_2 e^{-k\delta_\omega/2} &= k(U_2 - c)(-A_0 e^{k\delta_\omega/2} + B_0 e^{-k\delta_\omega/2}) \\ &\quad - \frac{\Delta U}{\delta_\omega}(A_0 e^{k\delta_\omega/2} + B_0 e^{-k\delta_\omega/2}). \end{aligned}$$

## 2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$-\frac{\Delta U}{\delta_\omega} A_0 e^{-k\delta_\omega/2} + \left[ 2k(U_1 - c) - \frac{\Delta U}{\delta_\omega} \right] B_0 e^{k\delta_\omega/2} = 0$$
$$\left[ 2k(U_2 - c) + \frac{\Delta U}{\delta_\omega} \right] A_0 e^{k\delta_\omega/2} + \frac{\Delta U}{\delta_\omega} B_0 e^{-k\delta_\omega/2} = 0$$

## 2D PARALLEL FLOW CONCEPTS

Broken-line profile mixing layer

$$4(k\delta_\omega)^2(c - \bar{U})^2 - \left[ (k\delta_\omega - 1)^2 - e^{-2k\delta_\omega} \right] \Delta U^2 = 0$$

$$k\delta_\omega \mapsto 2k, c/\bar{U} \mapsto c$$

$$4k^2(c - 1)^2 - R^2 \left[ (2k - 1)^2 - e^{-4k} \right] = 0$$

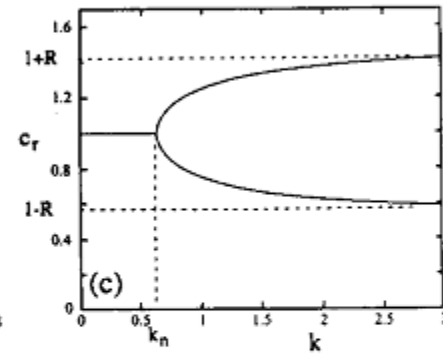
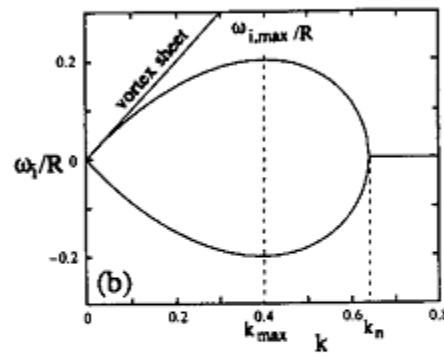
$$c \equiv \frac{\omega}{k} = 1 \pm \frac{R}{2k} \left[ (2k - 1)^2 - e^{-4k} \right]^{1/2}$$

## 2D PARALLEL FLOW CONCEPTS

### Broken-line profile mixing layer

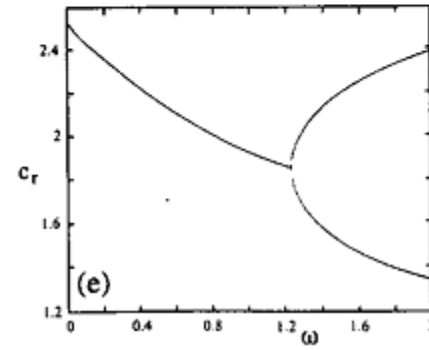
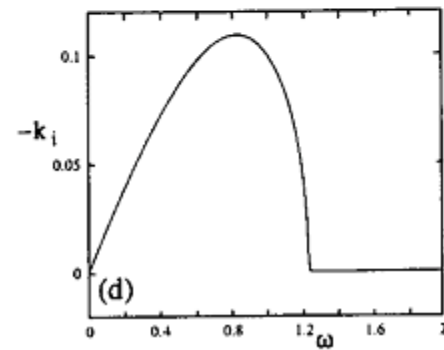
Temporal approach

$$2k_n - 1 = e^{-2k_n}$$



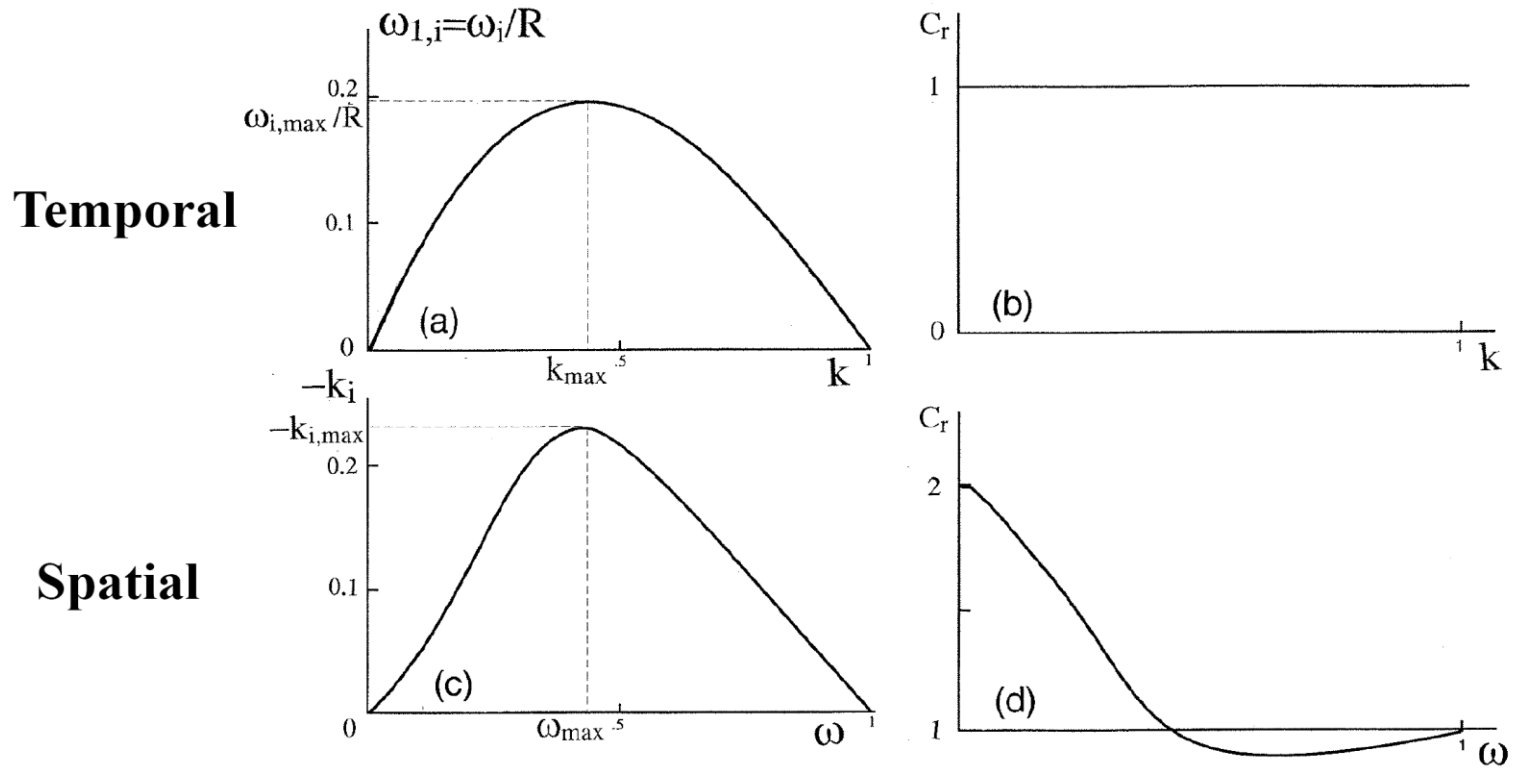
Spatial approach

$$R = 0.5$$



## 2D PARALLEL FLOW CONCEPTS

### Hyperbolic tangent mixing layer



Michalke (1964, 65)

# Solving a spatial instability problem

## ex: Rayleigh equation

## Back to temporal stability analysis!

How to solve Rayleigh equation for real  $k$  and complex  $\omega$ ?

We fix  $k$ , we need to find all  $\omega$  and  $\psi$  such that

$$k \left( U \left( \frac{d^2}{dy^2} - k^2 \right) - U''(y) \right) \psi = \omega \left( \frac{d^2}{dy^2} - k^2 \right) \psi$$
$$\psi(-L) = \psi(L) = 0$$

Formally,

$$\mathcal{A}\psi = c\mathcal{E}\psi$$

$$c = \omega/k$$

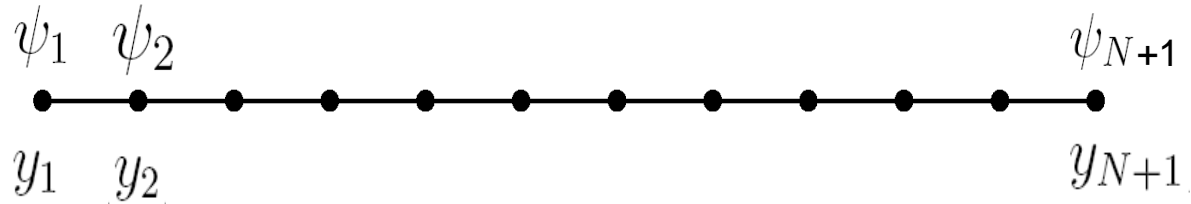
Discretize

$$\mathbf{A}\Psi = c\mathbf{E}\Psi$$

Generalized eigenvalue problem

# How to solve Rayleigh equation for real $k$ and complex $\omega$ ?

Finite differences of order 1



$$\Psi = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \vdots \\ \psi(y_N) \\ \psi(y_{N+1}) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \\ \psi_{N+1} \end{pmatrix} \quad \Psi'' = \begin{pmatrix} \psi''(y_1) \\ \psi''(y_2) \\ \vdots \\ \psi''(y_N) \\ \psi''(y_{N+1}) \end{pmatrix}$$

# How to solve Rayleigh equation for real k and complex $\omega$ ?

Finite differences

$$\begin{pmatrix} \psi_2'' \\ \psi_3'' \\ \psi_4'' \\ \vdots \\ \psi_{N-3}'' \\ \psi_{N-2}'' \\ \psi_{N-1}'' \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_{N-3} \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}$$

Sparse matrix but low order!

# How to solve Rayleigh equation for complex $k$ and real $\omega$ ?

We fix  $\omega$ , we need to find all  $k$  and  $\psi$  such that

$$k \left( U \left( \frac{d^2}{dy^2} - k^2 \right) - U''(y) \right) \psi = \omega \left( \frac{d^2}{dy^2} - k^2 \right) \psi$$
$$\psi(-L) = \psi(L) = 0$$

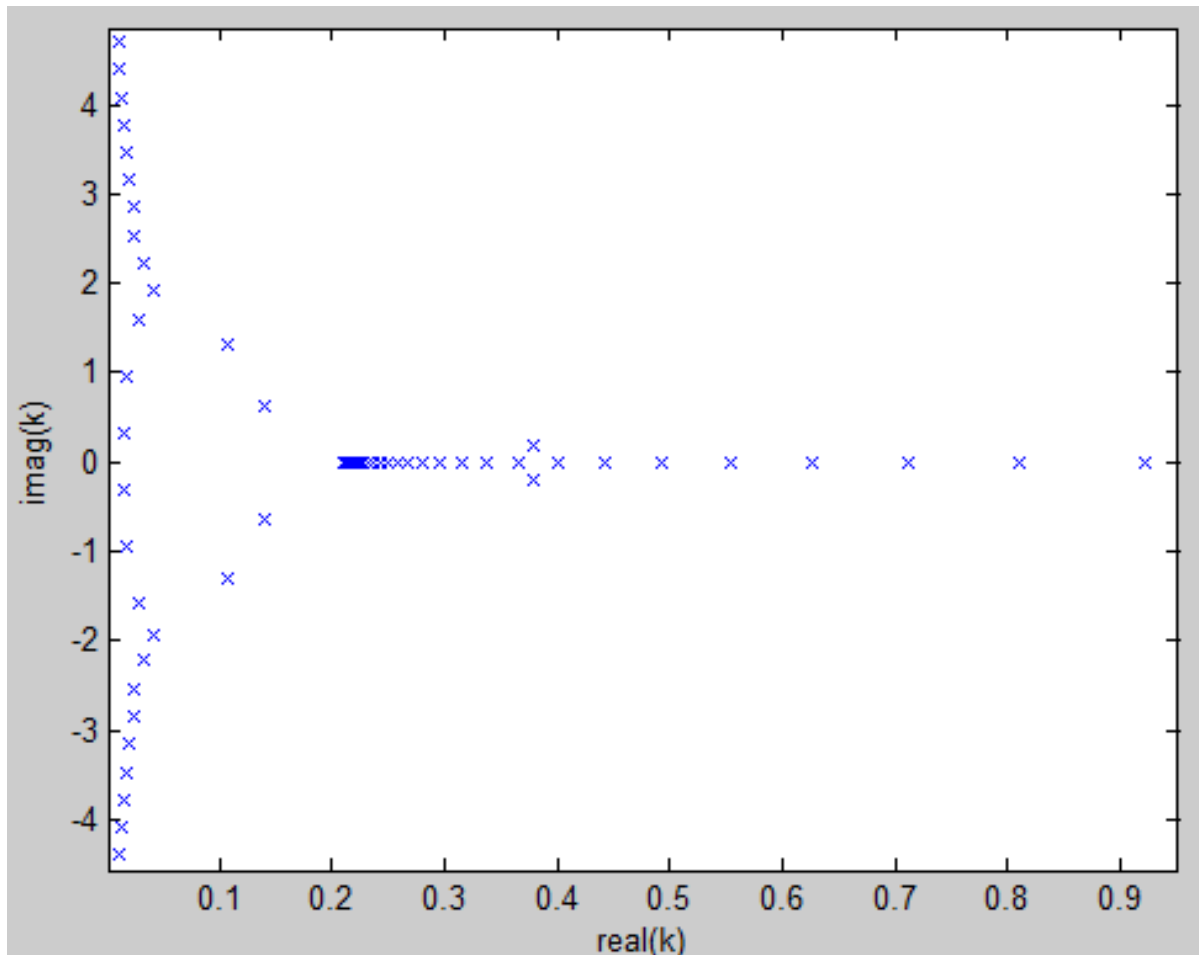
Formally,

$$(A_0(\omega, y) + kA_1(\omega, y) + k^2A_2(\omega, y) + k^3A_3(\omega, y)) \psi = 0$$

Polynomial eigenvalue problem

# Many more eigenvalues (for Rayleigh equation: 3 x more!)

$$U=1+0.9*\tanh(y); \omega=0.4; L=5$$



Which of these waves are unstable?

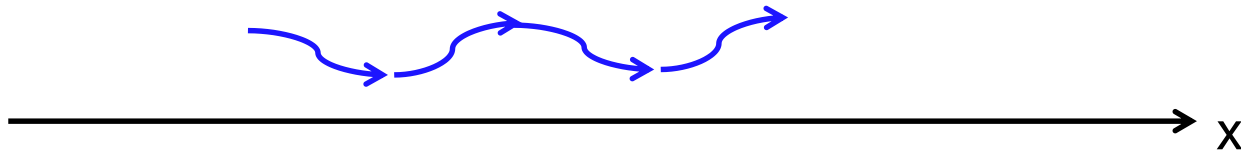
$\text{Im}(k) < 0$ ?

$\text{Im}(k) > 0$ ?

Recall :  $\exp(i(kx - \omega t))$

The stability of a spatial wave can be only determined if one knows in which direction it propagates!

$k^+$  waves propagate towards positive  $x$



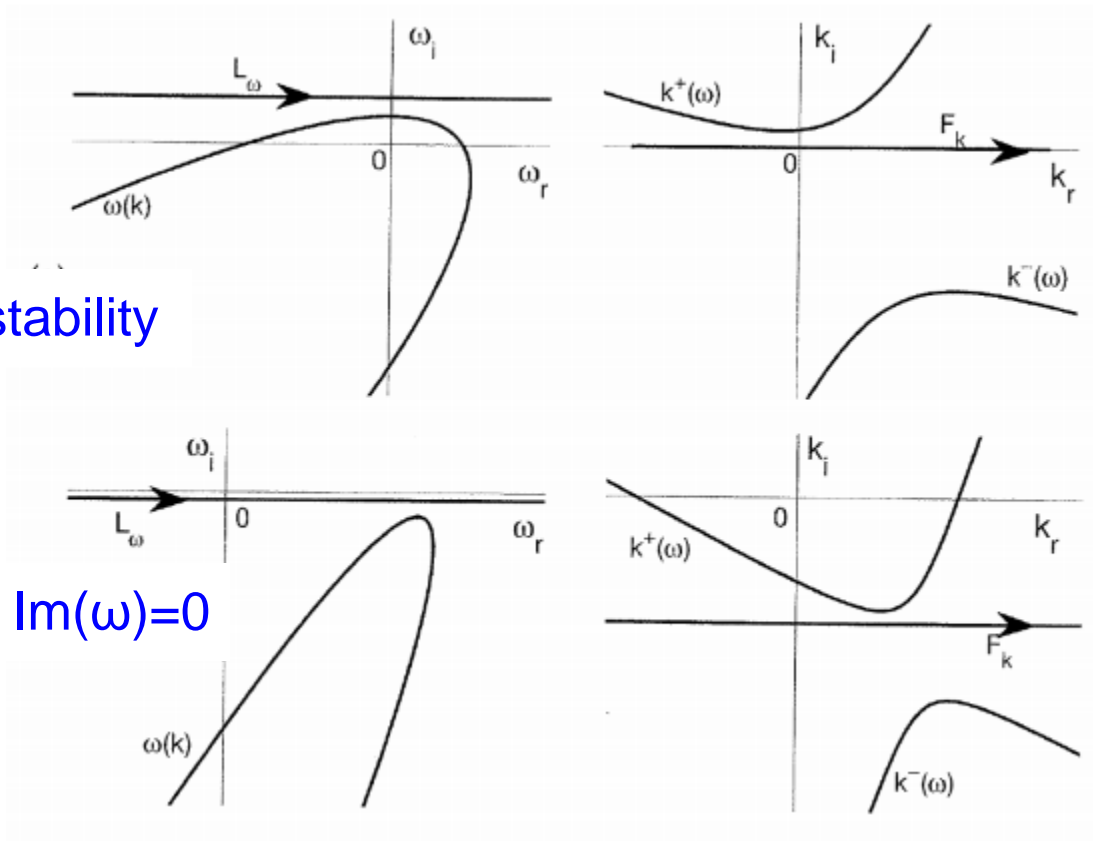
$k^-$  waves propagate towards negative  $x$



However, determining this direction of propagation is particularly difficult, except in the case of a temporally stable flow.

The addition of a positive imaginary offset to the frequency makes the temporal problem stable!

This separates the spatial waves into  $k^+$  and  $k^-$  waves.



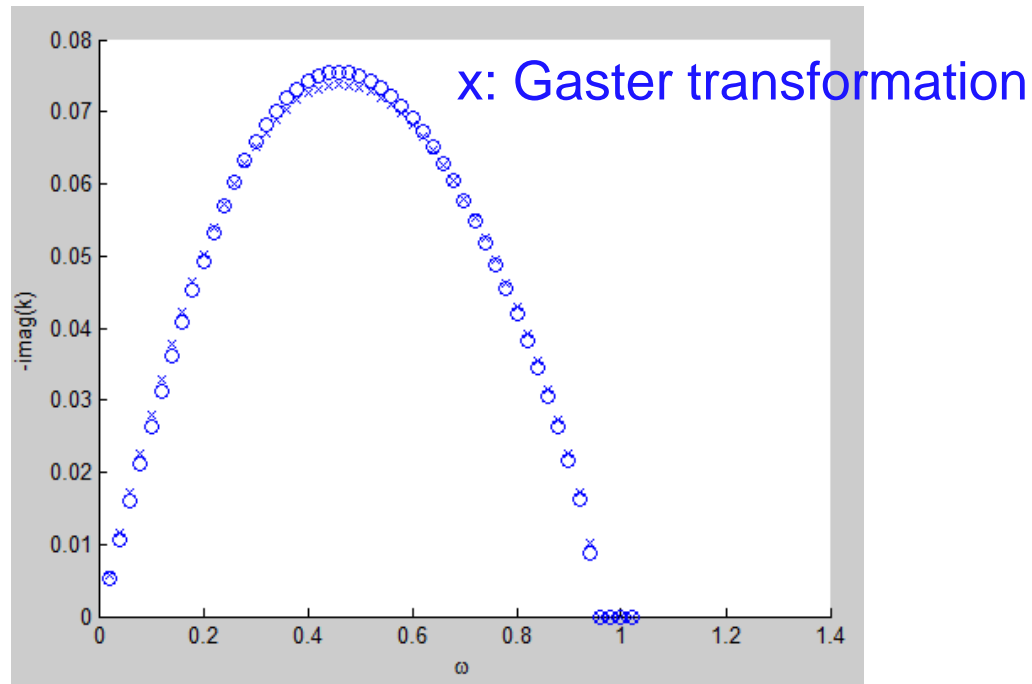
offset spatial stability

spatial stability:  $\text{Im}(\omega)=0$

The branches are then followed by continuity

# Validity of Gaster transformation?

$R=0.4$



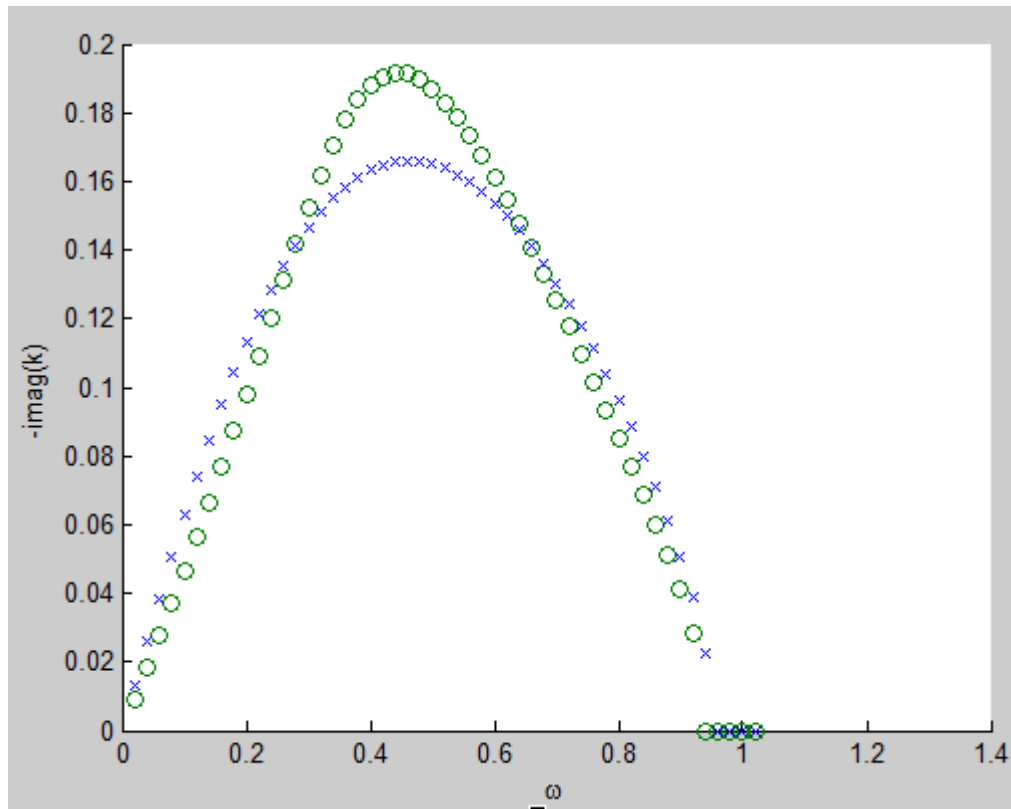
$$R \ll 1$$

$$-k_i(\omega, R) \sim R\omega_{1,i}(\omega)$$

# Validity of Gaster transformation?

R=0.9

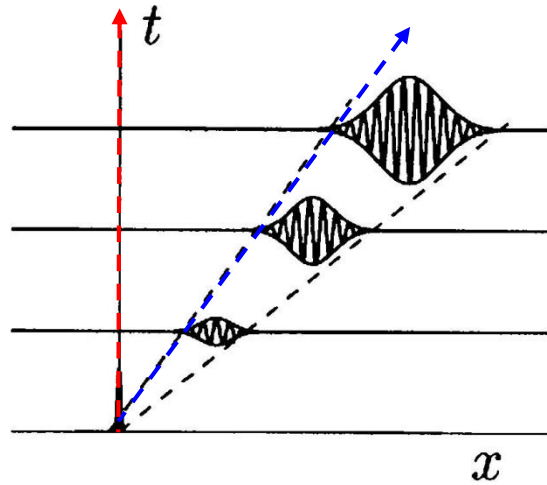
x: Gaster transformation



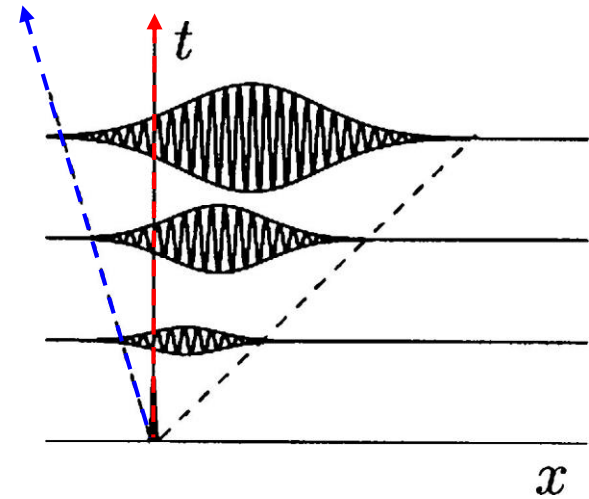
$$-k_i(\omega, R) \sim R\omega_{1,i}(\omega)$$

# Spatio-temporal instability theory

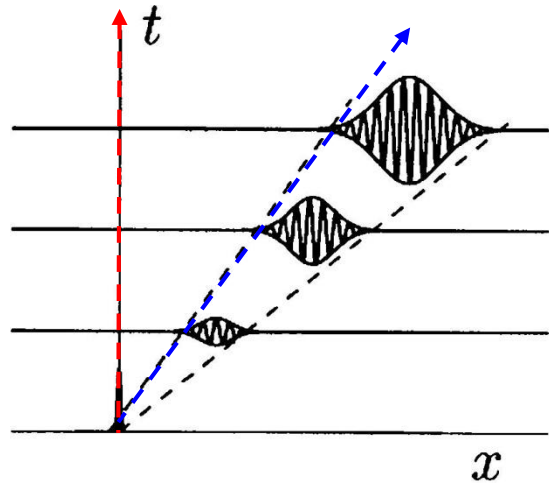
☞ Convective instability



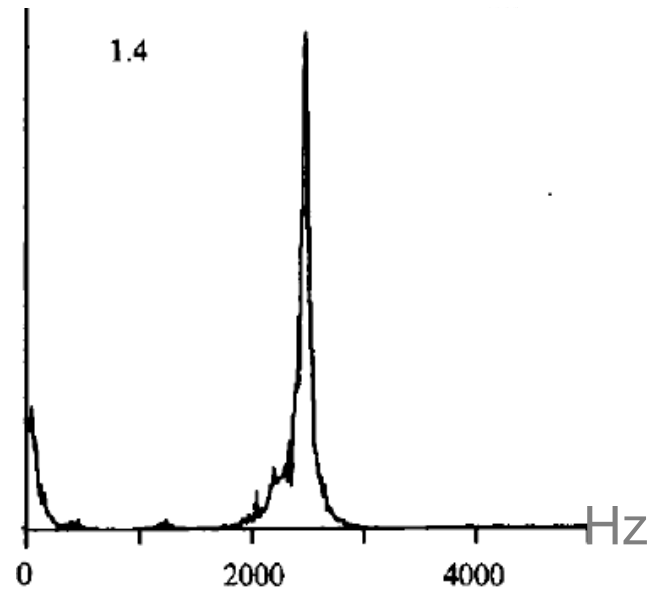
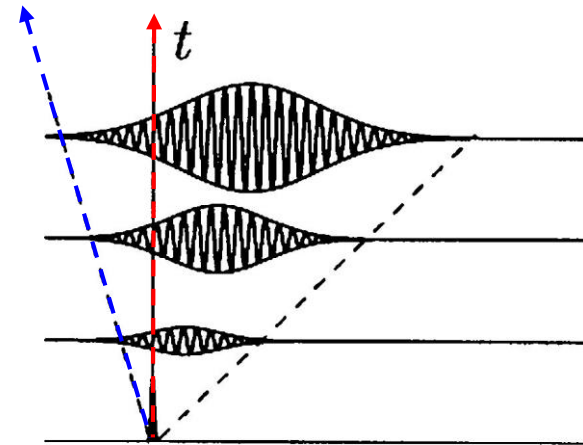
☞ Absolute instability



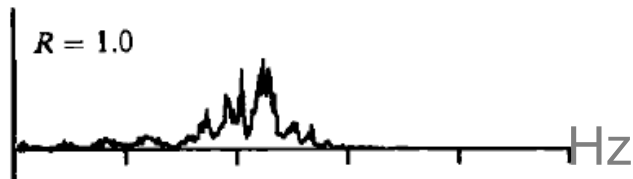
# 👉 Convective instability



# 👉 Absolute instability



Spectrum



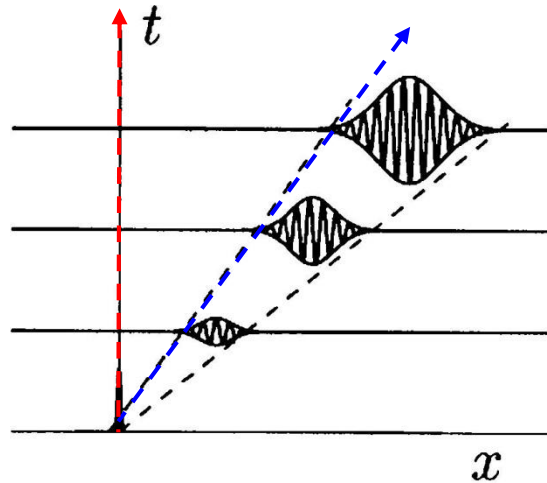
👉 amplifier

👉 oscillator

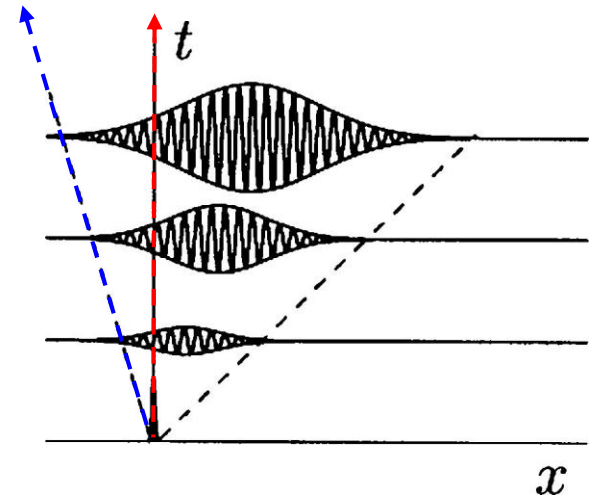
Mixing layer experiments by Strikowsky and Niccum (1991)

# Spatio-temporal instability theory

☞ Convective instability



☞ Absolute instability



We need to generalize the concept of group velocity since  $\omega$  (and why not  $k$ ) is complex

For neutral waves, the group velocity is  $d\omega/dk$

Here this quantity is the derivative of a complex function with respect to a complex variable. Cauchy-Riemann conditions apply.

# Spatio-temporal spectral analysis

Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k, \omega) e^{i(kx - \omega t)} dk d\omega$$

$$\hat{u}(k, \omega) = (2\pi)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x, t) e^{-i(kx - \omega t)} dx dt$$

Direct Fourier Transform

# Spatio-temporal spectral analysis

## Inverse Fourier Transform

$$u(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k, \omega) e^{i(kx - \omega t)} dk d\omega$$

Use dispersion relation  $\omega(k)$ !

# Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

# Carrier/enveloppe

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

# Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

# Spectral analysis

Fourier transform:

$$u(x, t) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx - \omega(k)t)} dk + c.c.$$

Carrier/enveloppe :

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$

Enveloppe :

$$A(x, t) = \int_0^{\infty} \hat{u}(k) e^{i(k - k_0)x - i(\omega - \omega_0)t} dk.$$

# Spectral analysis at time=0

Fourier transform:

$$u(x, 0) = \frac{1}{2} \int_0^{+\infty} \hat{u}(k) e^{i(kx)} dk + \text{c.c.}$$

$\hat{u}(k)$  is given by Fourier transform at time  $t=0$

Envelope :

$$A(x, 0) = \int_0^{\infty} \hat{u}(k) e^{i(k-k_0)x} dk + \text{c.c.}$$

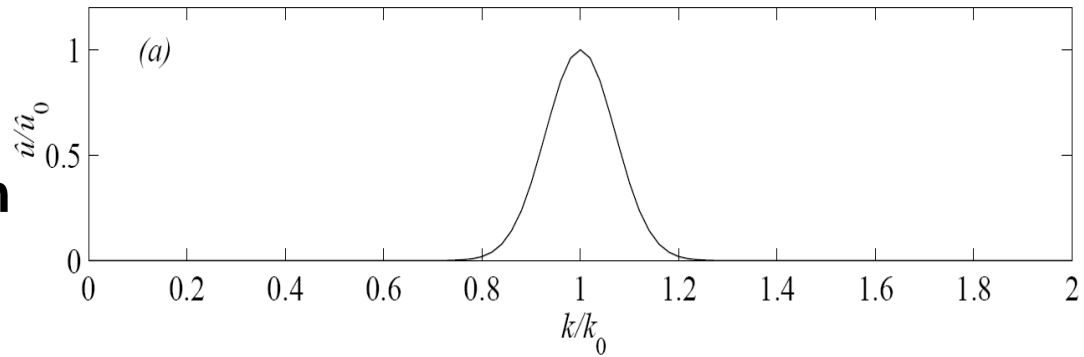
# Spectral analysis

Gaussian spectrum:  $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

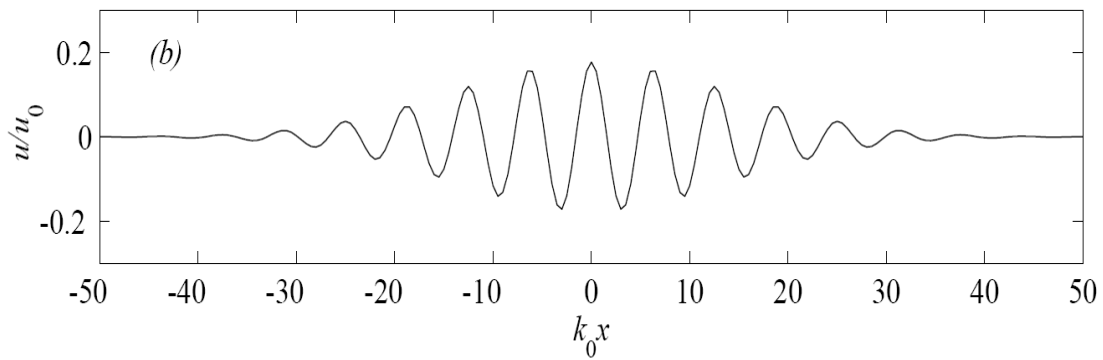
Initial envelope :  $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

# Gaussian spectrum

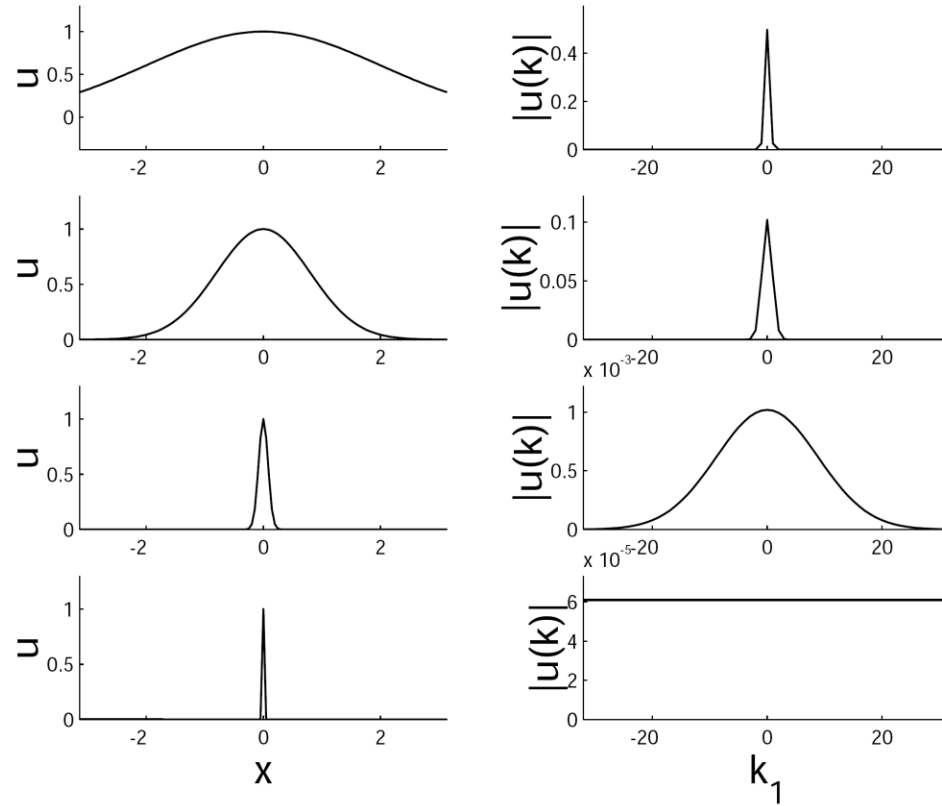
**spectrum**



**wave**



# Gaussian wavepackets



$u_0(x)$	$\hat{u}_0(k_1)$
$\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$
$\delta(x)$	$\frac{1}{2\pi}$

# Spectral analysis

Initial envelope :

$$A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$$

Gaussian spectrum:

$$\hat{u}(k) = u_0 e^{-\sigma^2 (k-k_0)^2}$$

# Spectral analysis

Initial envelope :  $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum:  $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

Evolution of envelope :  $A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$

# Spectral analysis

Initial envelope :  $A(x, 0) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{x^2}{4\sigma^2}}$

Gaussian spectrum:  $\hat{u}(k) = u_0 e^{-\sigma^2(k-k_0)^2}$

Evolution of envelope :  $A(x, t) = \int_0^\infty \hat{u}(k) e^{i(k-k_0)x - i(\omega - \omega_0)t} dk.$

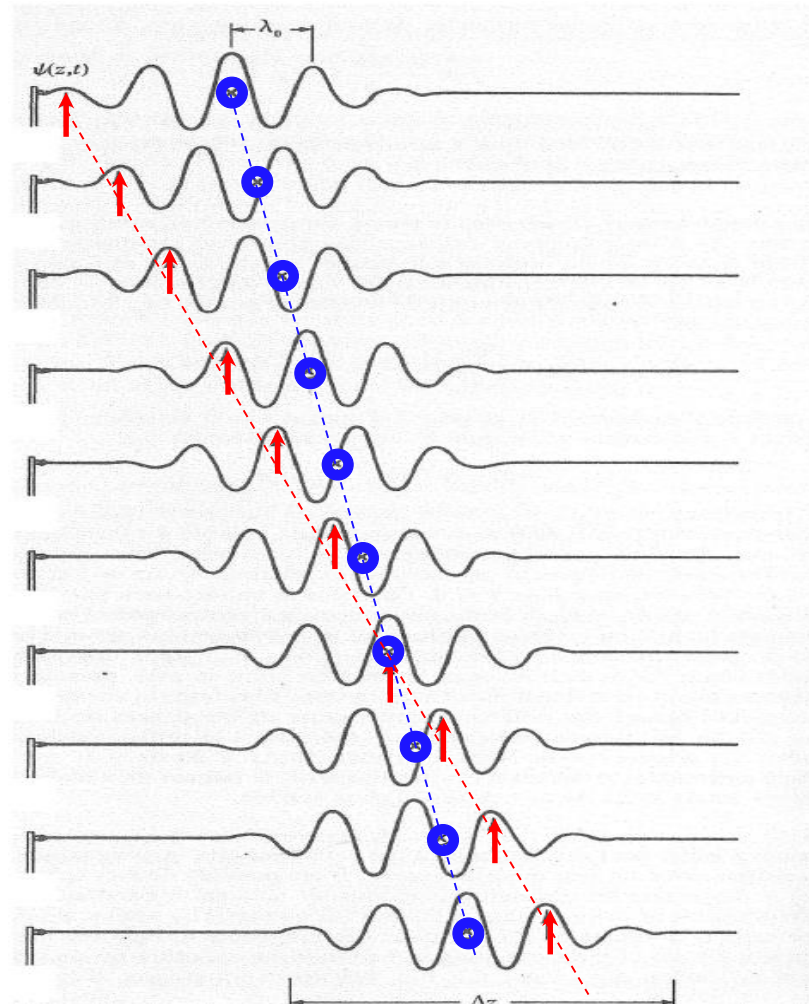
Definition group velocity  $\omega - \omega_0 = c_g(k - k_0), \quad c_g = \frac{\partial \omega}{\partial k}(k_0)$

# Spectral analysis

Definition of group velocity  $\omega - \omega_0 = c_g(k - k_0)$ ,  $c_g = \frac{\partial \omega}{\partial k}(k_0)$

$$A(x, t) = \frac{u_0 \sqrt{\pi}}{2\sigma} e^{-\frac{(x - c_g t)^2}{4\sigma^2}}$$

# Group velocity



Wavepacket

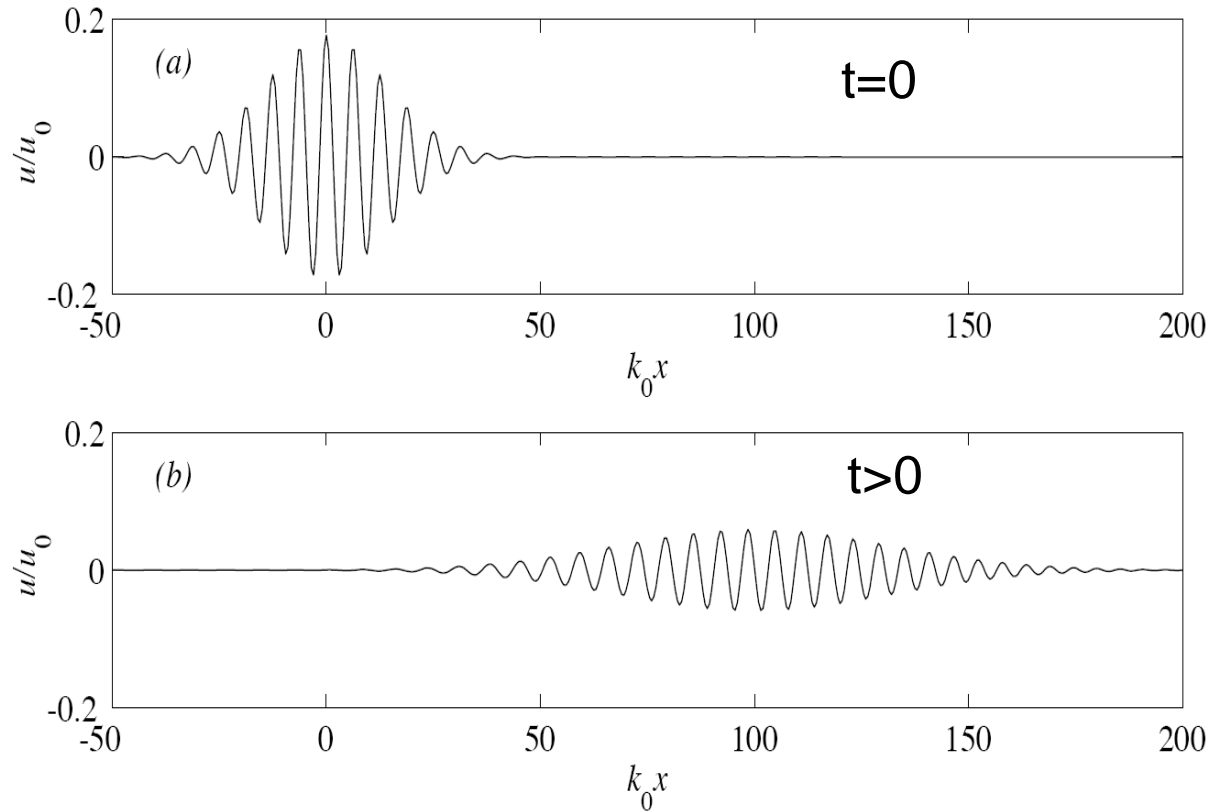
# Spectral analysis

Higher order  
development

$$\omega - \omega_0 = c_g(k - k_0) + \frac{\omega_0''}{2}(k - k_0)^2$$
$$c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0)$$

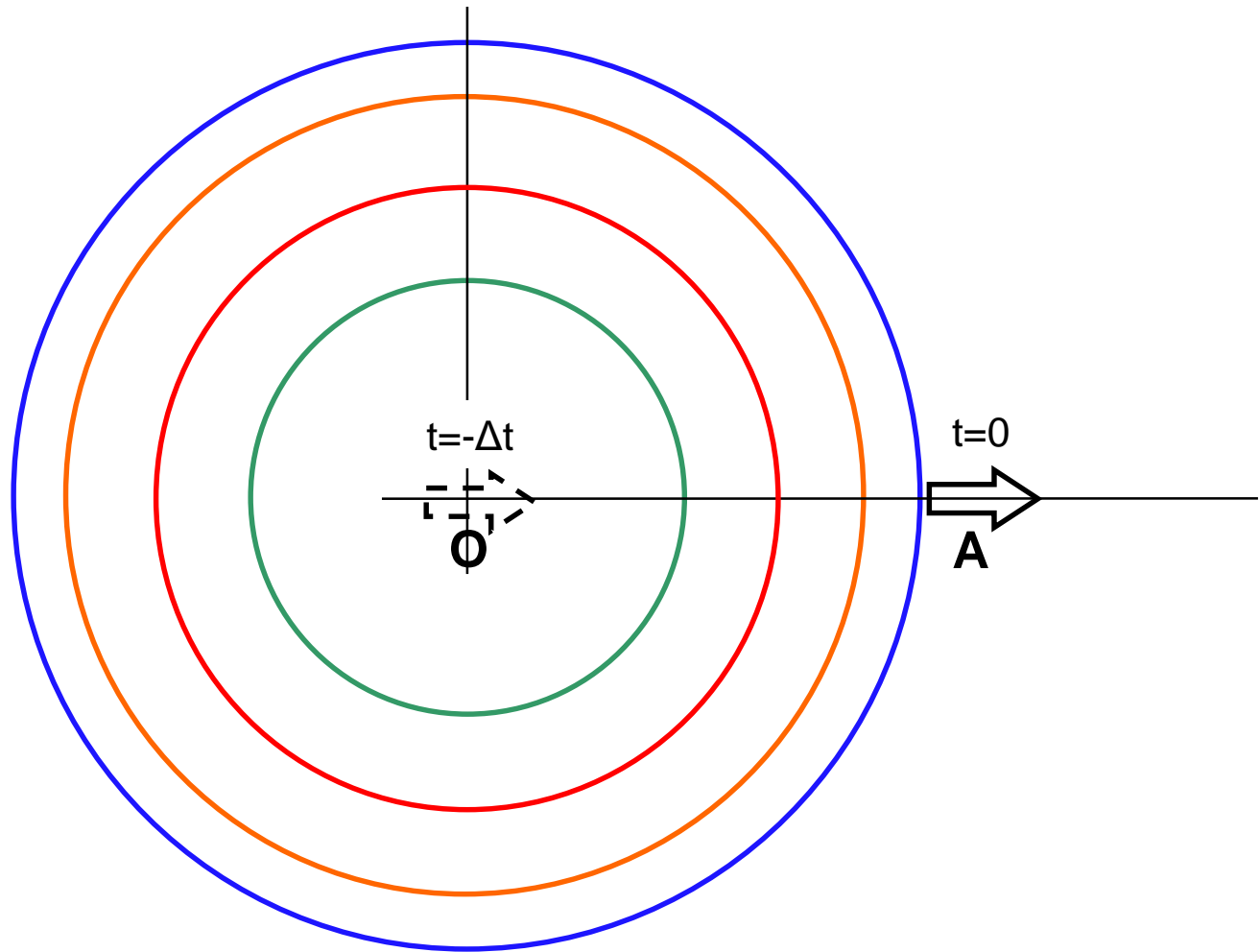
$$A(x, t) = \frac{u_0}{2} \sqrt{\frac{\pi}{\sigma^2 + \frac{1}{2}i\omega_0''t}} \exp\left(-\frac{(x - c_g t)^2}{4(\sigma^2 + \frac{1}{2}i\omega_0''t)}\right)$$

# Wave packet dispersion

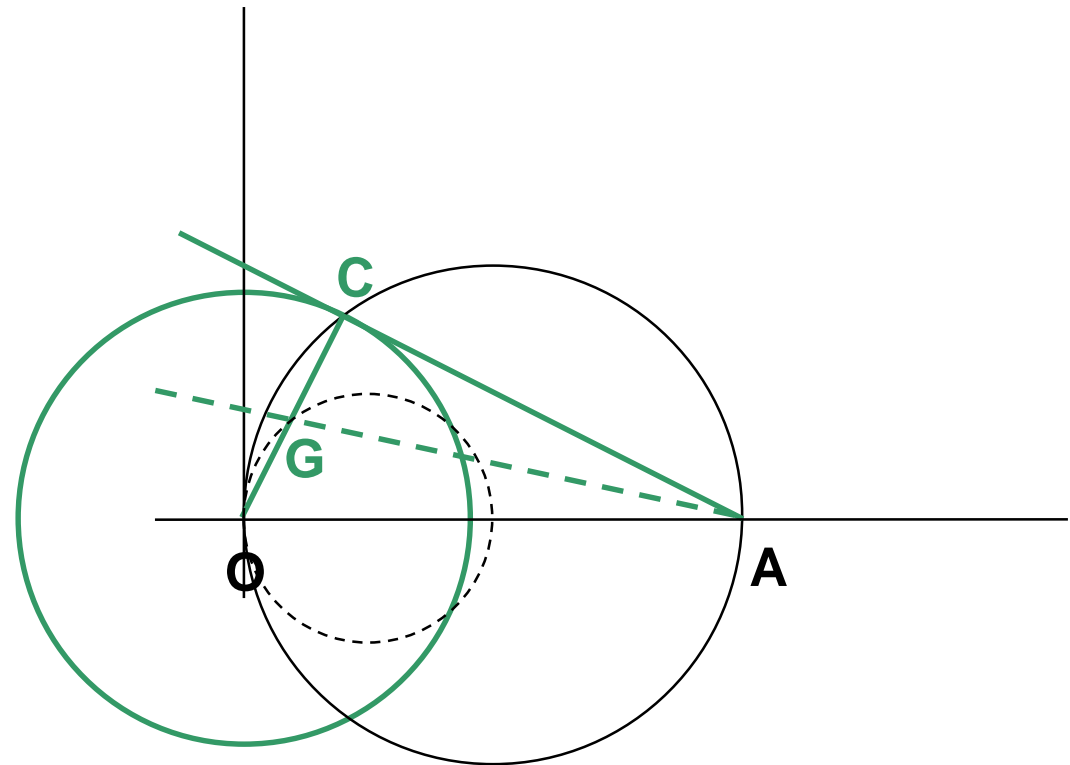


Onde correspondant à l'enveloppe  $\hat{u}(k)$  pour  $\sigma^{-1}k_0 = 0,1$  et  $\omega_0'' = 4c_g/k_0$  : (a), instant initial  $t = 0$  ; (b),  $c_g t = 100/k_0$ .

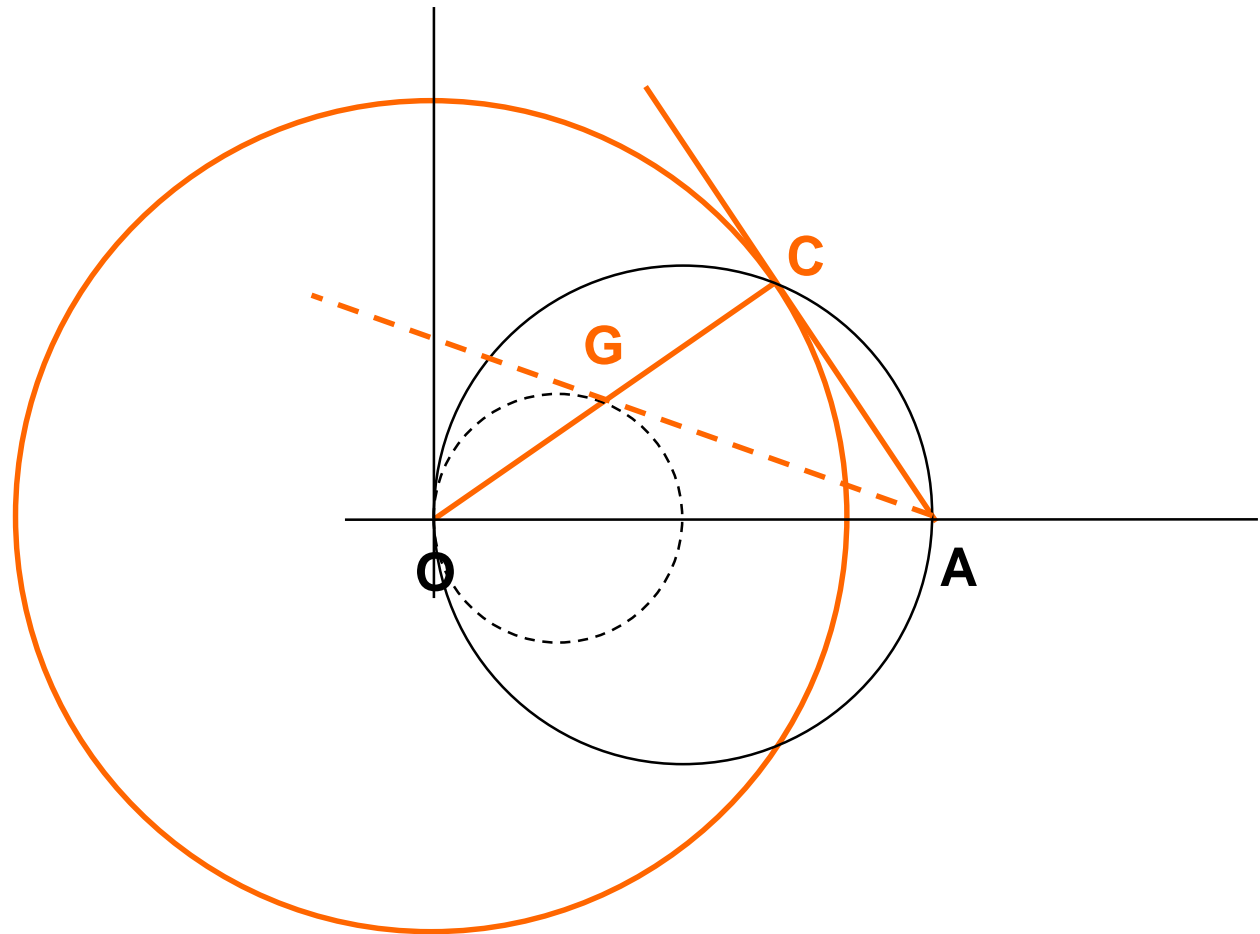
# Kelvin's wake



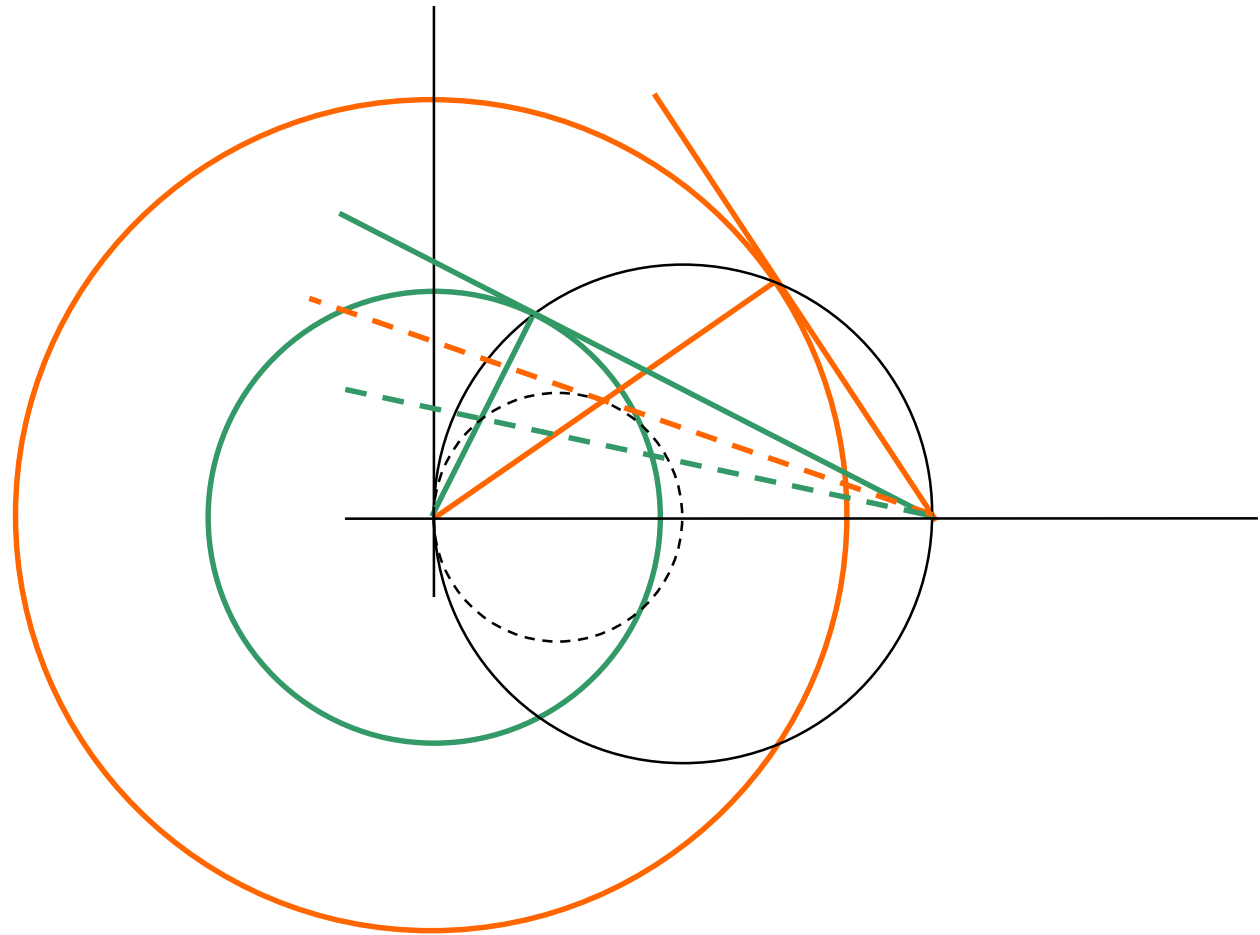
# Gravity waves created by a ship



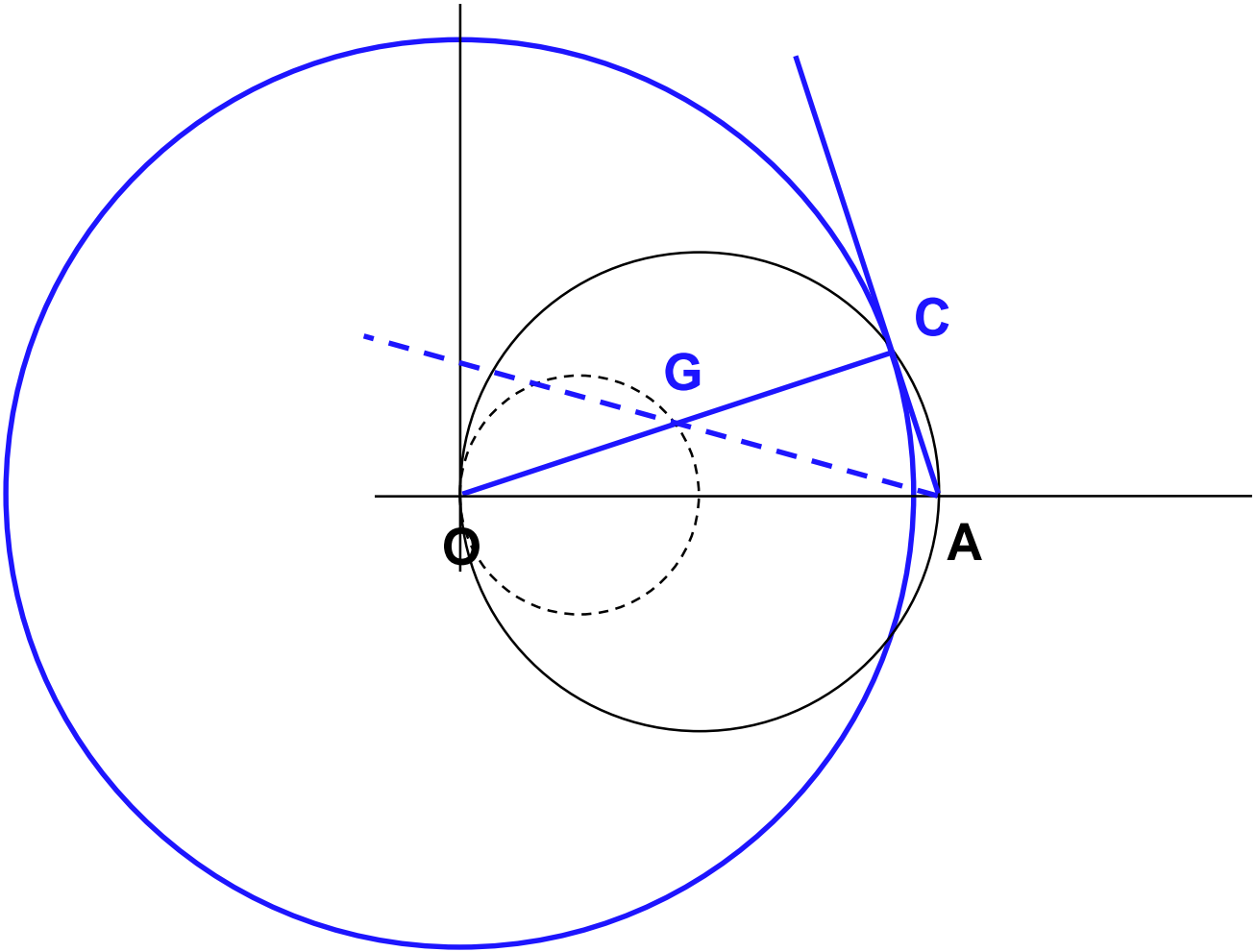
# Gravity waves created by a ship



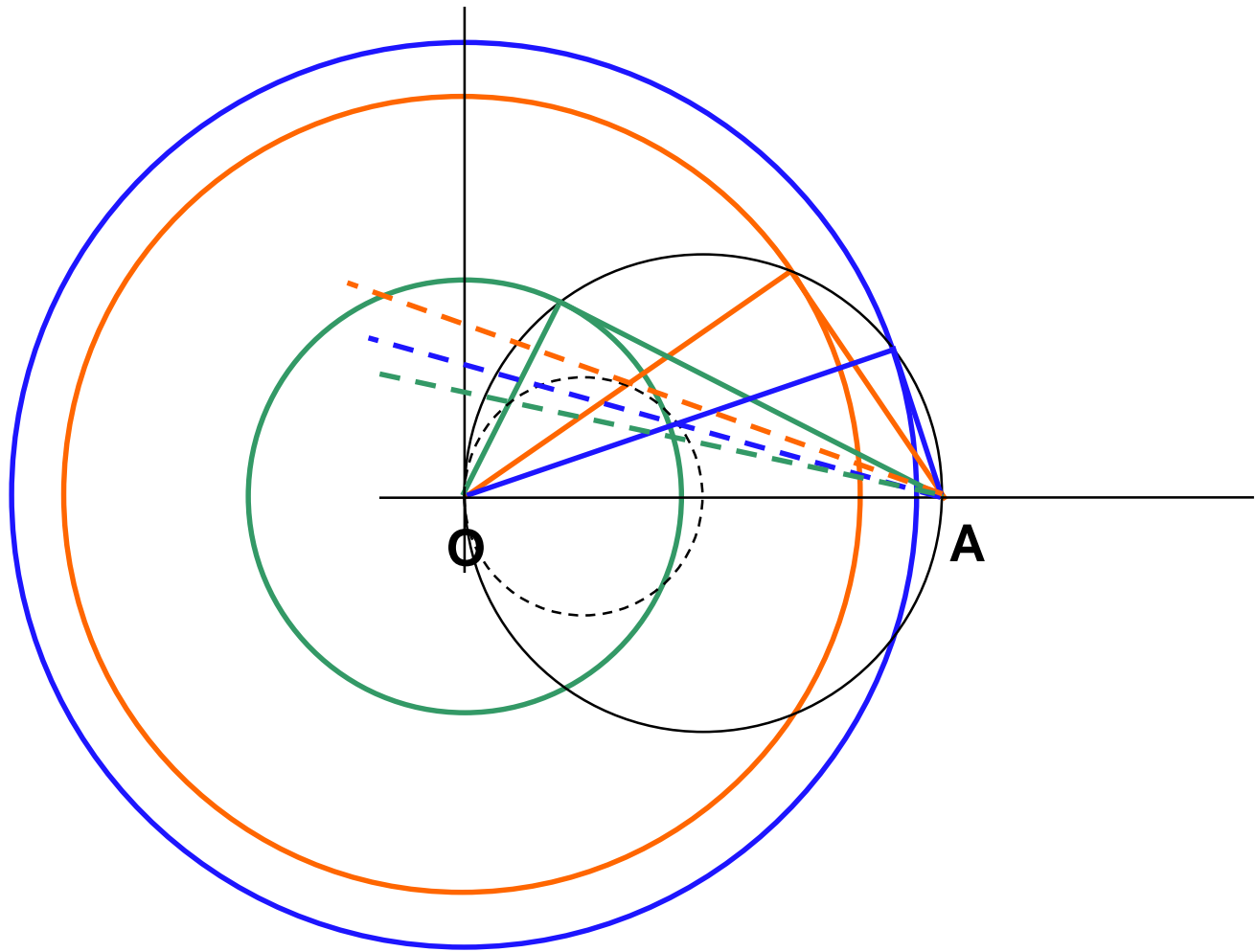
# Gravity waves created by a ship



# Gravity waves created by a ship



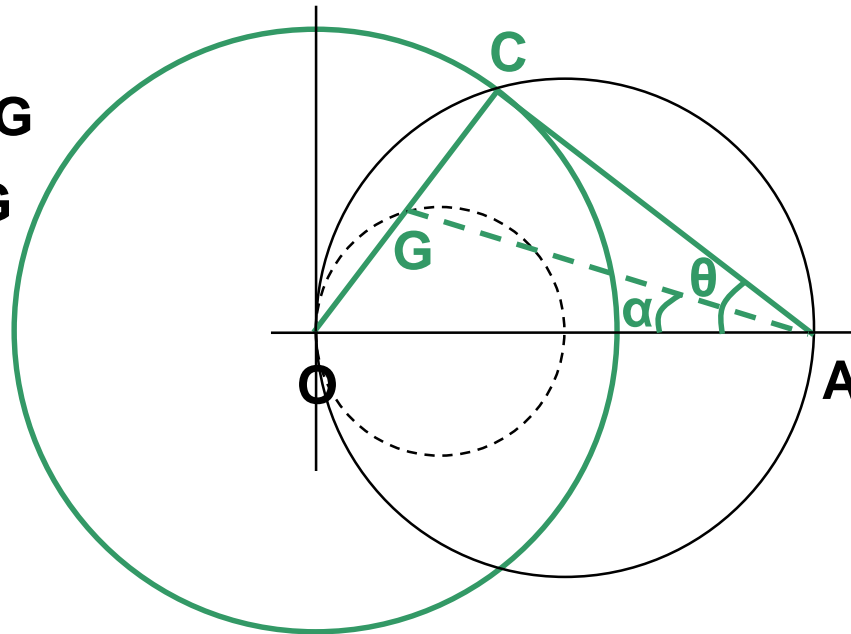
# Gravity waves created by a ship



# Gravity waves created by a ship

$$\sin(\alpha)/OG = \cos(\theta)/AG$$

$$\sin(\theta - \alpha)AG = GC = OG$$

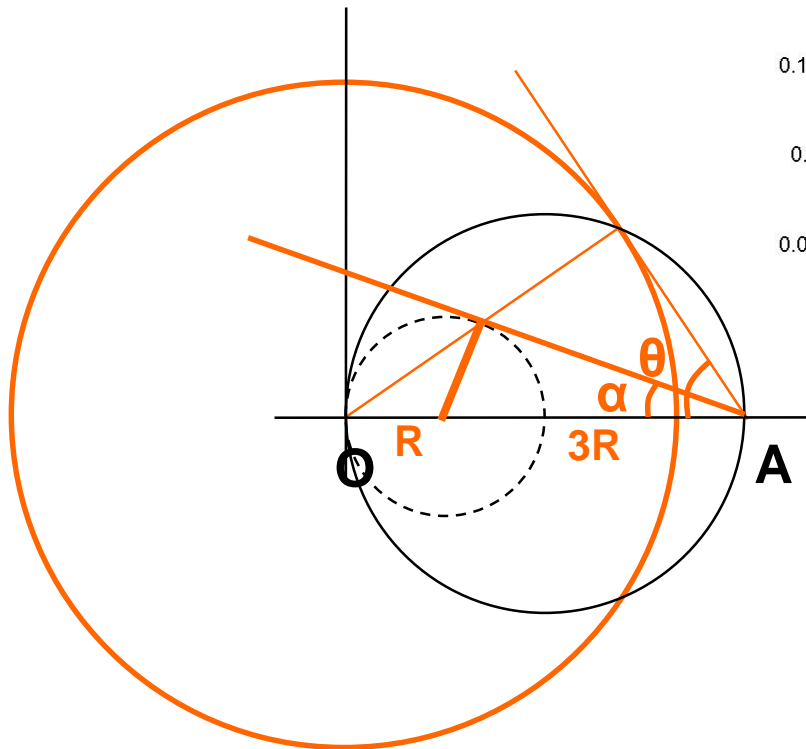


$$\Rightarrow \sin(\alpha) = \cos(\theta) \sin(\theta - \alpha)$$

$$\Rightarrow \sin(\alpha) = \cos(\theta) (\sin(\theta) \cos(\alpha) + \cos(\theta) \sin(\alpha))$$

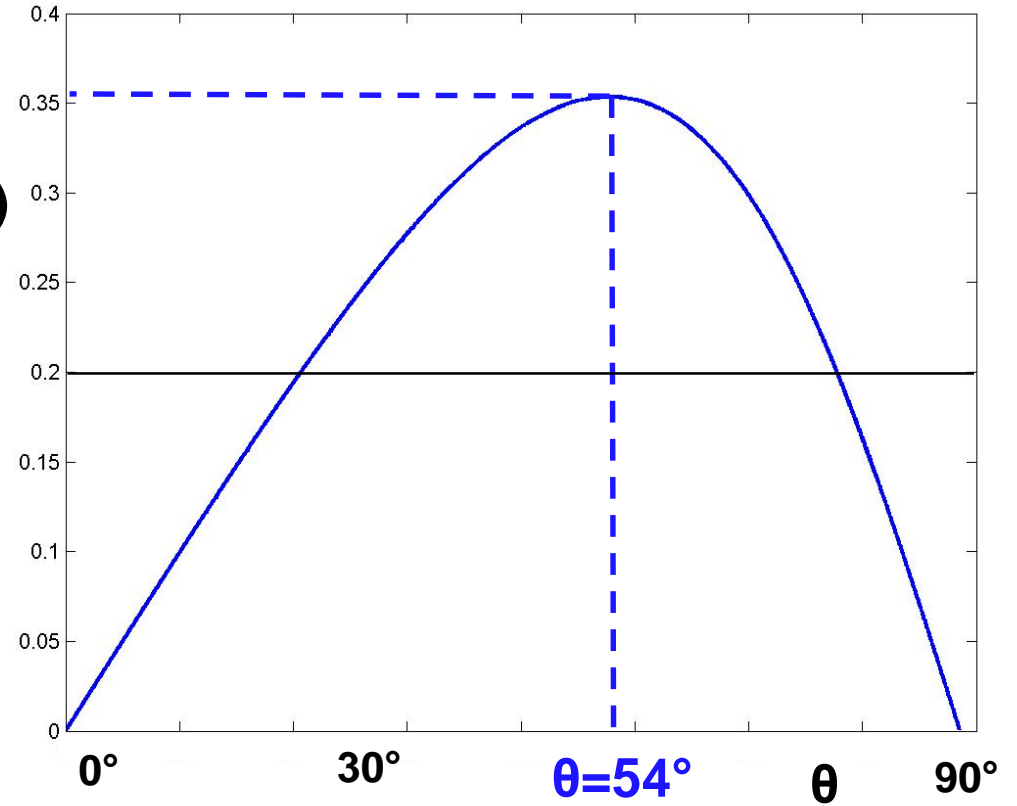
$$\Rightarrow \tan(\alpha) = \cos(\theta) \sin(\theta) / (1 + \cos^2(\theta))$$

# Gravity waves created by a ship

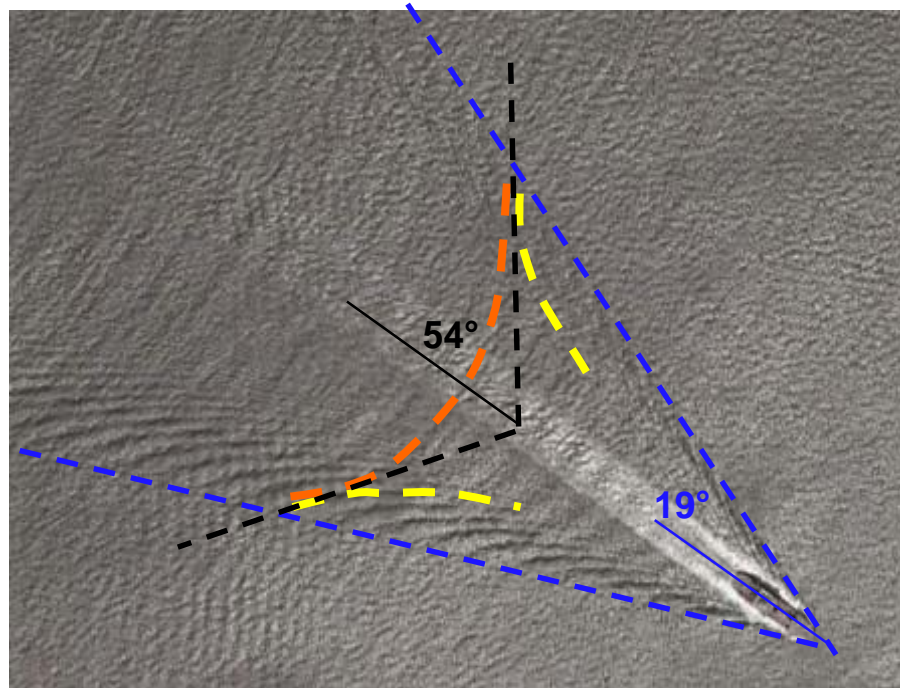


$\alpha = 19^\circ$

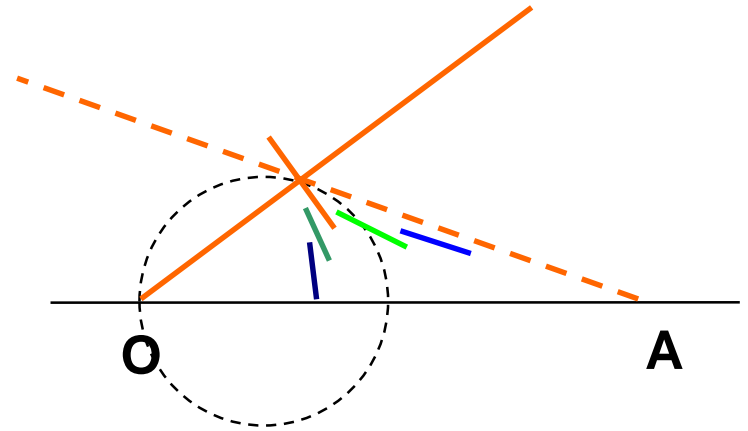
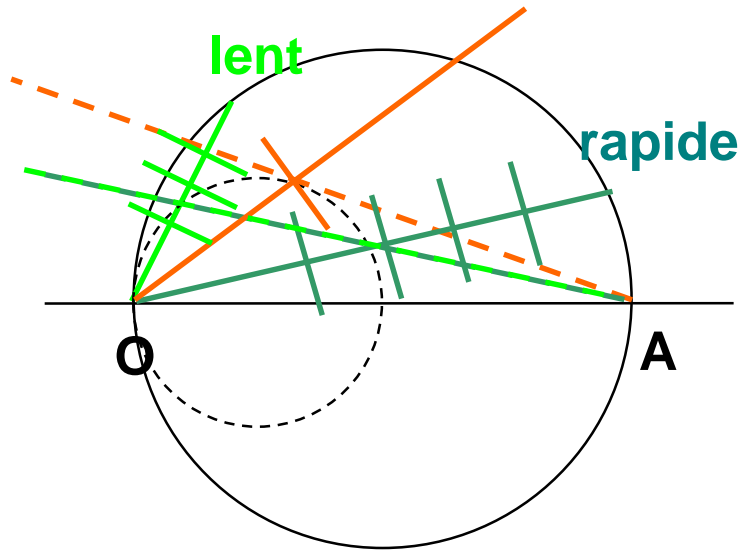
$\tan(\alpha)$



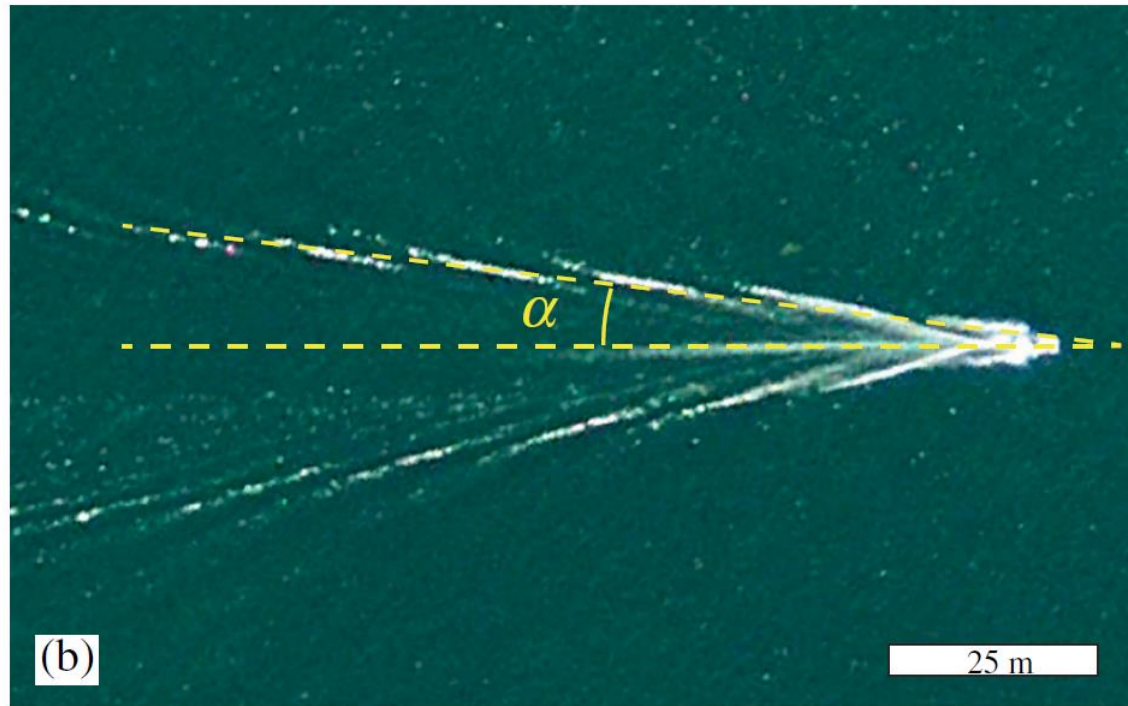
# Gravity waves created by a ship



# Gravity waves created by a ship

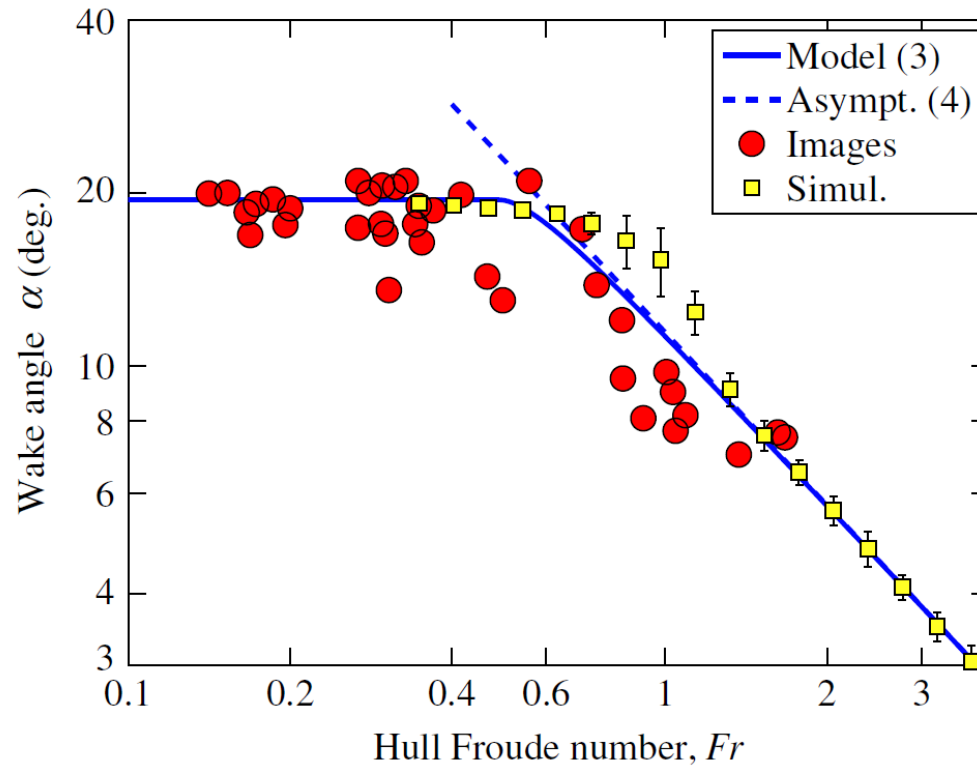


But observations show



Moisy and Rabaud 2013

# But observations show

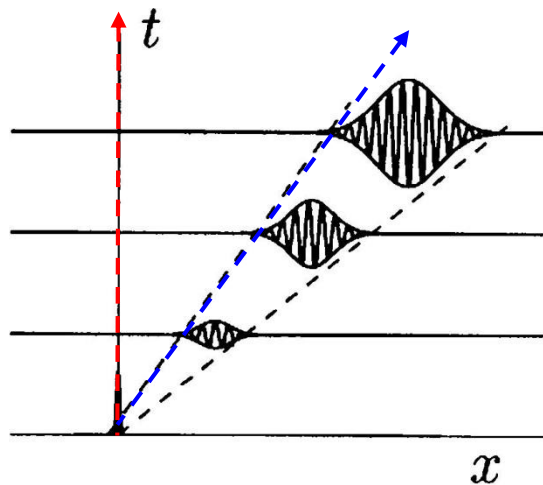


Moisy and Rabaud 2013

# Generalization: Spatio-temporal instability theory

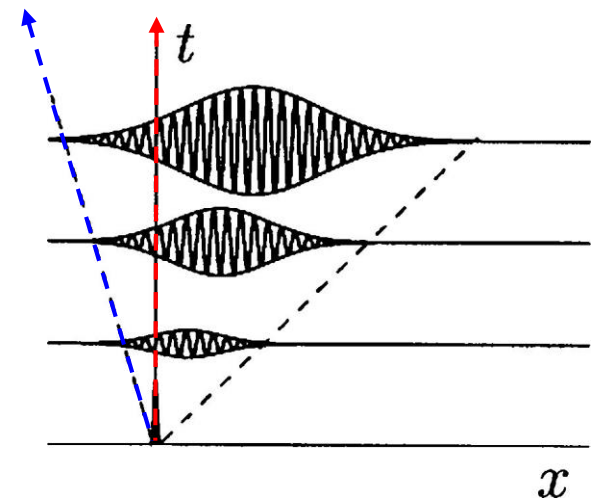
First find the zero group velocity wave:  $d\omega/dk=0 \Rightarrow (k_0, \omega_0)$   
and consider the sign of  $\text{Im}(\omega_0)$

👉 Convective instability



$$\text{Im}(\omega_0) < 0$$

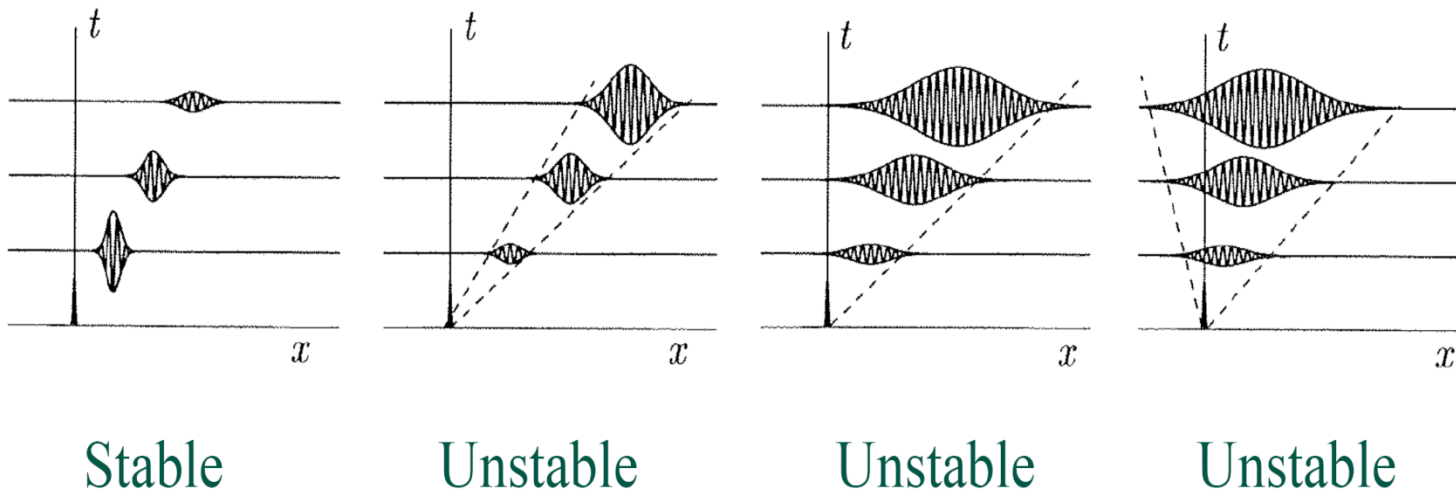
👉 Absolute instability



$$\text{Im}(\omega_0) > 0$$

# LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Green's function or impulse response



Briggs (1964) Bers (1983)  
Huerre and Monkewitz (1985)

## LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Linearly stable flow

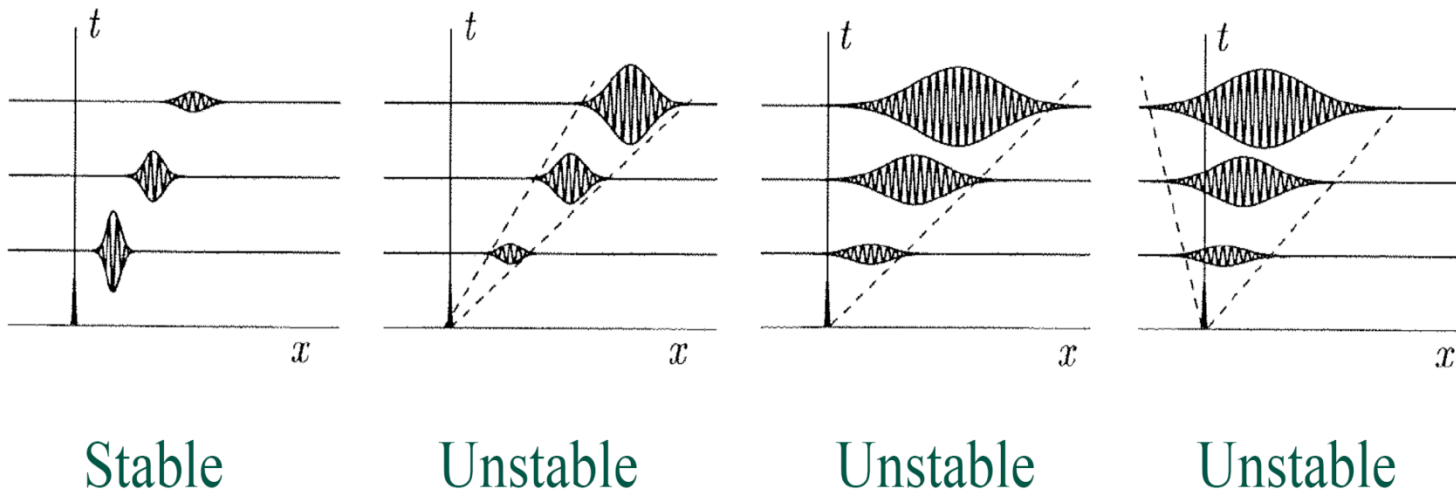
$$\lim_{t \rightarrow \infty} G(x, t) = 0 \quad \text{along all rays } x/t = \text{const.}$$

Linearly unstable flow

$$\lim_{t \rightarrow \infty} G(x, t) = \infty \quad \text{along at least one ray } x/t = \text{const.}$$

# LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Green's function or impulse response



Briggs (1964) Bers (1983)  
Huerre and Monkewitz (1985)

# LINEAR IMPULSE RESPONSE: ABSOLUTE/CONVECTIVE INSTABILITY

Convectively unstable flow

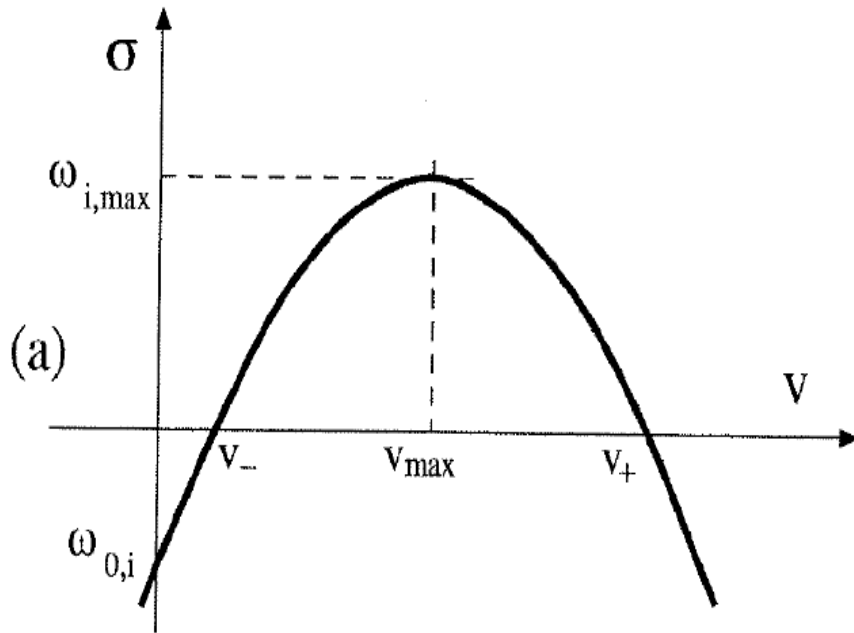
$$\lim_{t \rightarrow \infty} G(x, t) = 0 \quad \text{along the ray } x/t = 0$$

Absolutely unstable flow

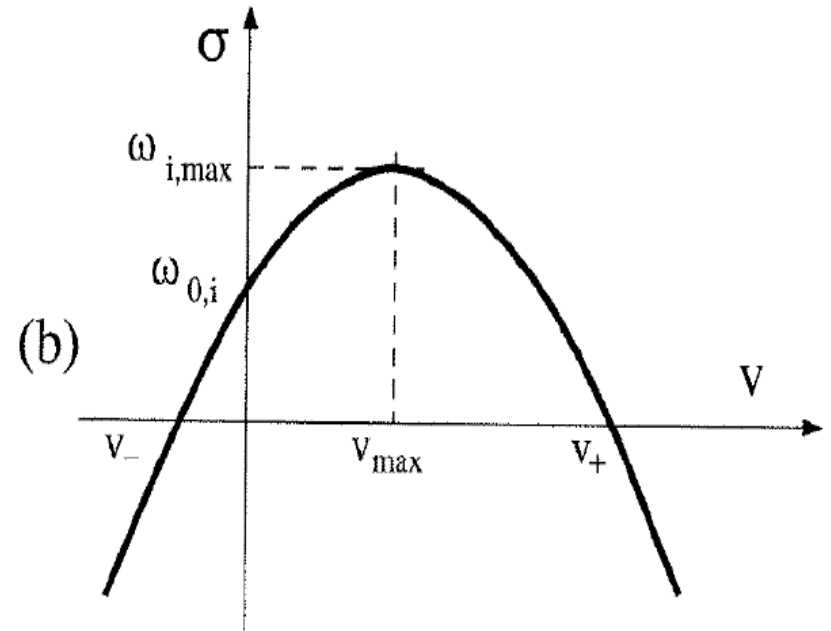
$$\lim_{t \rightarrow \infty} G(x, t) = \infty \quad \text{along the ray } x/t = 0$$

# ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

Temporal growth rate « at velocity  $v$  »



Convective instability



Absolute instability

# ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

## Important notions

Absolute wavenumber  $k_0$  and frequency  $\omega_0 = \omega(k_0)$  observed along ray  $v = 0$ , i.e. for a stationary observer, defined by

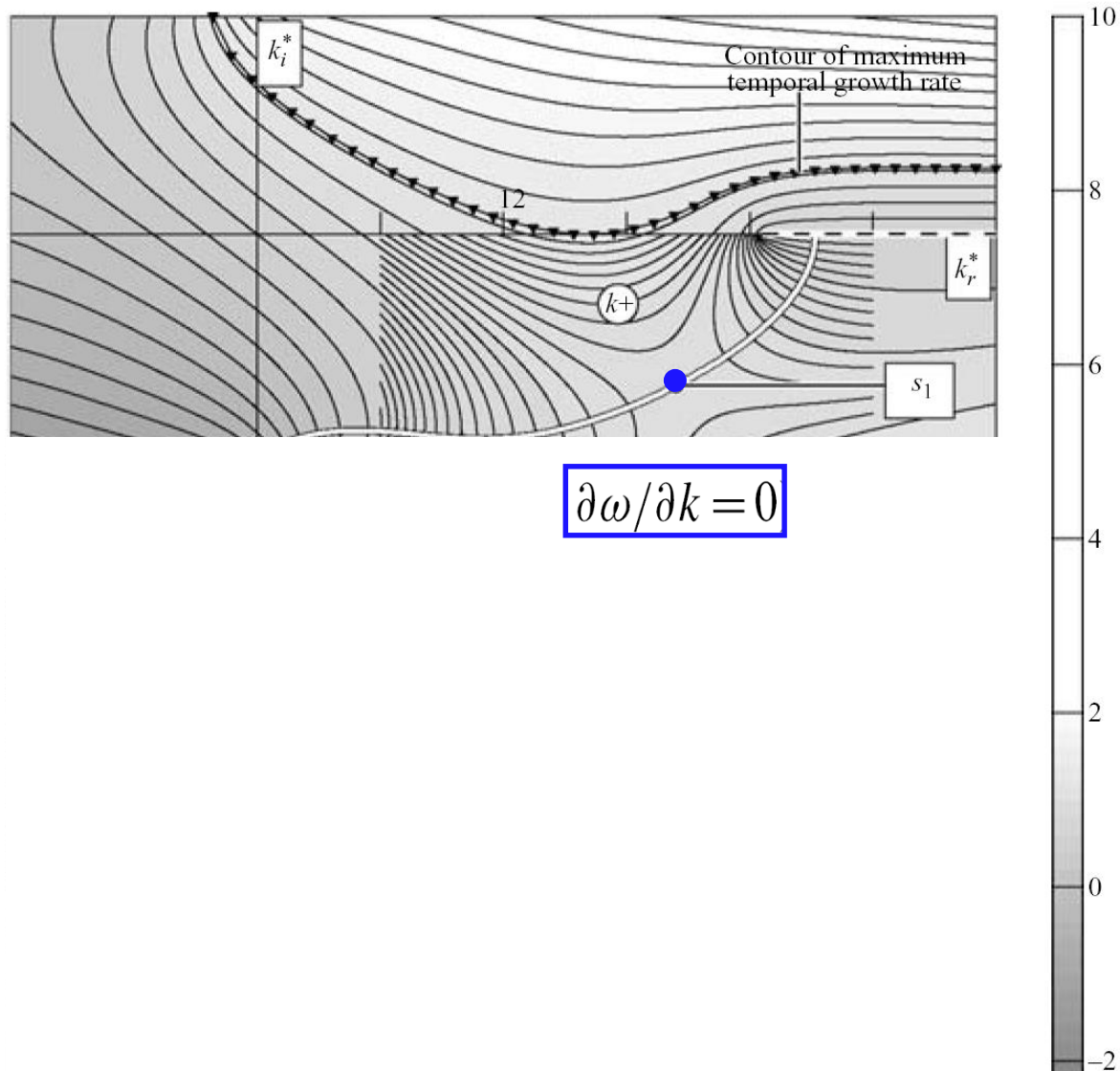
$$\frac{\partial \omega}{\partial k}(k_0) = 0$$

Absolute growth rate is

$$\sigma(0) = \omega_{0,i}$$

# Isovaleurs de $\omega_i$

Absolute frequency  $\omega_0$ : Saddle point condition



# ANALYSIS IN COMPLEX FOURIER SPACE: AU/CU CRITERION

## Instability criteria

$\omega_{i,max} < 0$                       linearly stable

$\omega_{i,max} > 0$                       linearly unstable

$\omega_{0,i} < 0$                       convectively unstable

$\omega_{0,i} > 0$                       absolutely unstable

# Hyperbolic tangent mixing layer

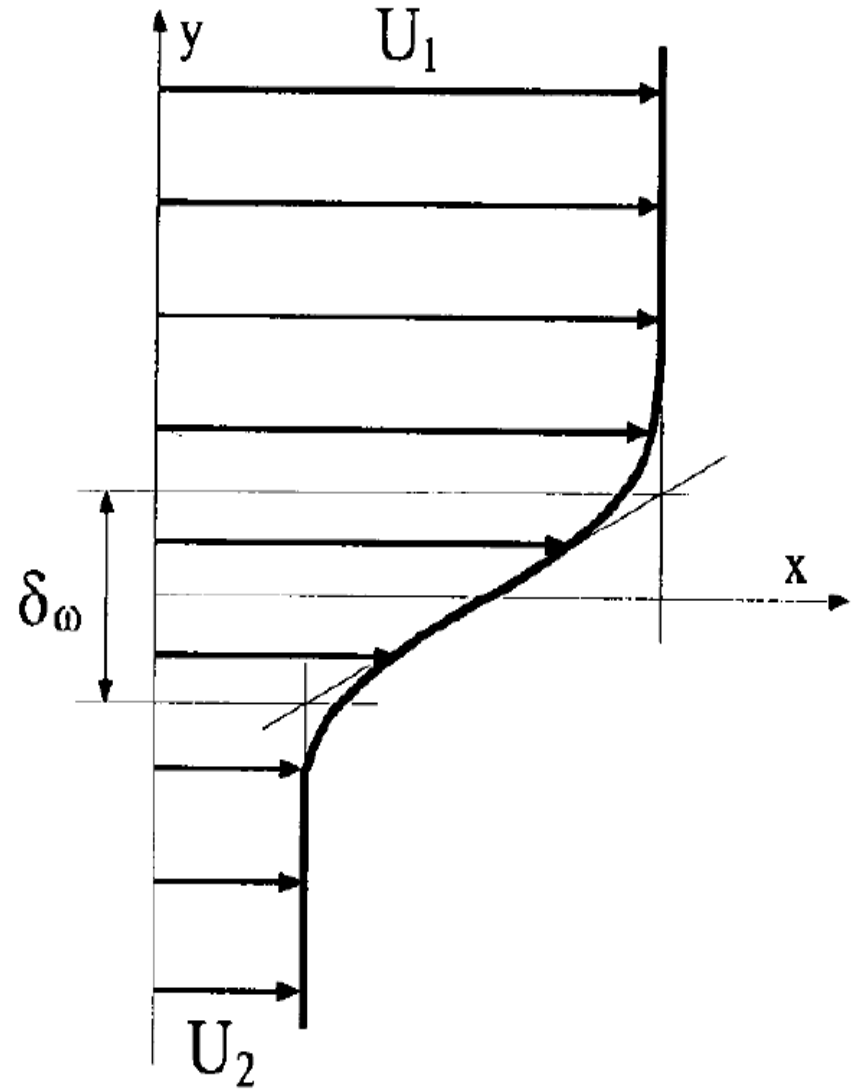
$$U(y) = \bar{U} + \frac{\Delta U}{2} \tanh\left(\frac{2y}{\delta_\omega}\right)$$

$$\delta_\omega(x) \equiv \frac{(U_1 - U_2)}{(dU/dy)_{\max}}$$

Velocity ratio

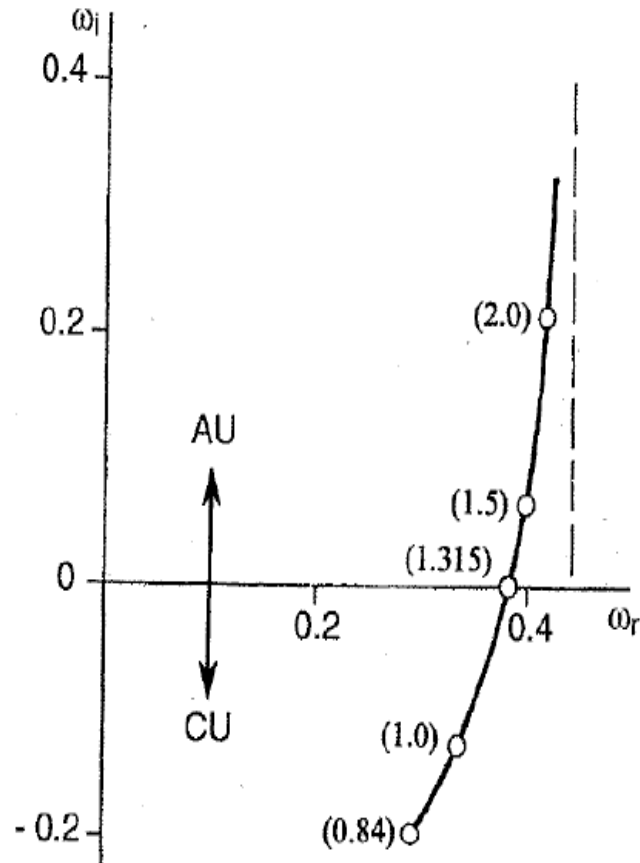
$$R \equiv \frac{U_1 - U_2}{U_1 + U_2} = \frac{\Delta U}{2\bar{U}}$$

$$U(y; R) = 1 + R \tanh y$$

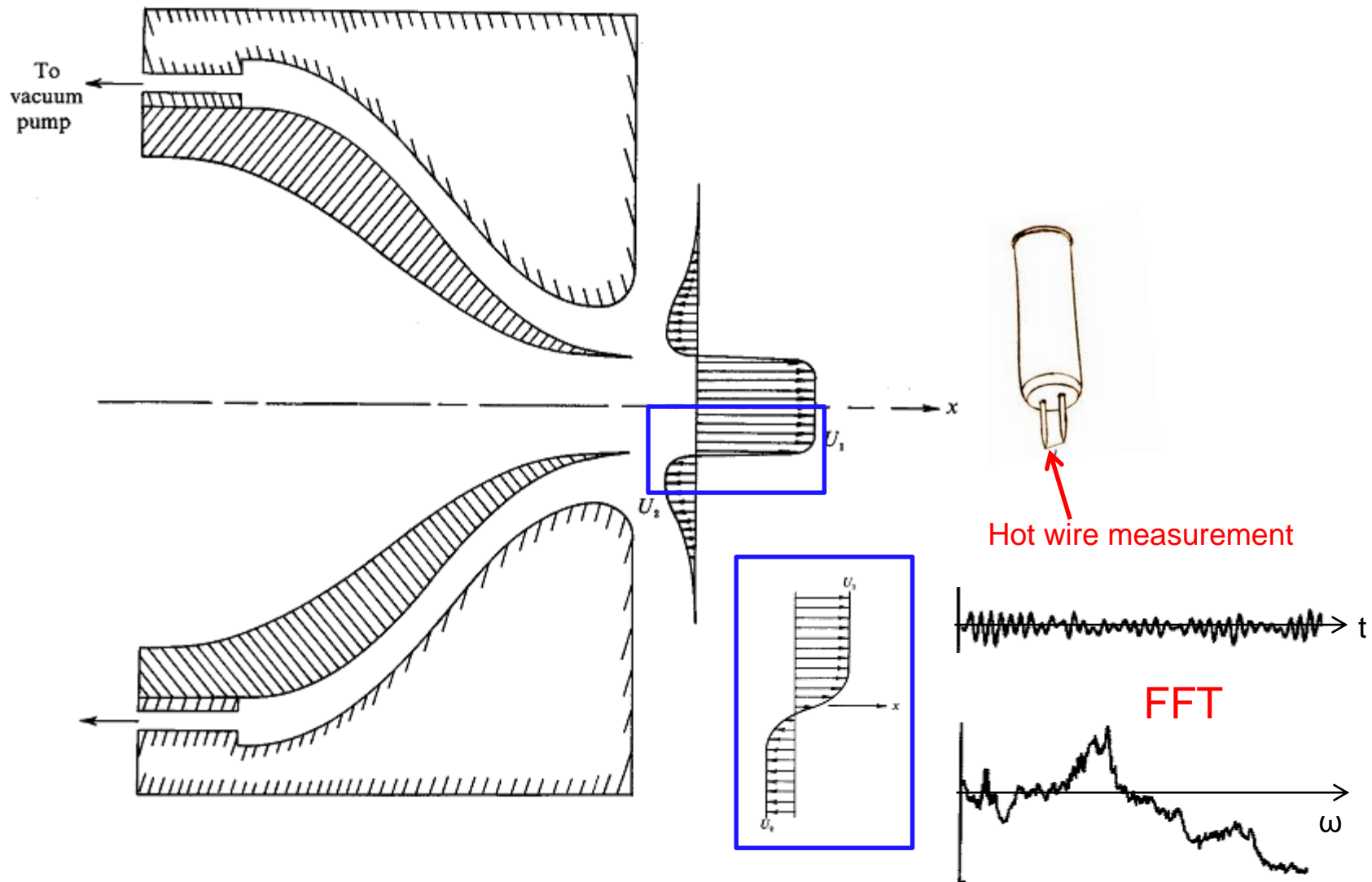


# APPLICATION TO MIXING LAYERS

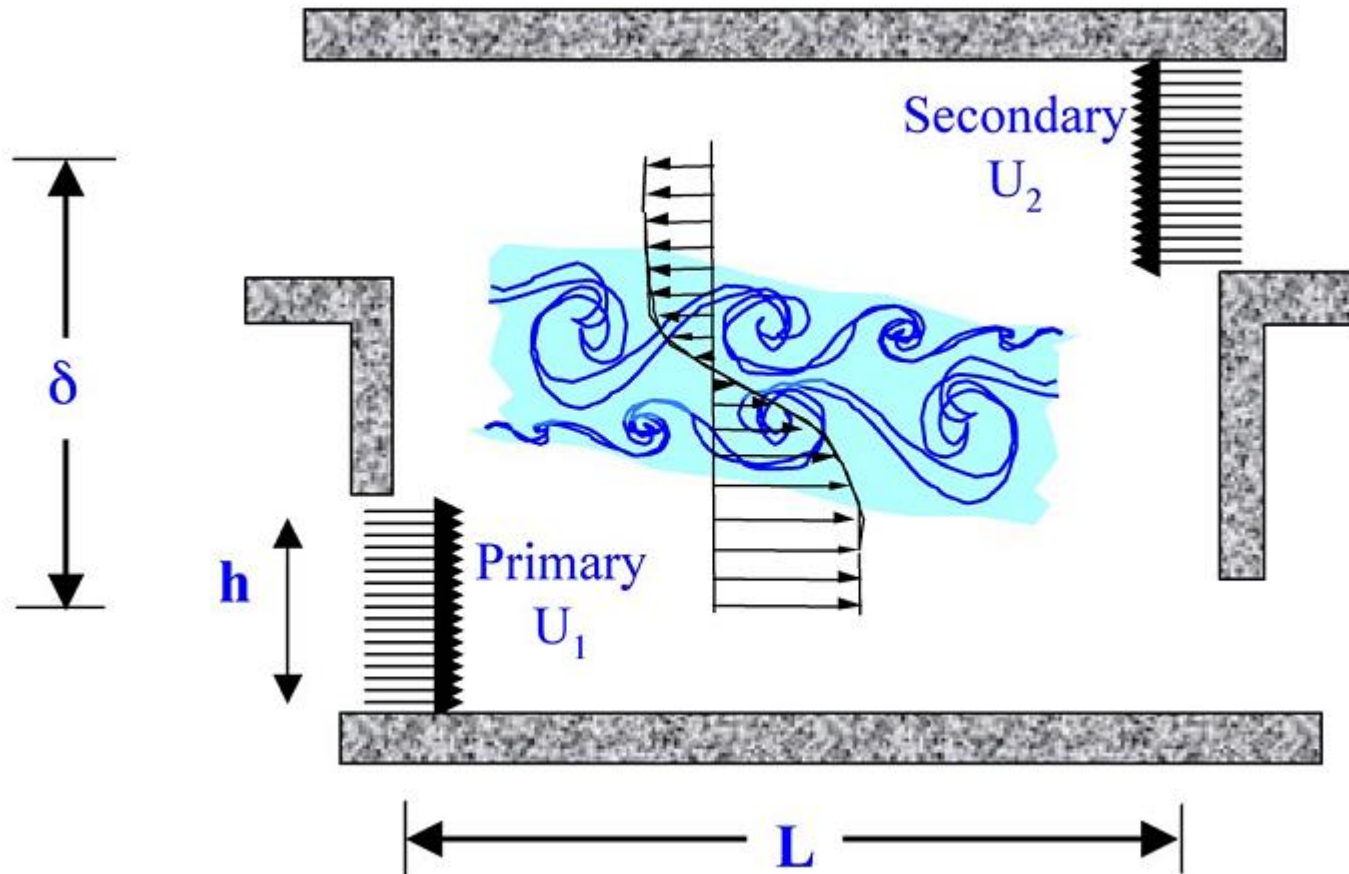
## Locus of complex absolute frequency



H.&Monkewitz (1985)

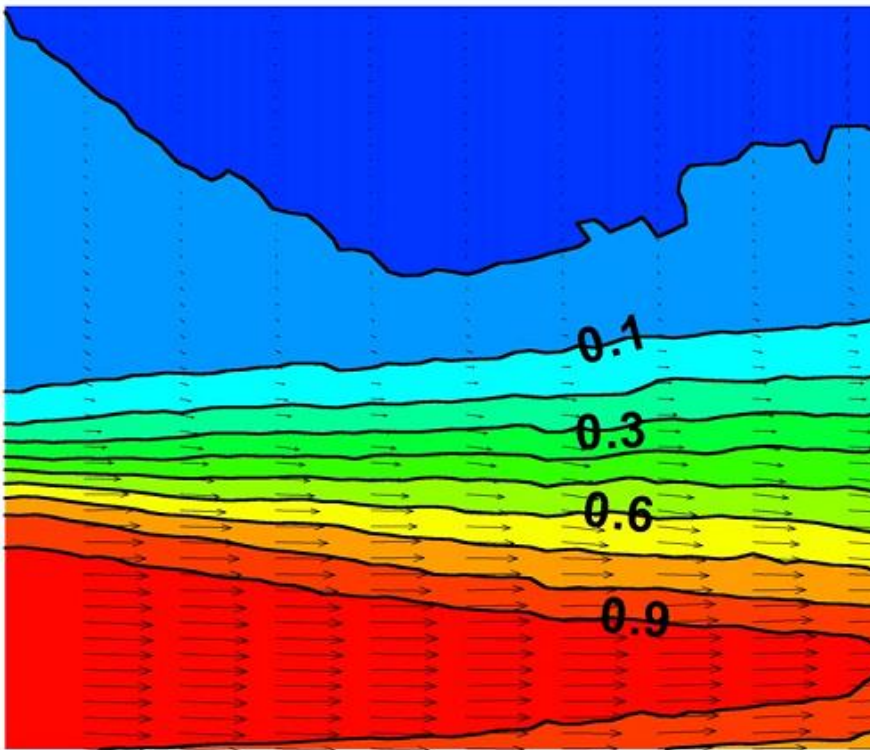


# Influence of countercurrent shear on turbulence level

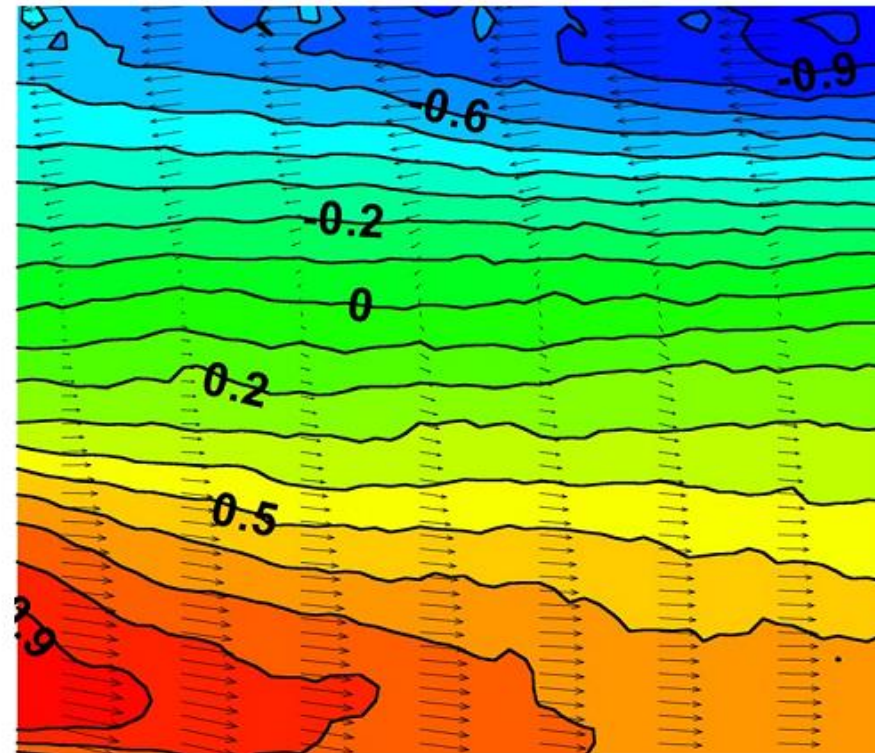


# Influence of countercurrent shear on turbulence level

Base flow



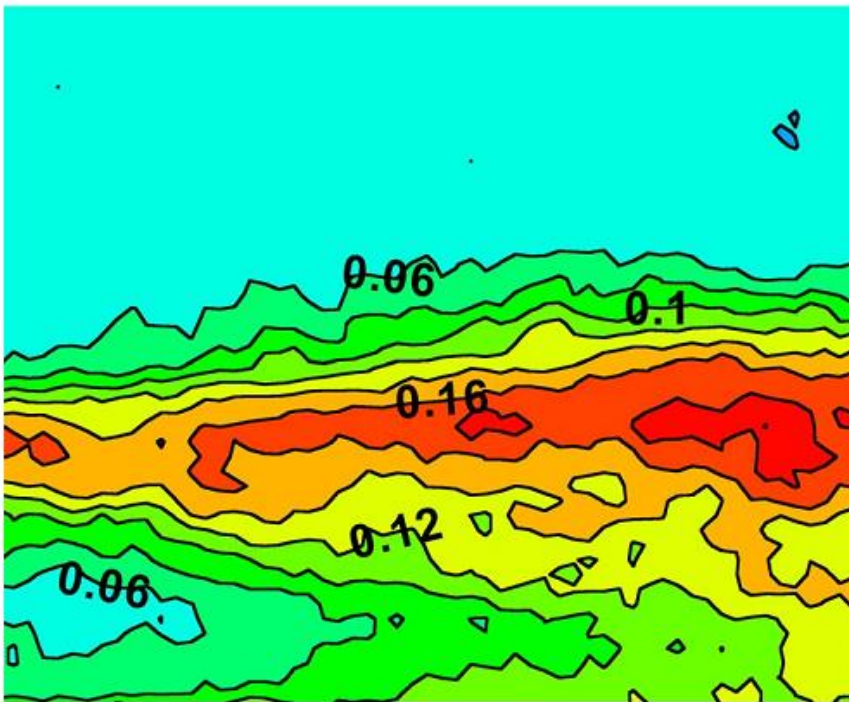
a) Single stream shear layer



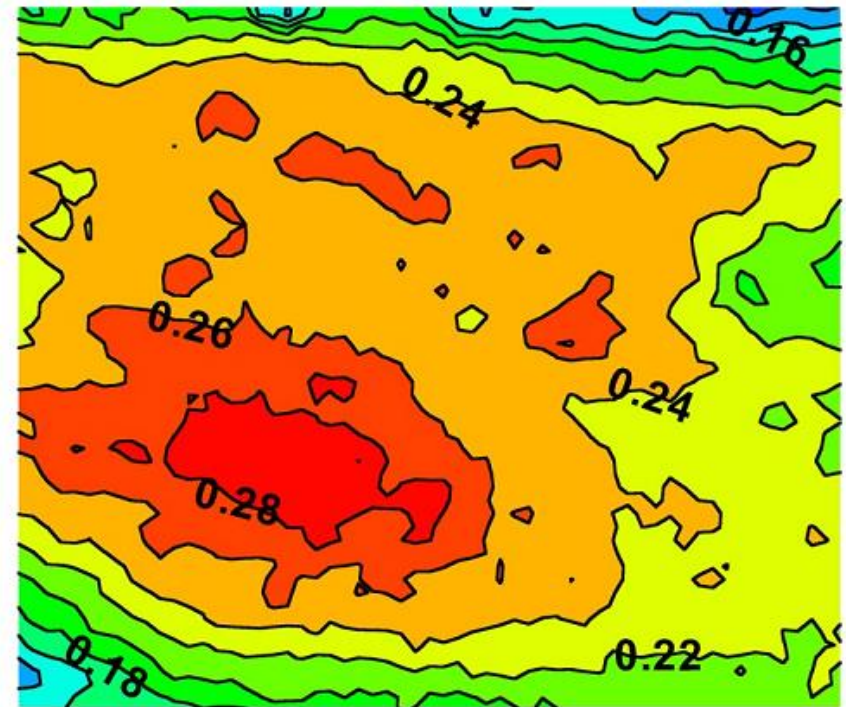
b) Countercurrent shear layer

# Influence of countercurrent shear on turbulence level

## Turbulence intensity

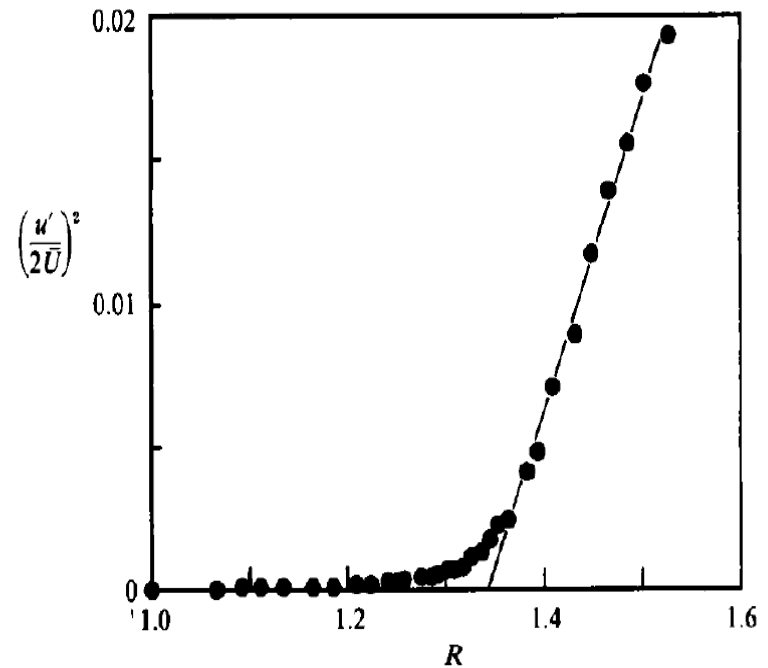
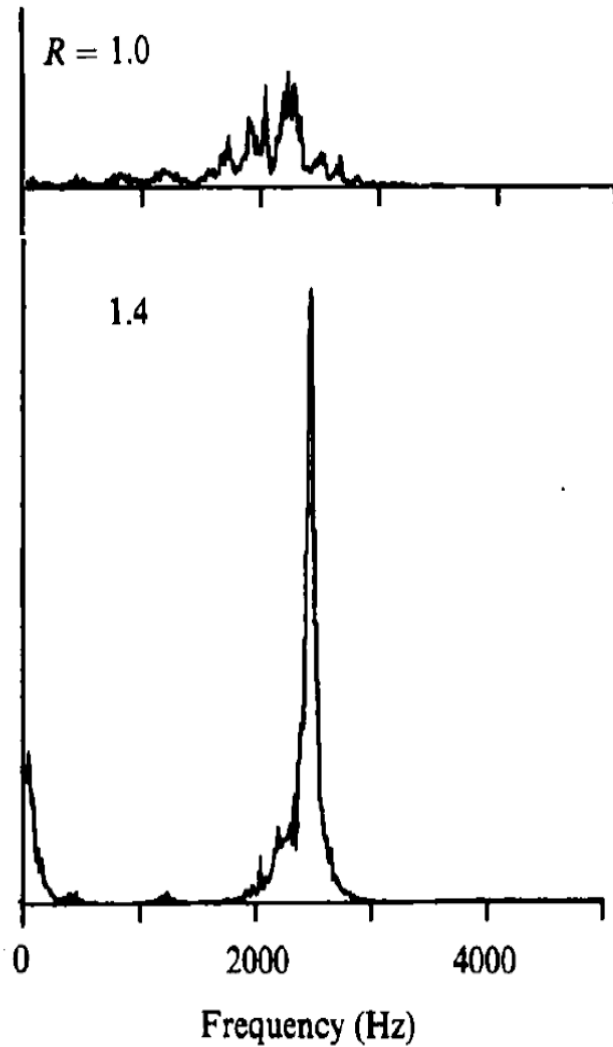


a) Single-stream shear layer



b) Countercurrent shear layer

# THE MIXING LAYER: SHIFT TO OSCILLATOR !



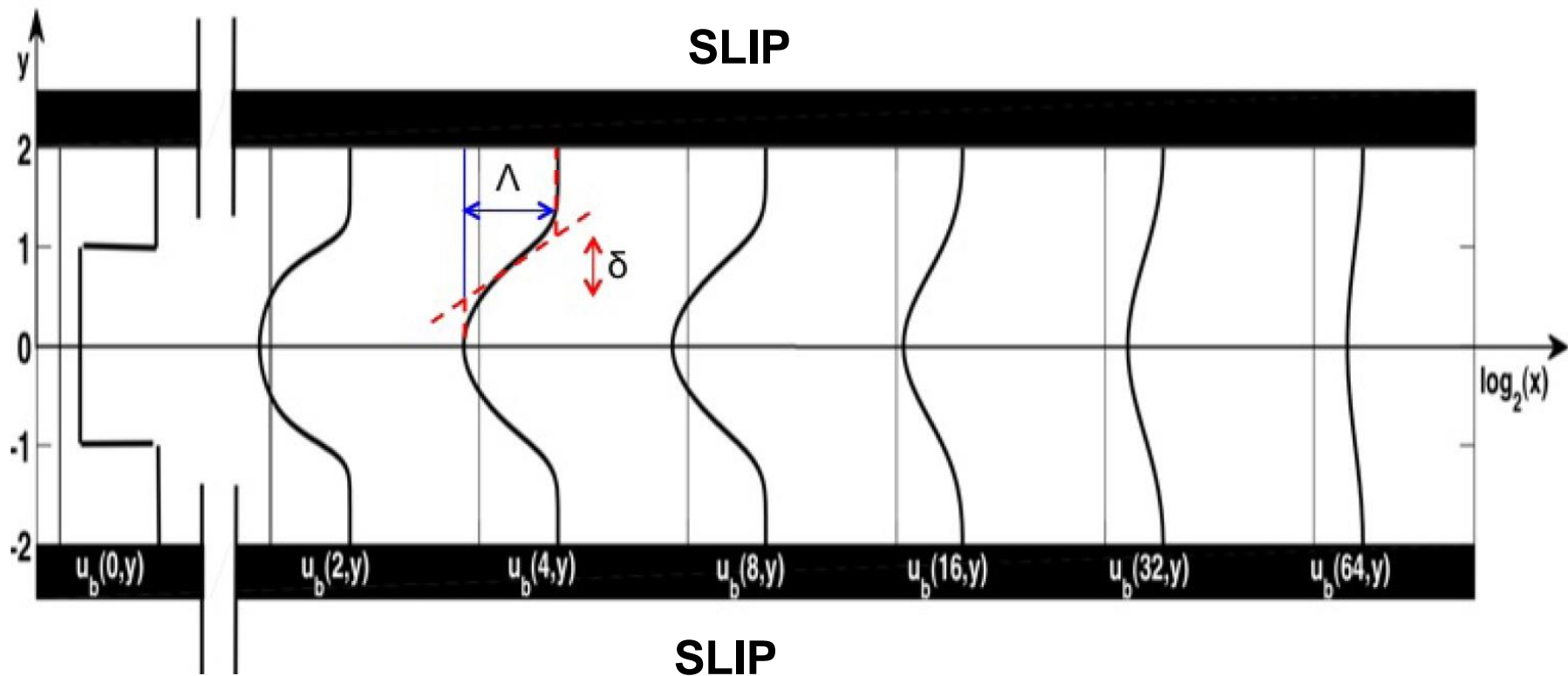
Strykowski & Niccum (1991)

# Direct Numerical Simulations with top-hat profile at inlet

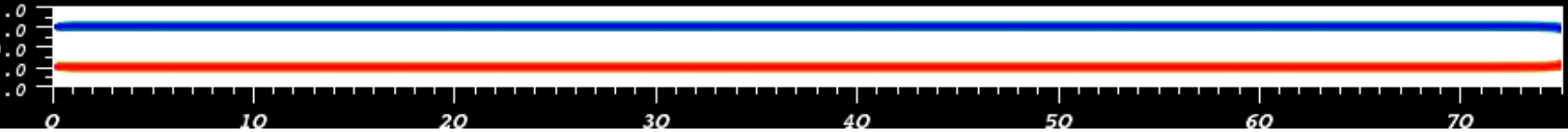
Viscous diffusion  $\longrightarrow$  Non-parallel flow

■  $\Lambda_{loc} = (U_{max} - U_{min}) / (U_{max} + U_{min})$

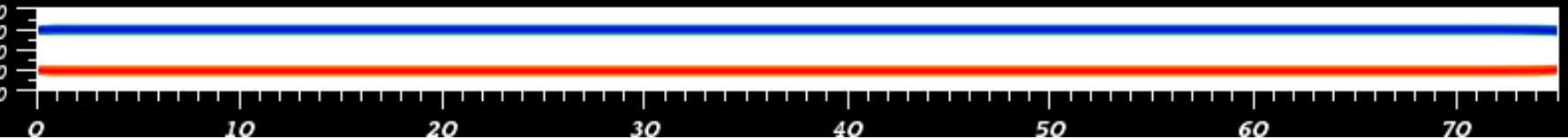
■  $\delta = (U_{max} - U_{min}) / (|dU/dy|_{max})$



# Vorticity field: $Re = 100$ , $h = 1$



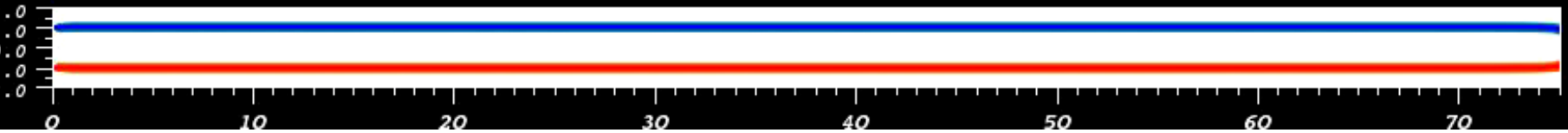
$$\Lambda = -0.739$$



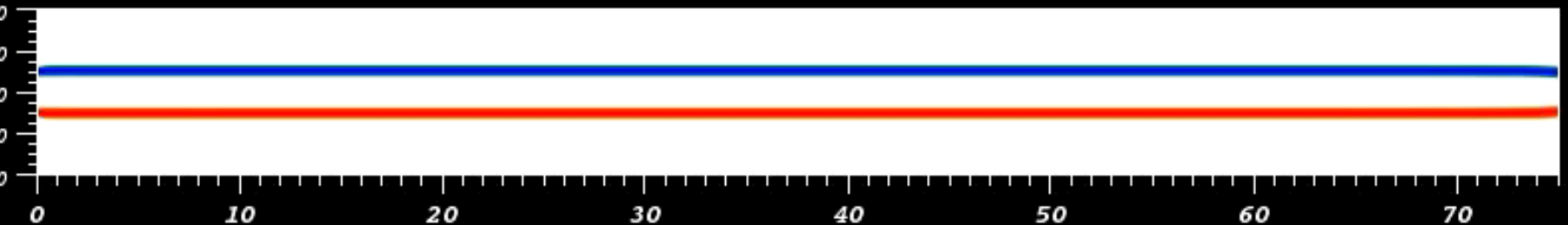
$$\Lambda = -0.667$$

An increase in  $\Lambda$  (more coflow) advects the perturbation

Vorticity field:  $Re = 100$ ,  $\Lambda = -0.739$



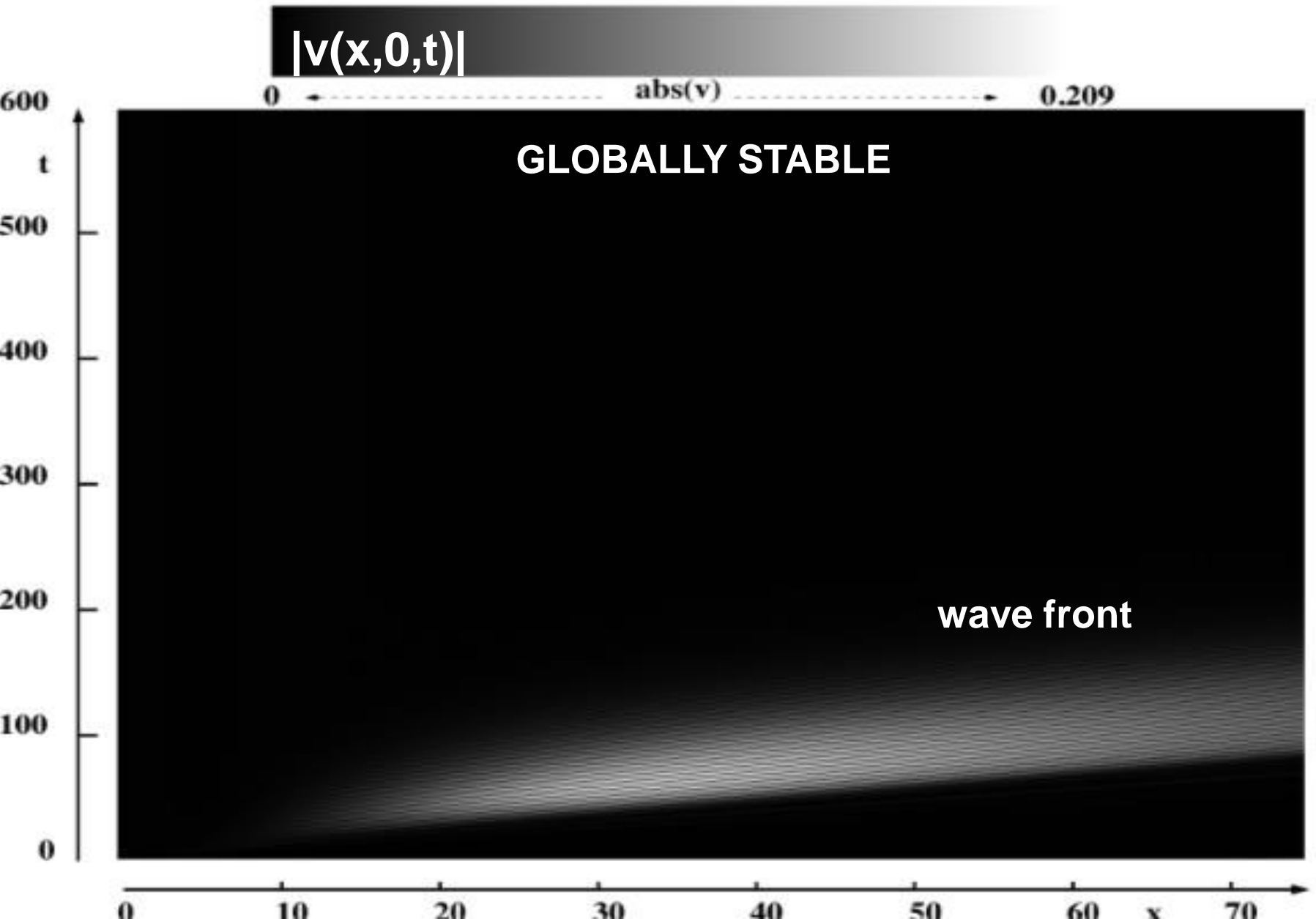
$h=1$



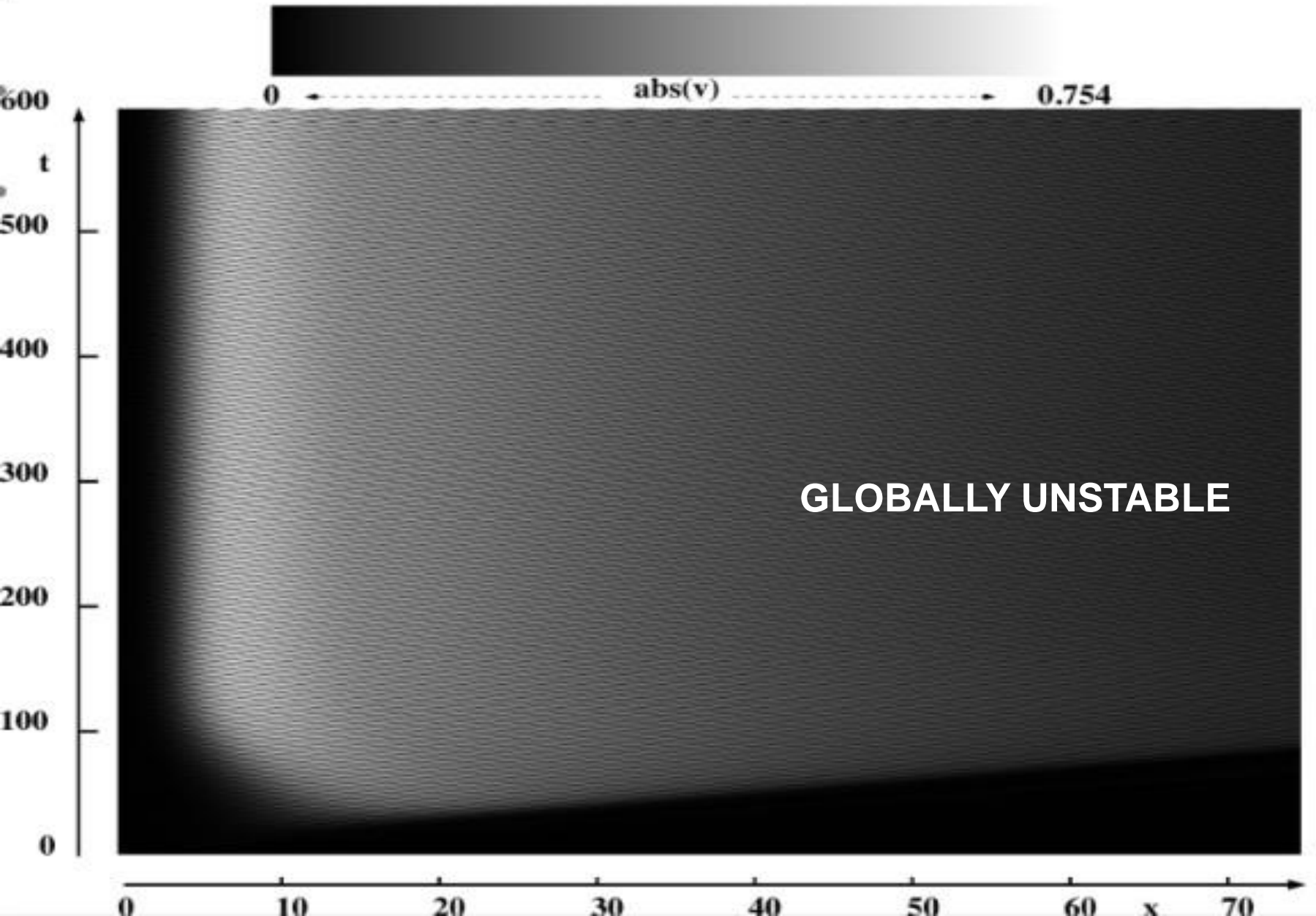
$h=3$

Destabilizing influence of confinement!

# Spatio-temporal diagram, $h=1$ and $\Lambda = -0.667$



# Spatio-temporal diagram, $h=1$ and $\Lambda = -0.739$



# THE BLUFF BODY WAKE: A TYPICAL FLOW OSCILLATOR



$Re = 140$   
Periodic  
flow

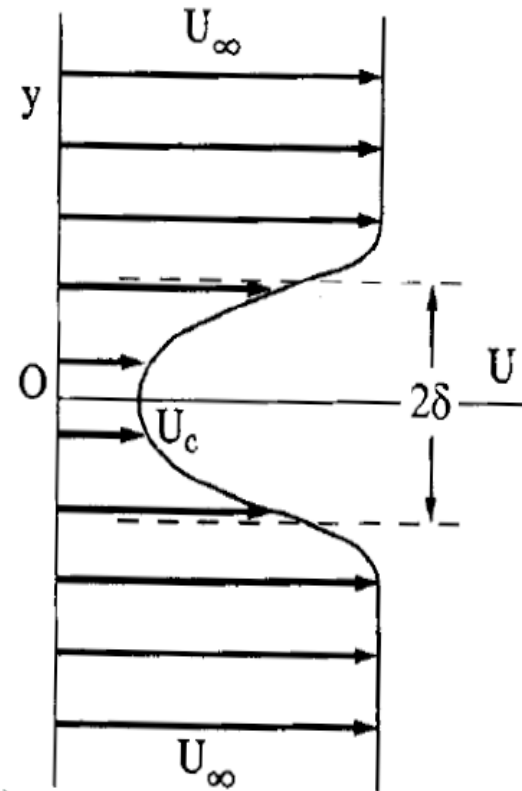
Taneda (1982)

# ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Family of wake profiles

$$U(y) = U_\infty + (U_c - U_\infty) U_1\left(\frac{y}{\delta}; N\right)$$

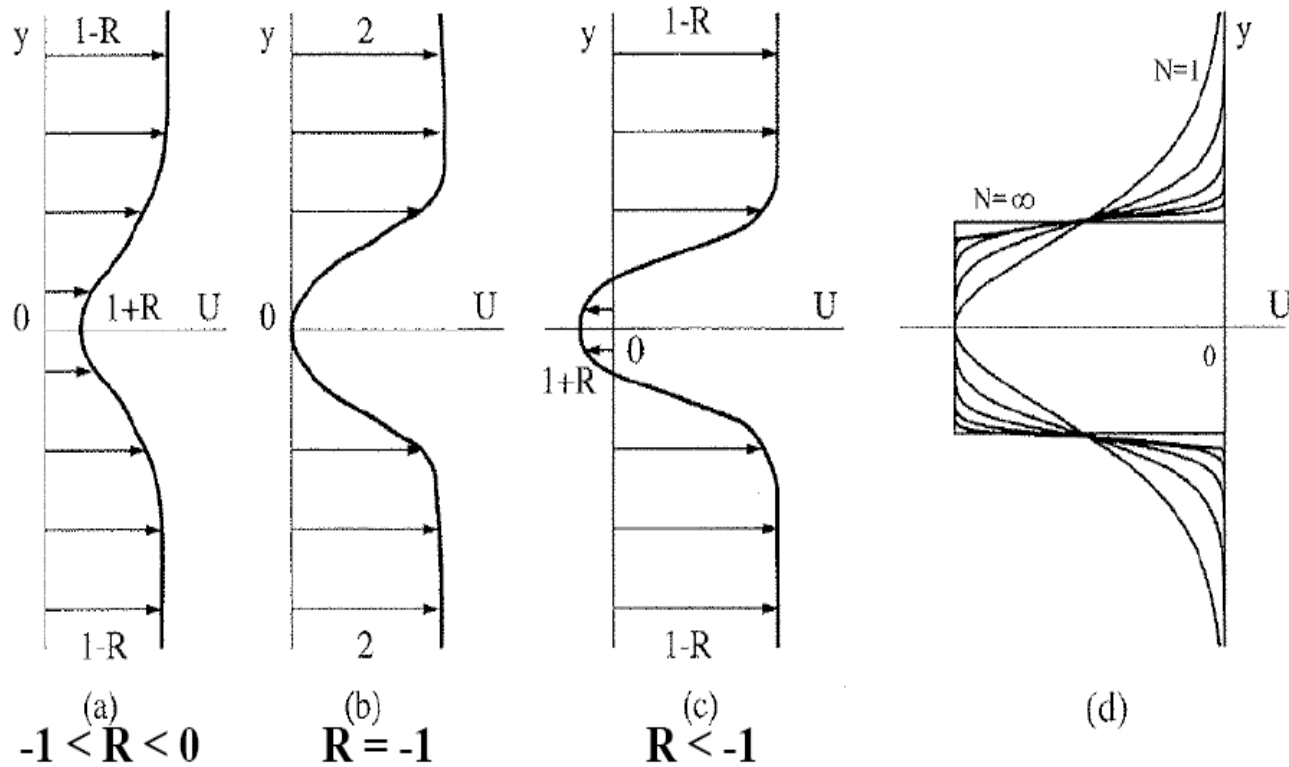
$$U_1(\xi; N) = [1 + \sinh^{2N}\{\xi \sinh^{-1}(1)\}]^{-1}$$



Monkewitz (1988)

# ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

## Family of wake profiles



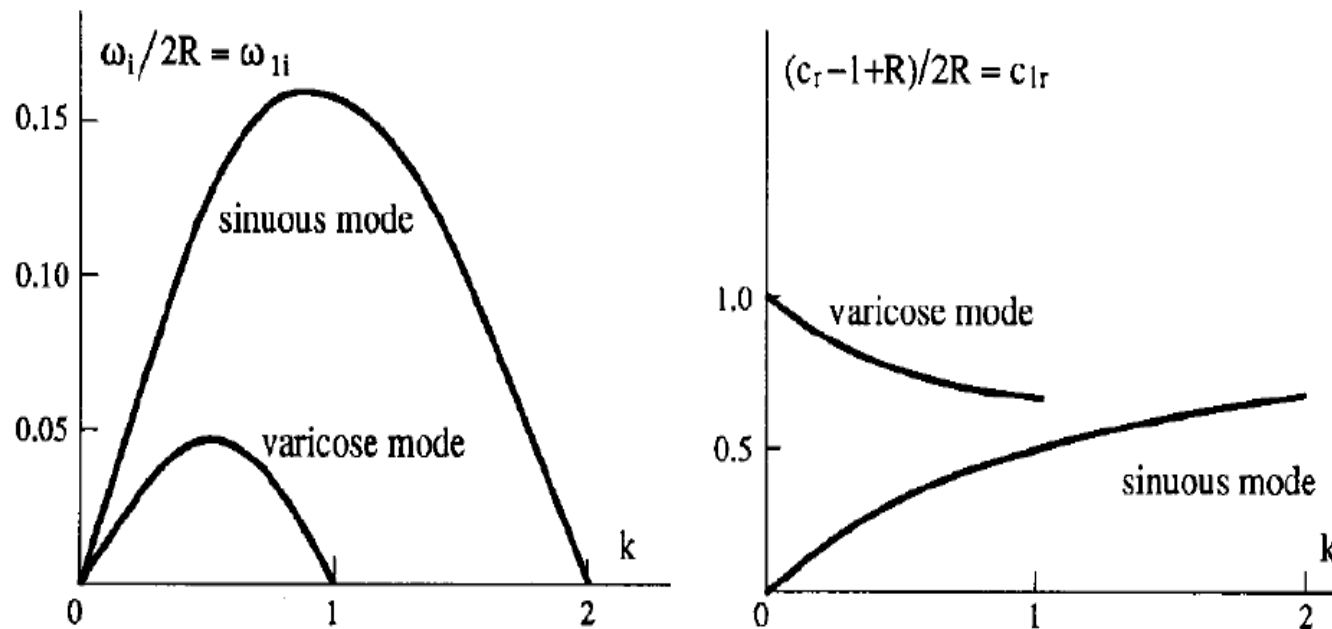
Effect of velocity ratio R

Effect of N

# ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL 2D PARALLEL FLOW CONCEPTS

$\text{sech}^2 y$  wake

Temporal approach

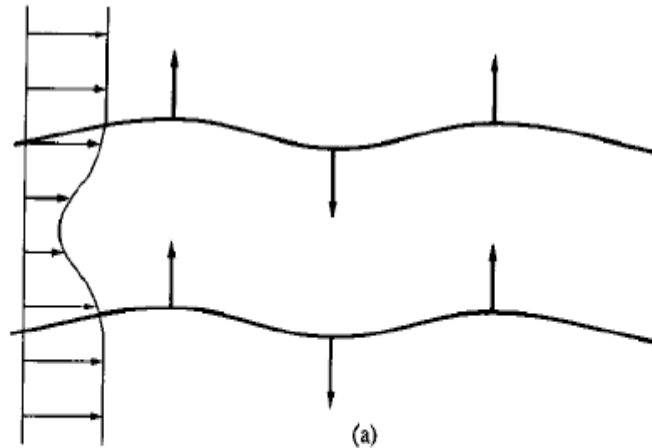


Betchov & Criminale (1966)

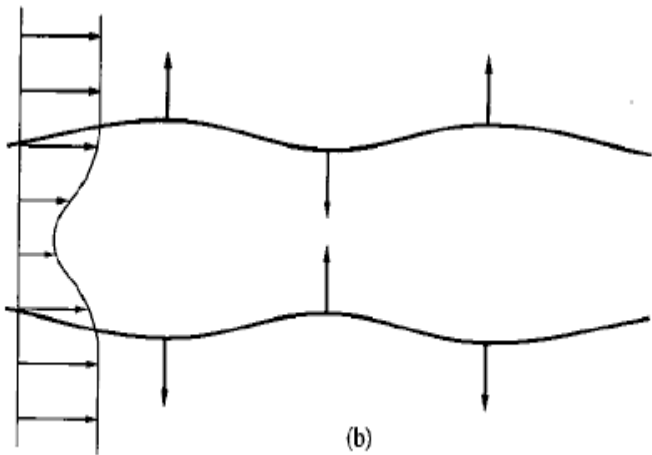
# 2D PARALLEL FLOW CONCEPTS

$\text{sech}^2 y$  wake

## Sinuuous and varicose modes



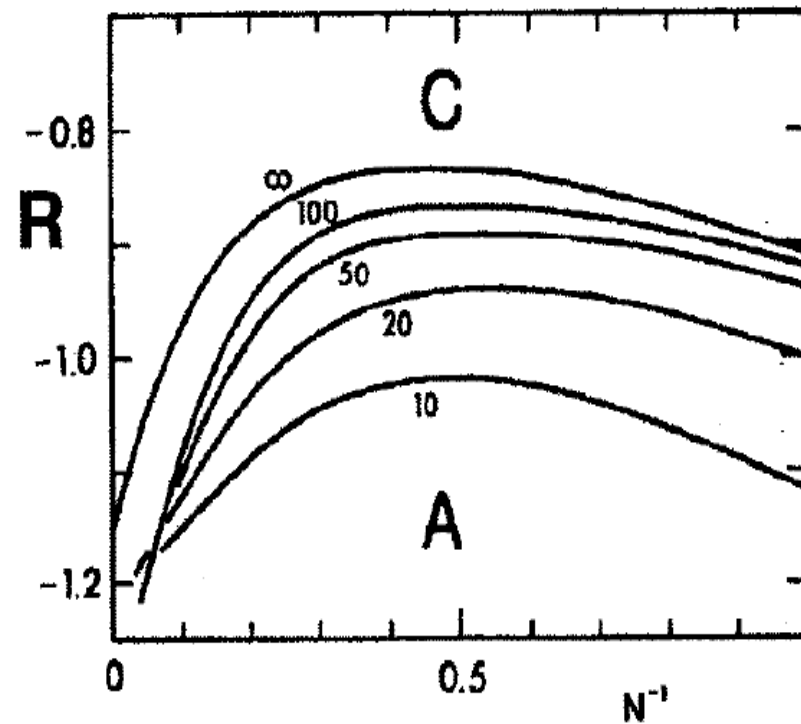
sinuous



varicose

# ABSOLUTE/CONVECTIVE INSTABILITY IN PARALLEL WAKES

Effect of steepness, velocity ratio and Reynolds number



Monkewitz (1988)

## LOCAL INSTABILITY BEHAVIOR OF CYLINDER WAKE

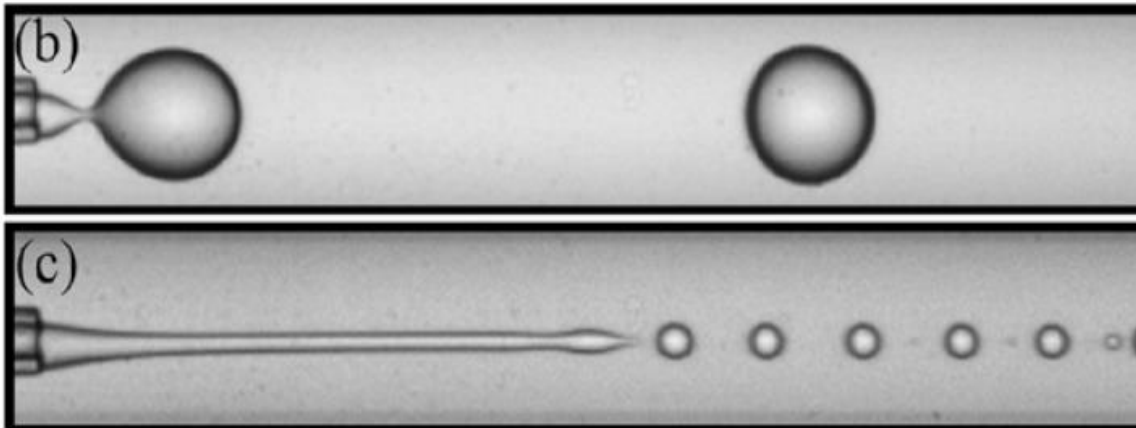
$$5 < Re < 25$$

**Convective instability**

$$25 < Re < 48.5$$

**Absolute instability**

# Dripping/Jetting transition linked to absolute/convective transition?



Absolutely unstable

Convectively unstable

Guillot et al. (2008), Utada et al. (2008)

## 5. Dispersion relation

$$\omega = Uk \pm \sqrt{\frac{\gamma k^2}{\rho} \left( k^2 - \frac{1}{R_0^2} \right) \frac{I'_0(kR)}{I_0(kR)}}$$

- **Unstable** if there exists one  $\omega$ ,  $\text{Im}(\omega) > 0$  at  $k < 1/R_0$
- **Neutral** if for all  $\omega$ ,  $\text{Im}(\omega) = 0$  at  $k > 1/R_0$
- **Stable (or damped)** if for all  $\omega$ ,  $\text{Im}(\omega) < 0$ :

The flow considered is not damped, we have neglected dissipation by neglecting viscosity

# Destabilisation d'un jet

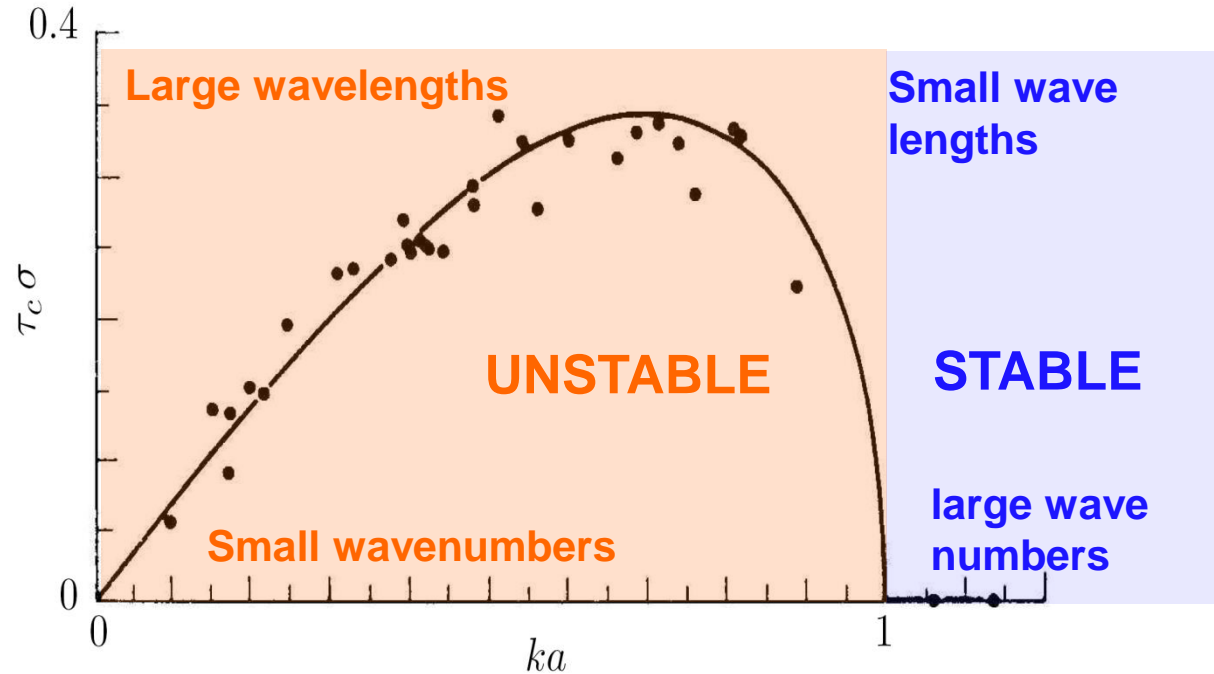
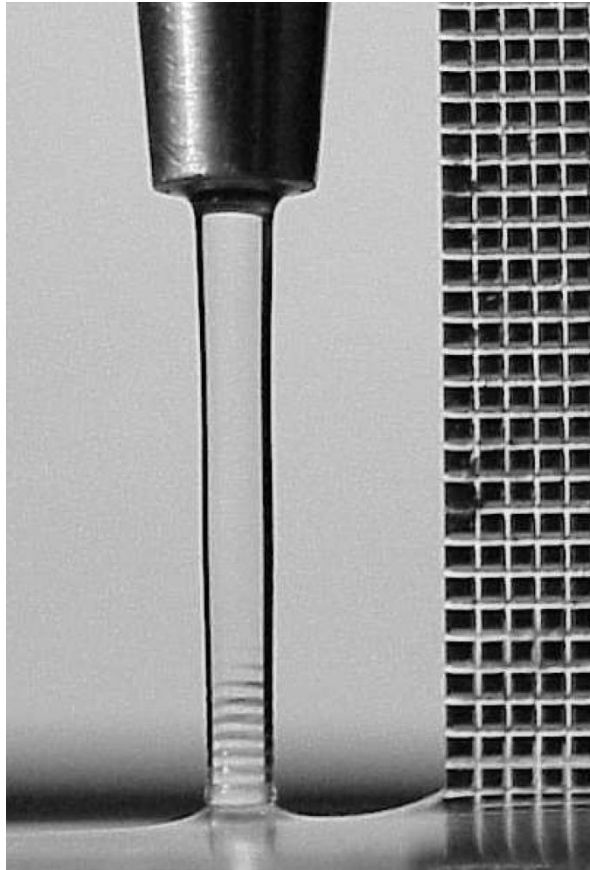


FIG. 2.10 – Taux de croissance  $\tau_c \sigma$ , avec  $\tau_c = \sqrt{\rho a^3 / \gamma}$ , de l'instabilité d'un filet fluide non visqueux, et points expérimentaux. D'après (Drazin & Reid 2004).



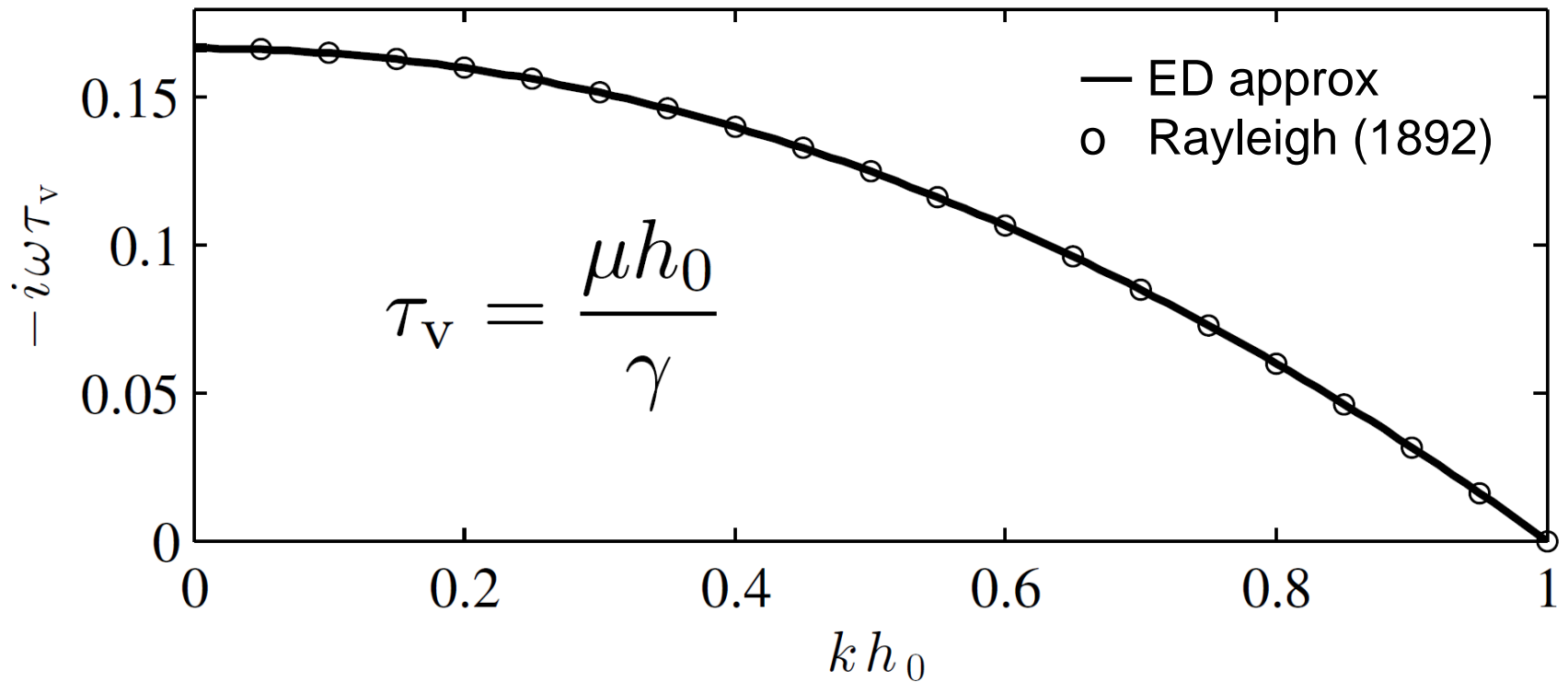
Surface tension is **destabilizing** as a consequence of the **radial** curvature  
Surface tension is **stabilizing** as a consequence of the **axial** curvature

Oh  $\gg$  1 – Viscosity dominated  
A very similar calculation yields  
(Rayleigh )

$$\omega = kU_0 + i \frac{\gamma}{2\mu R_0} \frac{(1 - (kR_0)^2)}{(kR_0)^2 (I_0(kR_0)^2 / I_1(kR_0)^2) - (1 + (kR_0)^2)}$$

$$Oh = \frac{\mu}{\sqrt{\rho\gamma R_0}}$$

# Eggers and Dupont (1994) equations Oh>>



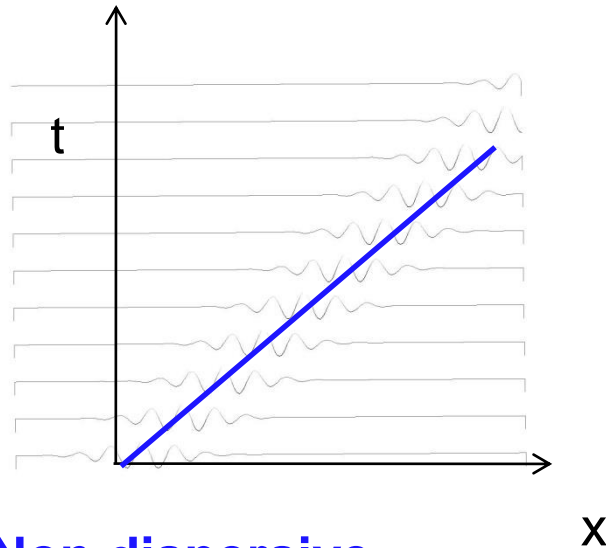
$$\omega = u_0 k + \frac{1}{6\tau_v} i \left( 1 - (kh_0)^2 \right)$$

# Nondimensionalize

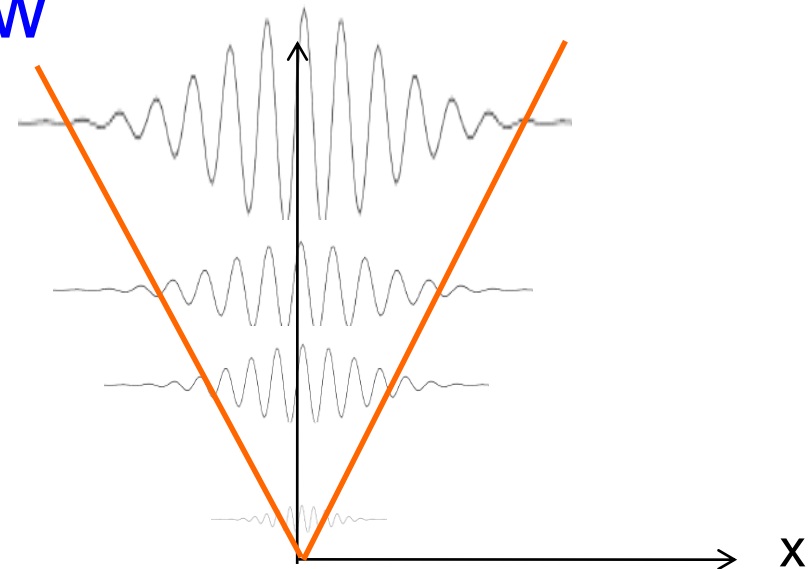
$$\omega = \text{Ca} k + \frac{i}{6}(1 - k^2)$$

$\text{Ca} = \mu u_0 / \gamma$  is the capillary number (a reduced velocity)

The instability waves simultaneously travel and grow



**Non dispersive propagation at the velocity of the interface**



**Propagating symmetric growing wavepacket**

## Find $k_0$

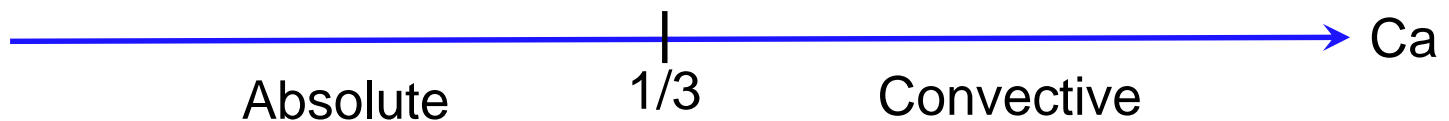
$$\omega = \text{Ca} k + \frac{i}{6}(1 - k^2)$$

$$\frac{d\omega}{dk}(k_0) = 0 \quad \Rightarrow \quad k_0 = -3i\text{Ca}$$

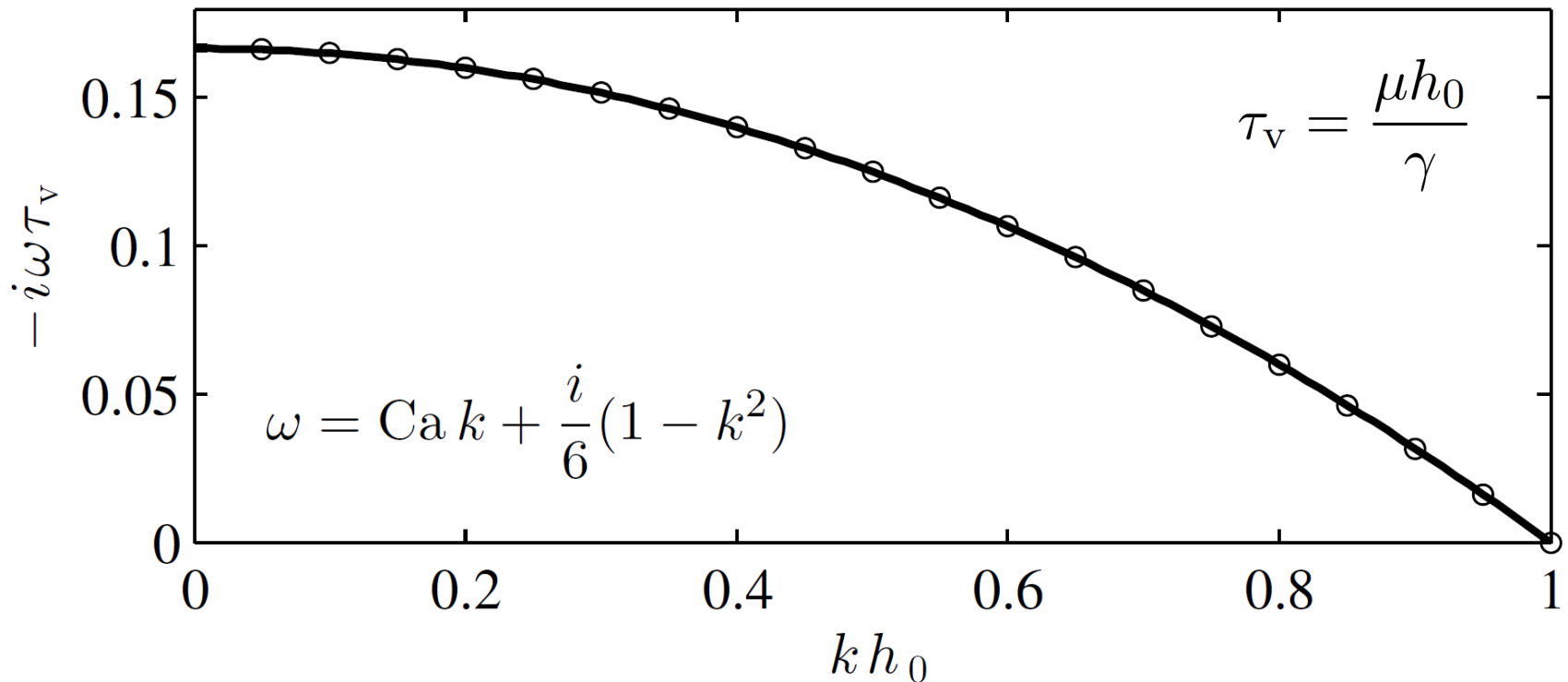
## Evaluate $\omega_0$

$$\omega_0 = \omega(k_0) = -\frac{3}{2}i\text{Ca}^2 + \frac{i}{6}$$

$$\text{Im}(\omega_0) = (1 - 9\text{Ca}^2)/6$$



Consider the governing equation of the viscous jet and impose boundary conditions



$$\frac{\partial \eta}{\partial t} = -Ca \frac{\partial \eta}{\partial z} + \frac{1}{6} \eta(z, t) + \frac{1}{6} \frac{\partial^2 \eta}{\partial z^2}$$

$$\eta(0, t) = \eta(l, t) = 0$$

Find eigenmodes  $\eta(x, t) = \bar{\eta}(z)e^{\lambda t}$

$$\frac{\partial \eta}{\partial t} = -\text{Ca} \frac{\partial \eta}{\partial z} + \frac{1}{6} \eta(z, t) + \frac{1}{6} \frac{\partial^2 \eta}{\partial z^2}$$

$$\eta(0, t) = \eta(l, t) = 0$$

$$\left( \lambda - \frac{1}{6} \right) \bar{\eta}(z) = -\text{Ca} \frac{d\bar{\eta}}{dz} + \frac{1}{6} \frac{d^2 \bar{\eta}}{dz^2}$$

$$\bar{\eta}(0) = \bar{\eta}(l) = 0$$

Fundamental solutions  $\bar{\eta}(z) = \hat{\eta} e^{\alpha z}$

$$\alpha^2 - 6\text{Ca}\alpha + 1 - 6\lambda = 0$$

$$\alpha_{1,2} = 3\text{Ca} \pm \frac{1}{2} \sqrt{36\text{Ca}^2 + 24\lambda - 4}$$

## Fundamental solutions $\bar{\eta}(z) = \hat{\eta}e^{\alpha z}$

$$\alpha^2 - 6\text{Ca}\alpha + 1 - 6\lambda = 0$$

$$\alpha_{1,2} = 3\text{Ca} \pm \frac{1}{2}\sqrt{36\text{Ca}^2 + 24\lambda - 4}$$

## Impose boundary conditions

$$\begin{bmatrix} 1 & 1 \\ e^{\alpha_1 l} & e^{\alpha_2 l} \end{bmatrix} \begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow e^{\alpha_2 l} - e^{\alpha_1 l} = 0$$

$$\Rightarrow \alpha_1 - \alpha_2 = i\frac{2\pi m}{l} \quad (\text{m integer})$$

$$\sqrt{36\text{Ca}^2 + 24\lambda_m - 4} = i\frac{2\pi m}{l}$$

Find eigenmodes  $\eta(x, t) = \bar{\eta}(z)e^{\lambda t}$

$$\lambda_m = -\frac{3}{2}\text{Ca}^2 + \frac{1}{6} - \frac{m^2\pi^2}{6l^2} = \underbrace{-i\omega_0}_{\text{circled}} - \frac{m^2\pi^2}{6l^2}$$

$>0$  if  $\text{Im}(\omega_0) > 0$

absolute frequency  $\omega(k_0)$

The flow including boundary conditions is globally unstable only if it is locally absolutely unstable