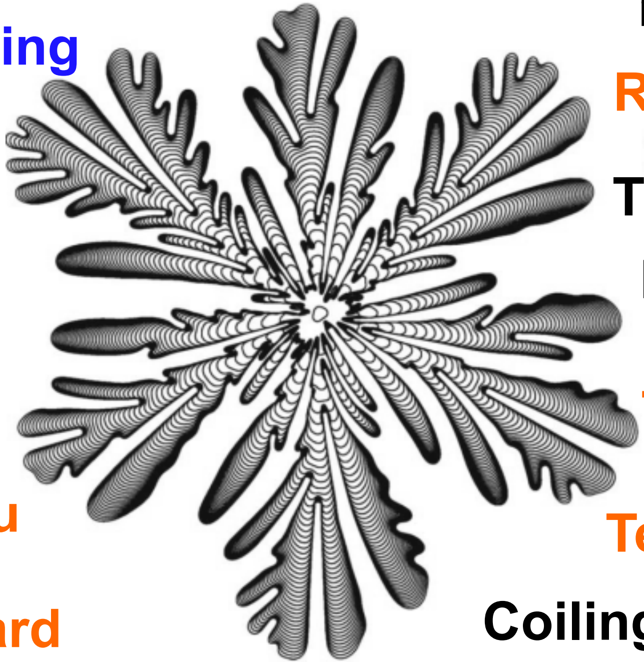
Most flows are unstable...



Vortex shedding
Saffman-Taylor
Flow separation
Tollmien-Schlichting
Rayleigh-Taylor
Lift-up and Streaks
Traffic waves
Meandering instability
Gravito-capillary waves
Taylor-Couette
Rayleigh-Plateau
Tearing instability
Coiling instability
Rayleigh-Benard
Benard-Marangoni
Kelvin-Helmholtz

2D PARALLEL FLOW CONCEPTS

Dispersion relation

2D vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi=\frac{1}{Re}\nabla^4\Psi$$

Basic flow + perturbation

$$\Psi(x,t)=\int U(y)dy+\psi(x,y,t)$$

Linear vorticity equation

$$\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial \psi}{\partial x}=\frac{1}{Re}\nabla^4\psi$$

2D PARALLEL FLOW CONCEPTS

Dispersion relation

Normal mode decomposition

$$\psi(x,y,t) = \mathcal{R}e\left\{\phi(y)\, e^{i(kx-\omega t)}\right\}$$

Orr-Sommerfeld equation

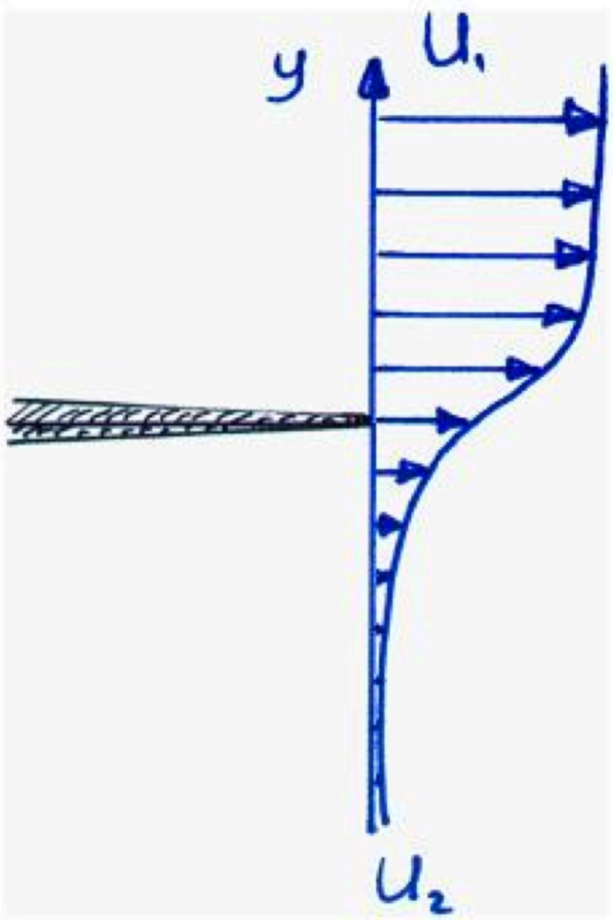
$$[U(y)-c][\phi'' - k^2\phi] - U''(y)\,\phi = \frac{1}{i\,k\,Re}\left(\frac{d^2}{dy^2} - k^2\right)^2 \phi$$

$$\phi(y) \Rightarrow 0 \quad \text{at } y = \pm\infty$$

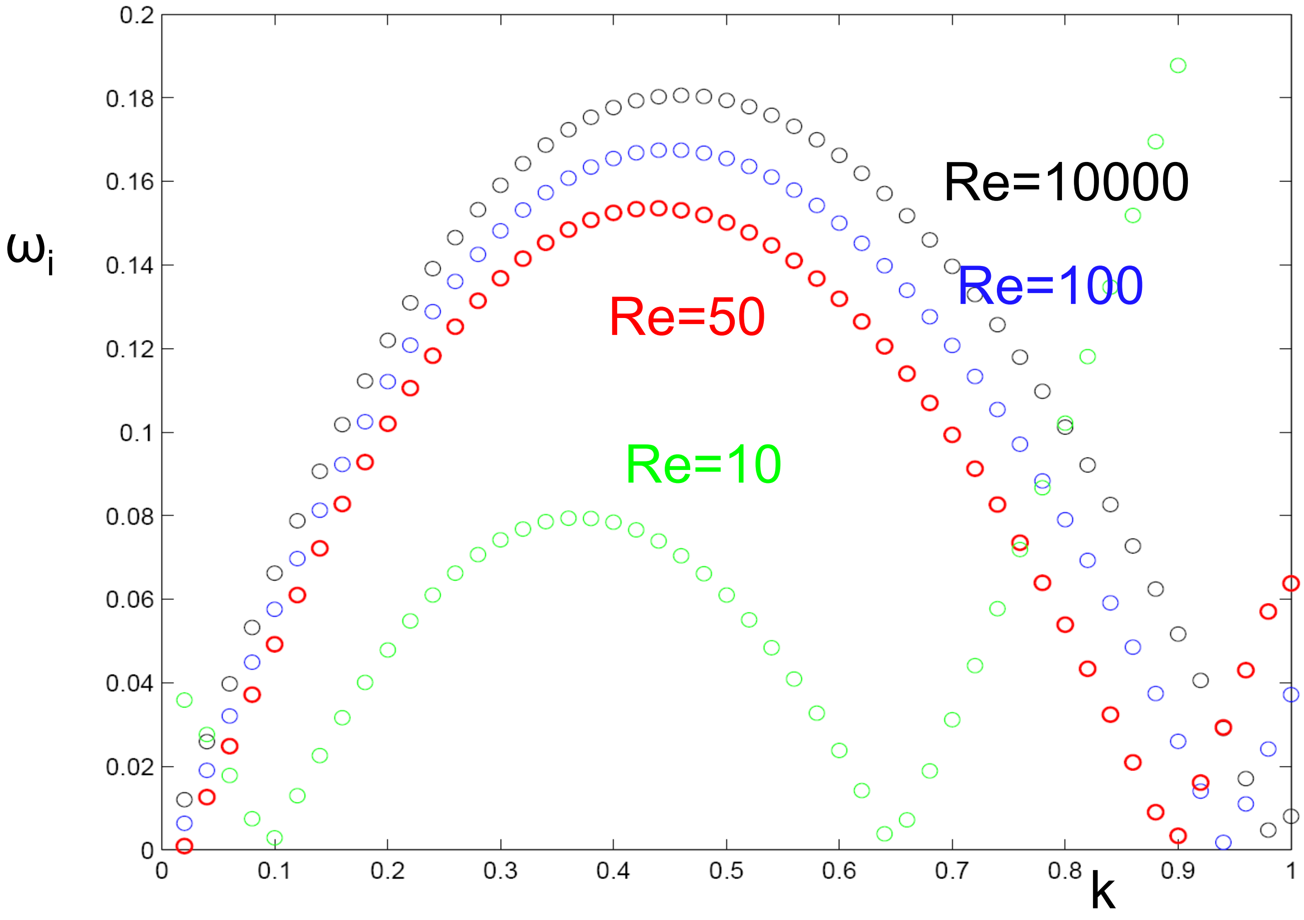
Dispersion relation

$$D(k,\omega;Re) = 0$$

Tangent hyperbolic mixing layer



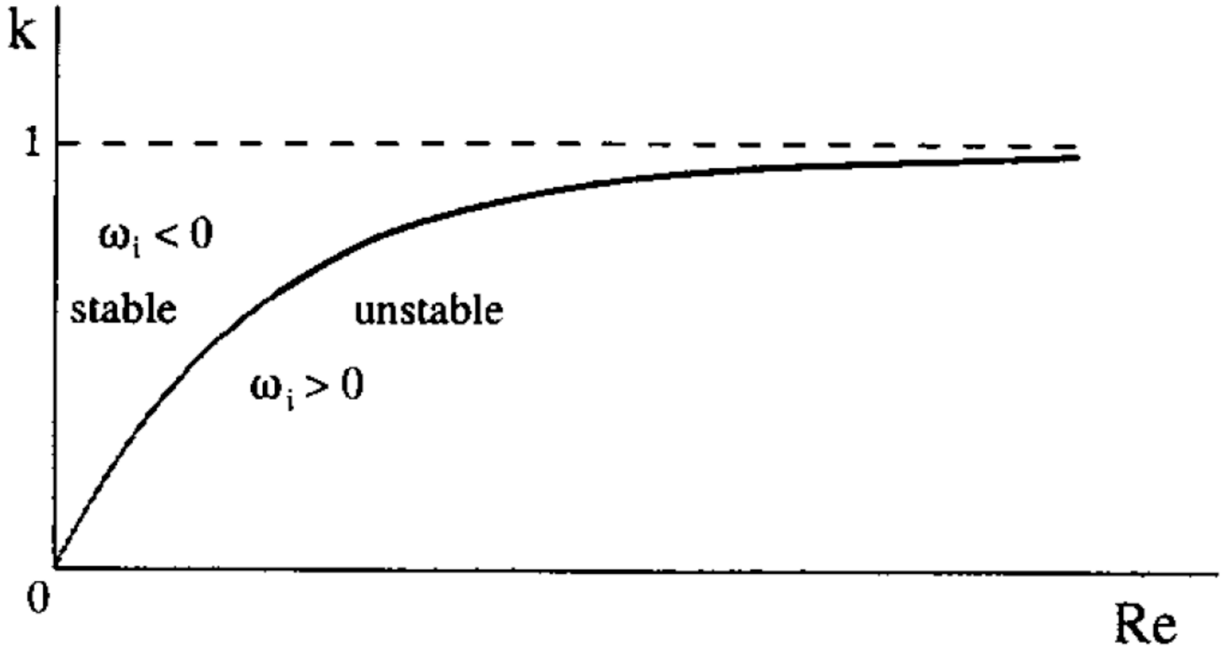
Viscosity has stabilizing influence on K-H instability



PARALLEL FLOW CONCEPTS

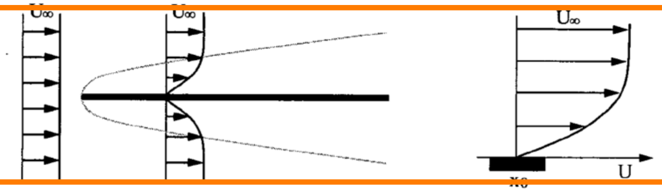
Viscous instabilities

Hyperbolic tangent mixing layer

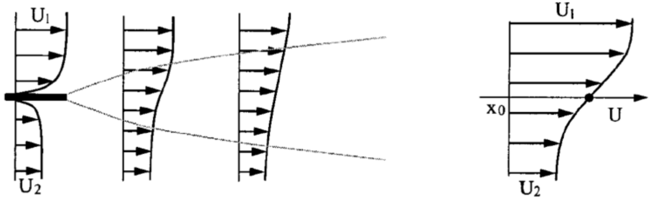


SPATIALLY DEVELOPING SHEAR FLOWS

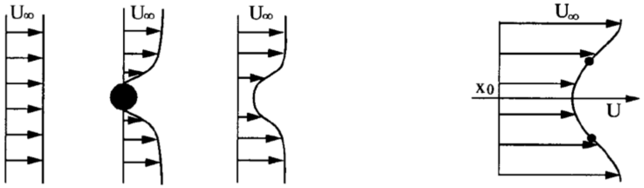
Flat plate boundary layer



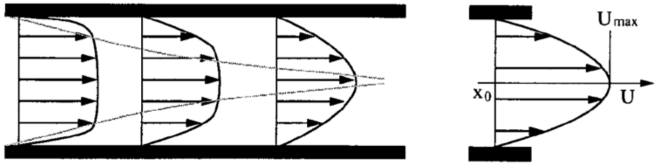
Mixing layer



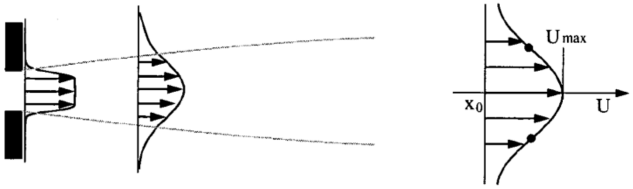
Cylinder wake



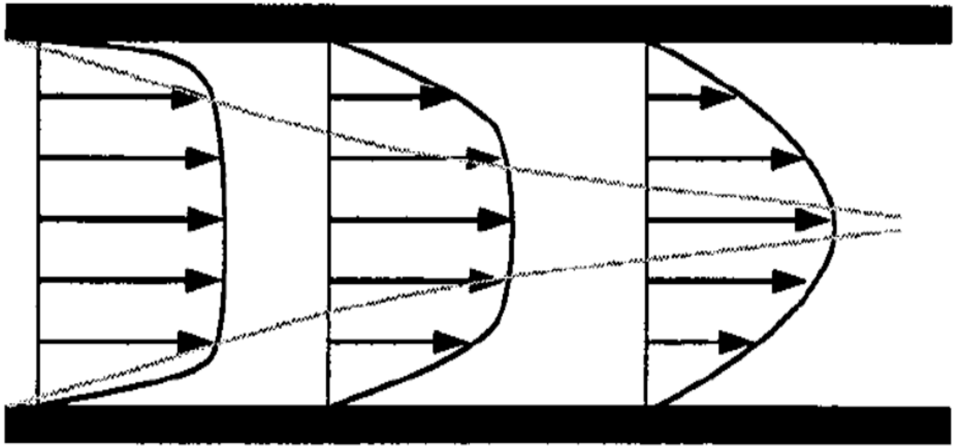
Plane channel flow

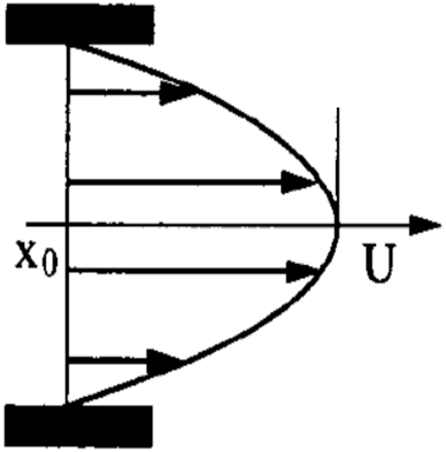


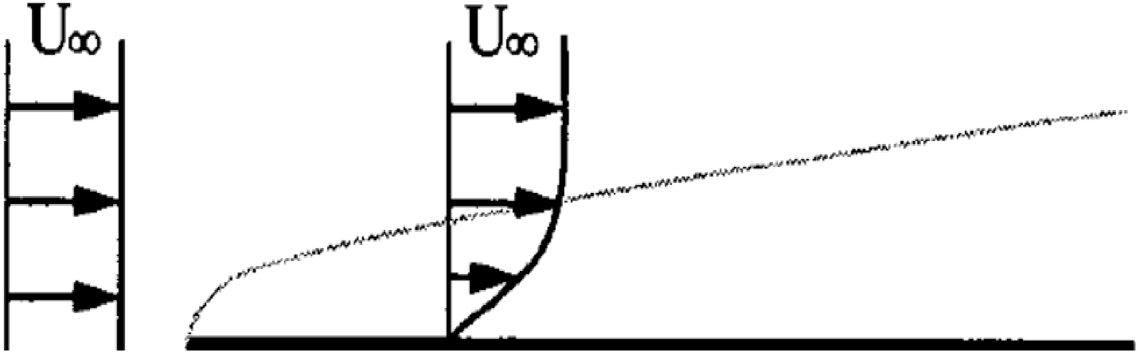
2D jet

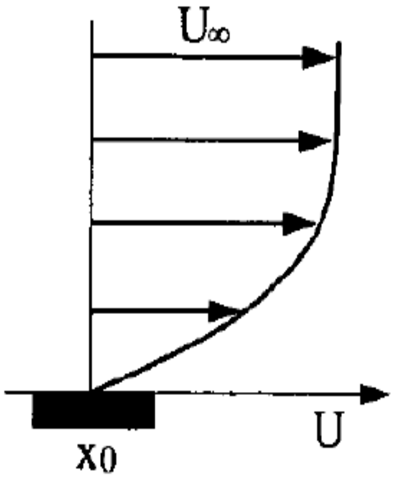


What about stable flows (no inflexion point)?

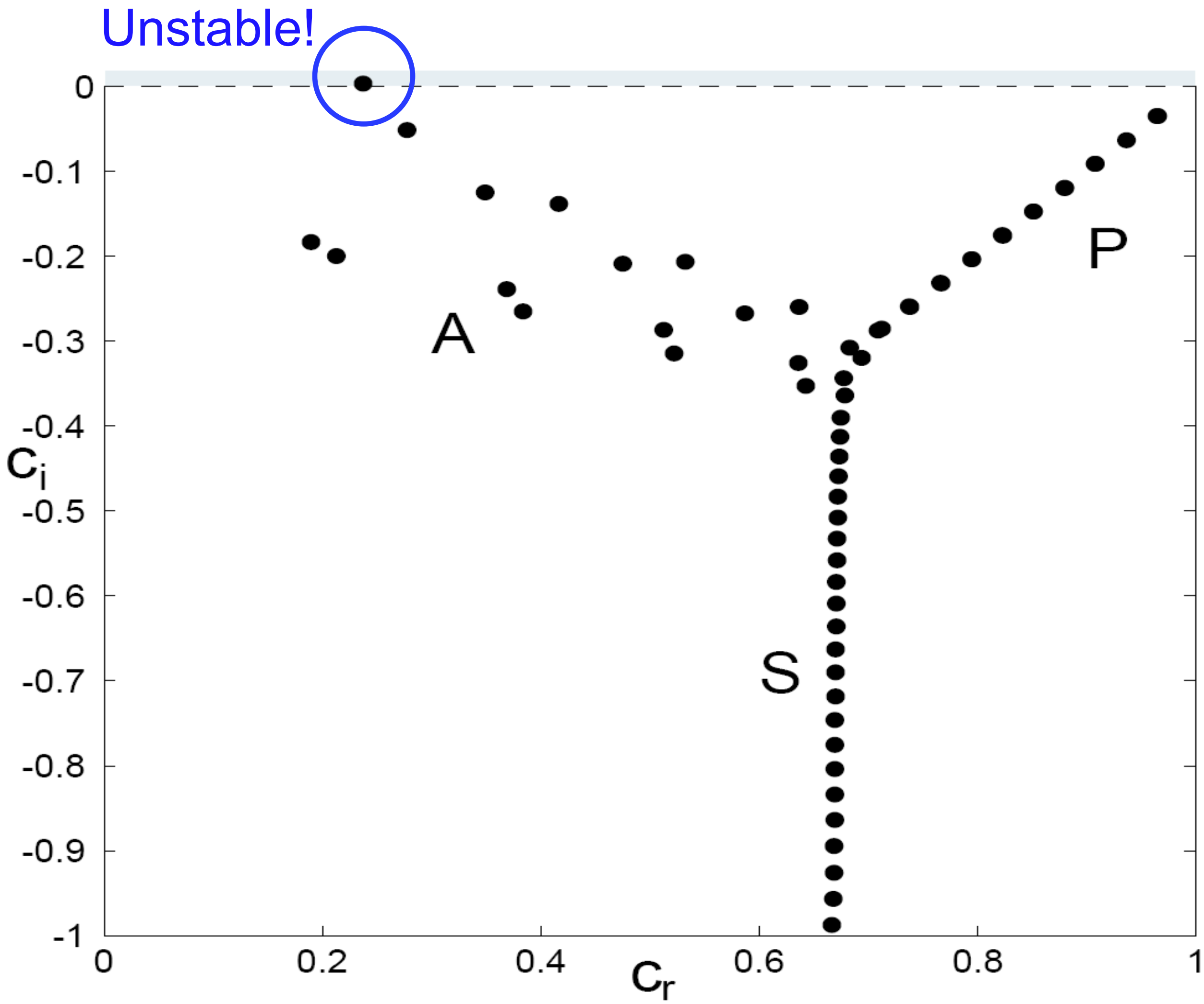




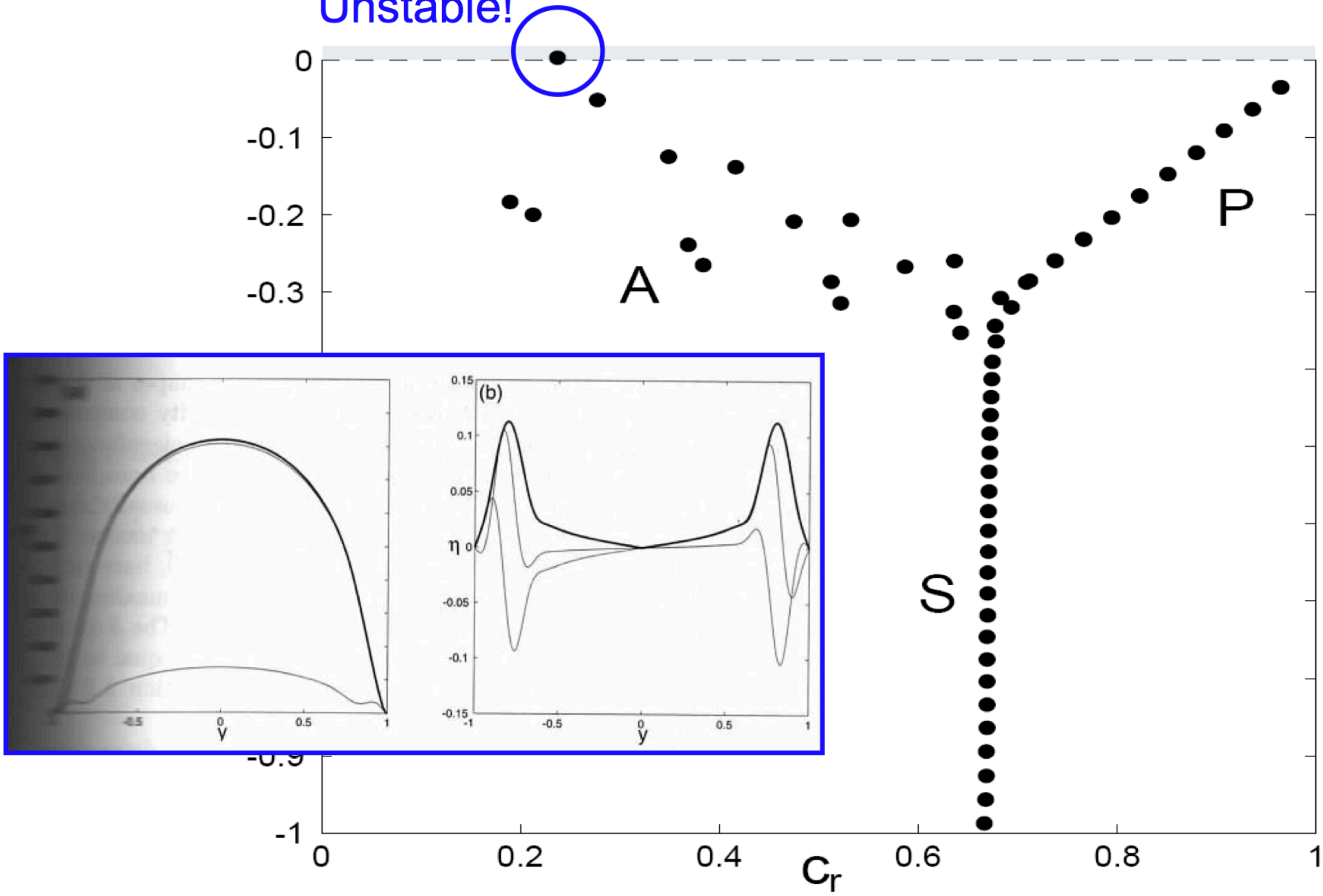




Plane poiseuille flow; Re=10000; α=1, β=0

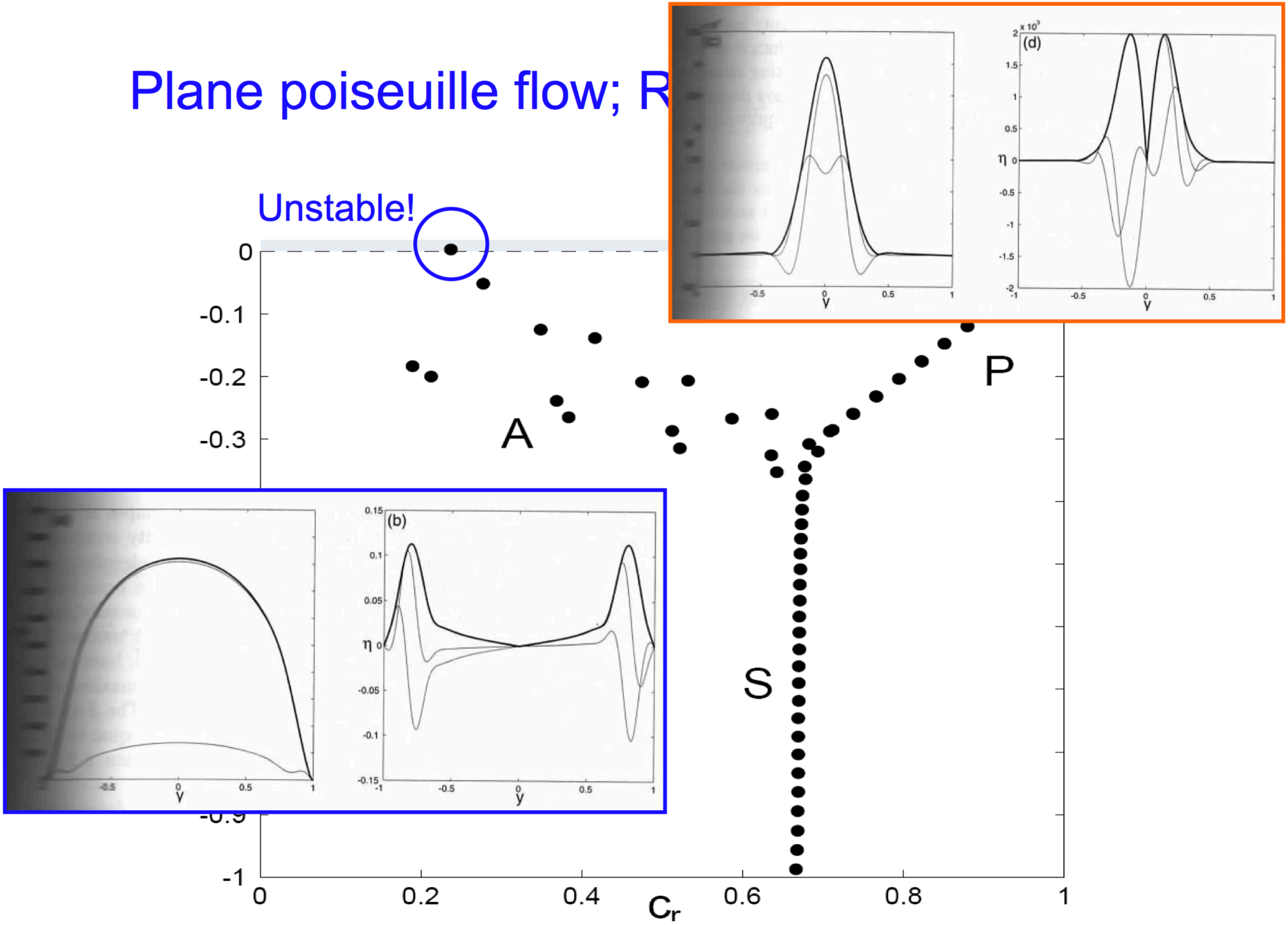


Plane poiseuille flow; Re=10000; α=1, β=0

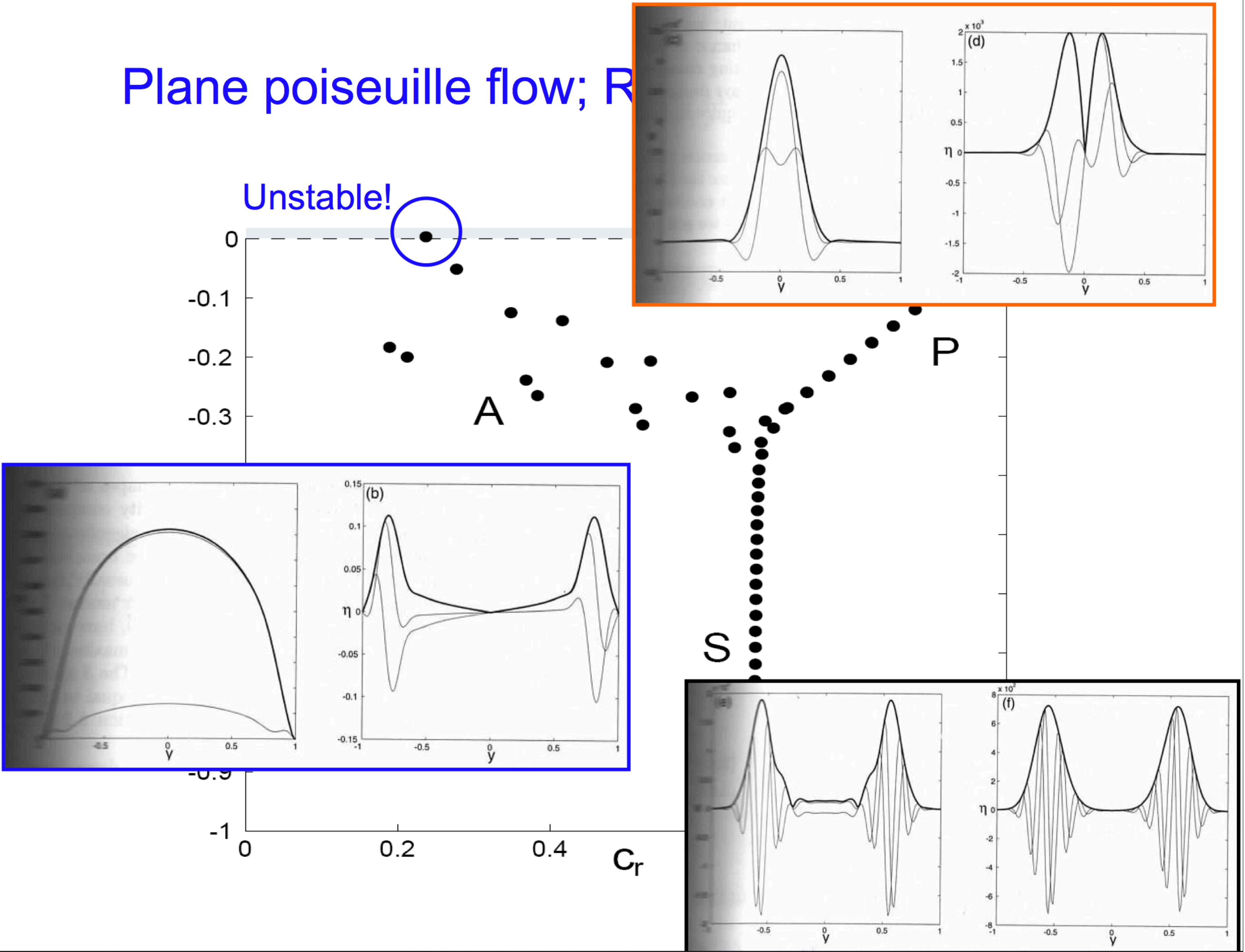


Unstable!

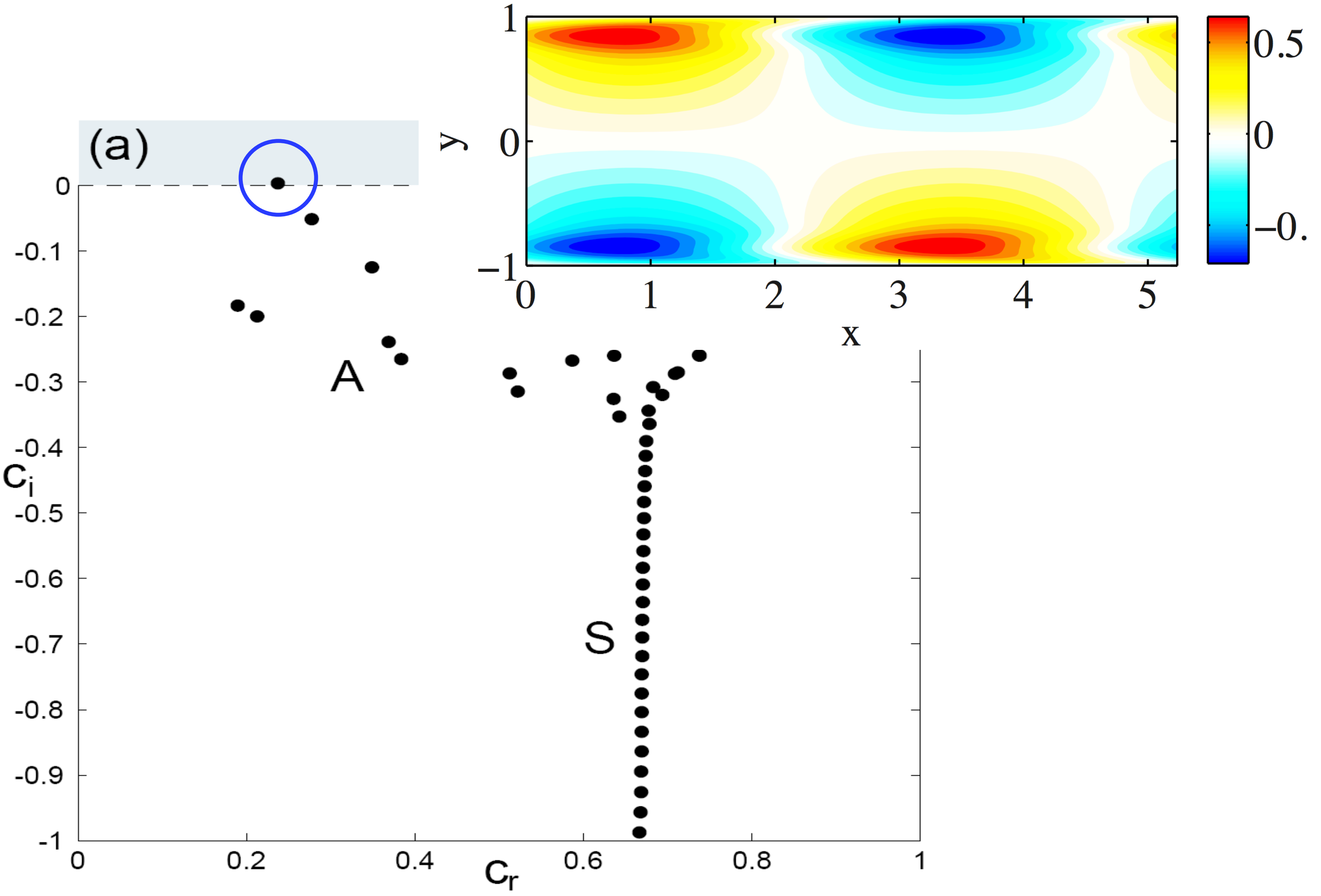
Plane poiseuille flow; R



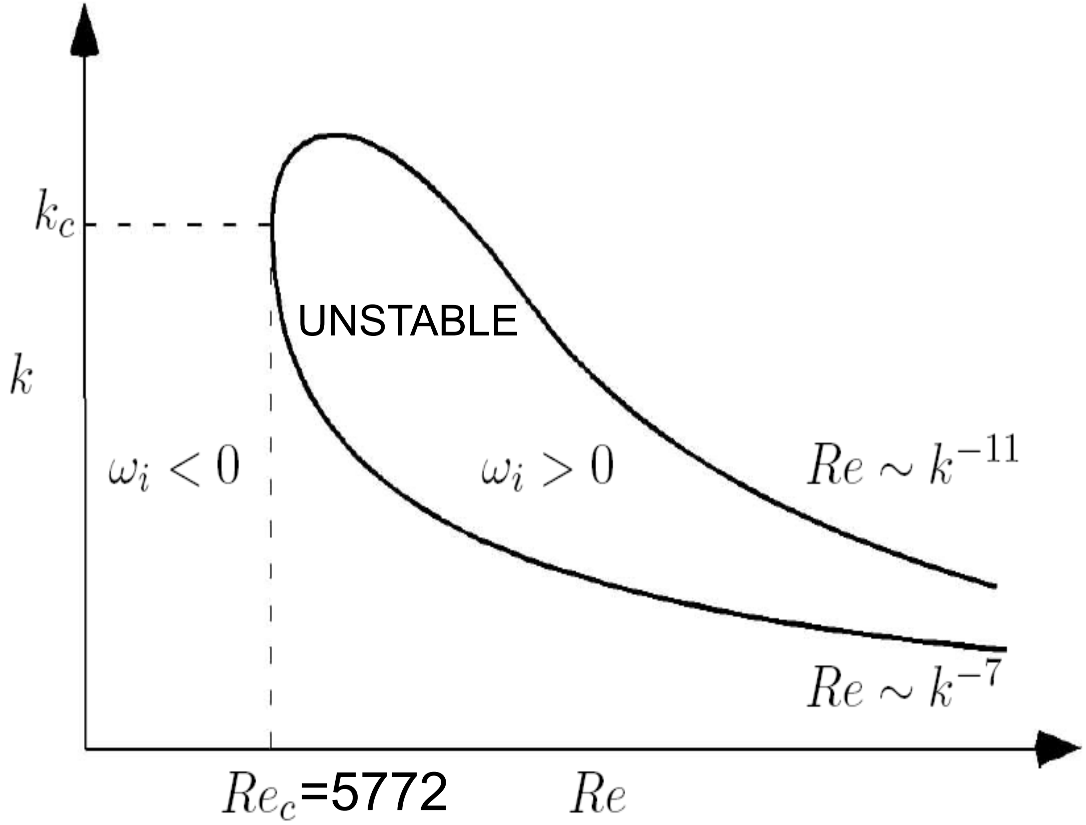
Plane poiseuille flow; R



Plane poiseuille flow; Re=10000; α=1, β=0

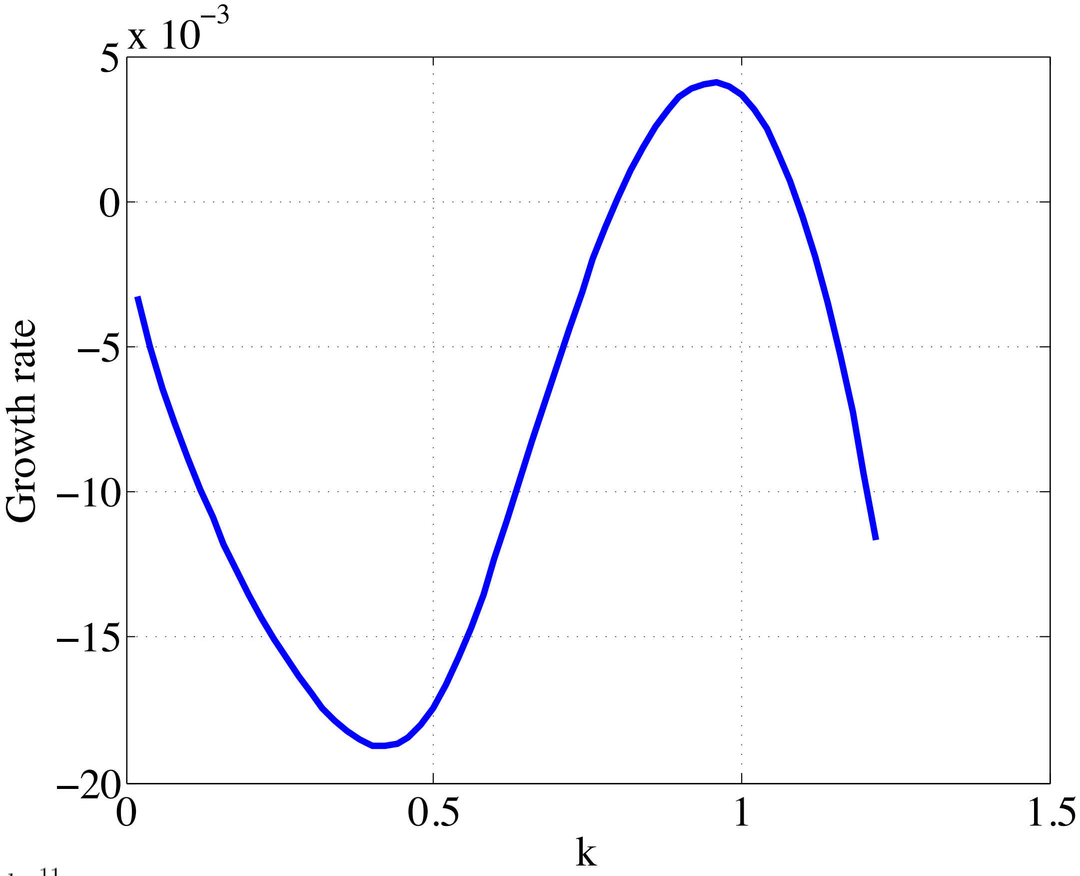


Neutral curve for plane poiseuille flow



– *Allure de la courbe de stabilité marginale dans le plan* $Re - k$ *pour l'écoulement de Poiseuille plan.*

Neutral curve for plane poiseuille flow



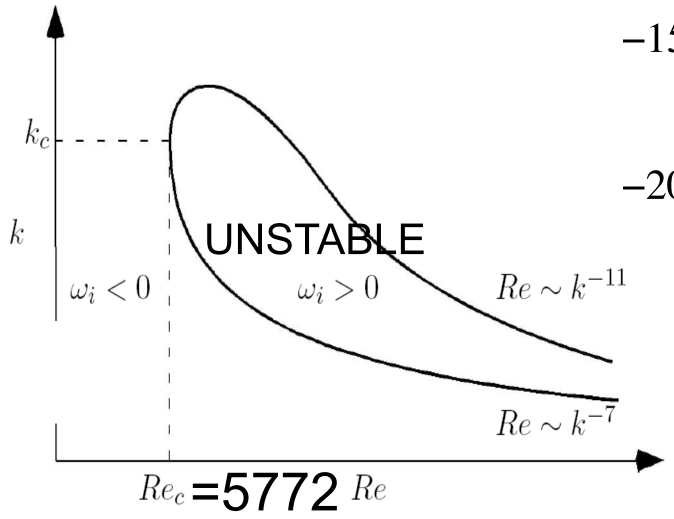


FIG. 5.7 – *Allure de la courbe de stabilité marginale dans le plan $Re - k$ pour l'écoulement de Poiseuille plan.*

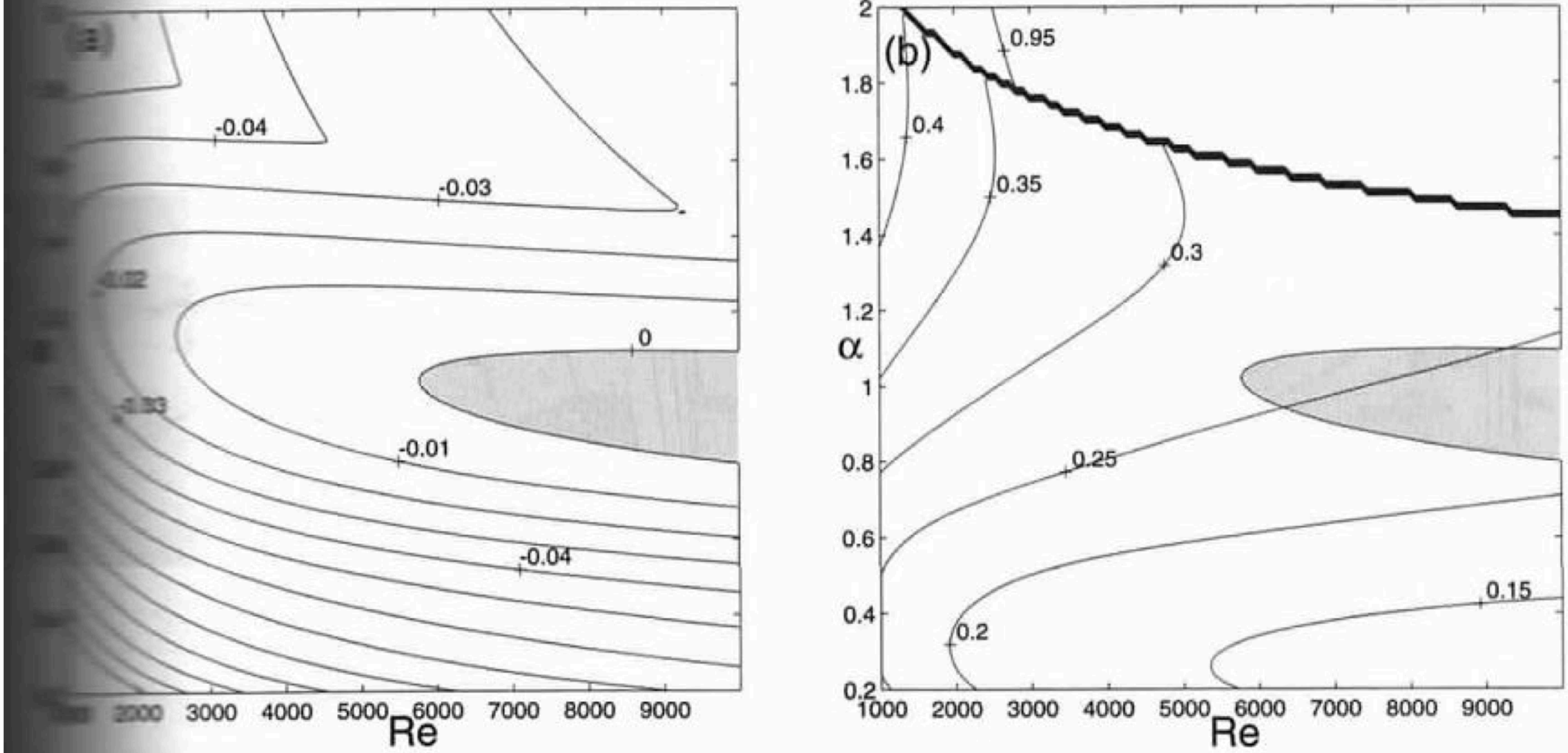
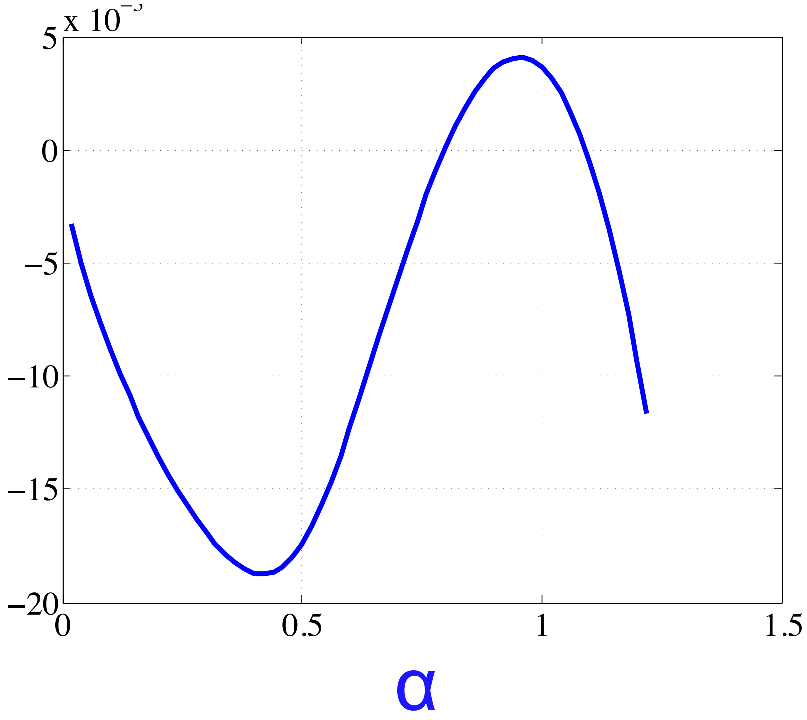


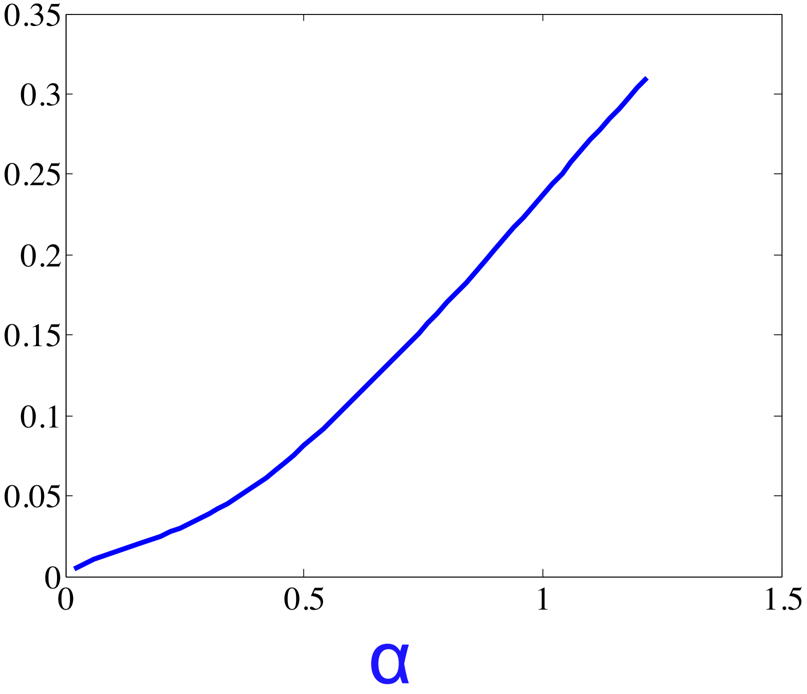
FIGURE 3.8. Neutral curve for plane Poiseuille flow: (a) contours of constant growth rate c_i; (b) contours of constant phase velocity c_r. The shaded area represents the region of parameter space where unstable solutions exist.

Neutral curve for plane poiseuille flow

Growth rate



Frequency



Boundary layers are destabilized by viscosity
Reynolds Orr equation

Perturbation kinetic energy: $e_c = \frac{1}{2}(u^2 + v^2 + w^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{y_1}^{y_2} \langle e_c \rangle \mathrm{d}y = \int_{y_1}^{y_2} \partial_y \overline{U} \tau_{xy} \mathrm{d}y - \frac{1}{Re} \int_{y_1}^{y_2} \langle \boldsymbol{\omega} . \boldsymbol{\omega} \rangle \mathrm{d}y$$

Production term Dissipation term

Boundary layers are destabilized by viscosity
Reynolds Orr equation

Perturbation kinetic energy: $e_c = \frac{1}{2}(u^2 + v^2 + w^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2}\langle e_c\rangle \mathrm{d}y = \int_{y_1}^{y_2} \partial_y \overline{U}\tau_{xy}\mathrm{d}y - \frac{1}{Re}\int_{y_1}^{y_2}\langle \boldsymbol{\omega}.\boldsymbol{\omega}\rangle \mathrm{d}y$$

Production term

Dissipation term

$$\tau_{xy} \equiv -\overline{uv} = -\tfrac{1}{2}|\hat{u}(y)||\hat{v}(y)|cos[\varphi_u(y) - \varphi_v(y)]\,\mathrm{e}^{2kc_it}$$

$$\hat{u}(y) \equiv |\hat{u}(y)|\,\mathrm{e}^{i\varphi_u(y)}\,, \qquad \hat{v}(y) \equiv |\hat{v}(y)|\,\mathrm{e}^{i\varphi_v(y)}\,.$$

It can be shown that Tollmien Schilcting waves, viscosity originates detuning of the phase

Concept of critical layer

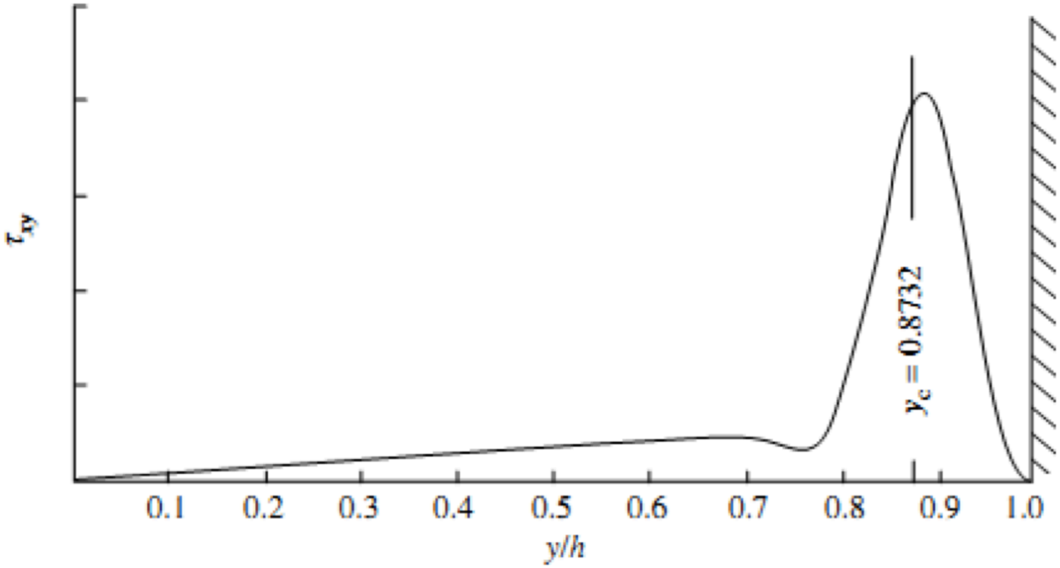


Figure 5.6 Transverse distribution of the Reynolds stress τ_{xy} (5.22) for plane Poiseuille flow (the vertical scale, which is related to the normalization condition, is arbitrary). y_c is the location of the critical layer where $c_r = \overline{U}(y_c)$. Re $= 10^4$, $kh = 1$, and $c = 0.24 + 0.0037\,\mathrm{i}$. Taken from Stuart (1963).

$1-y^2 = 0.24 \Rightarrow y = 0.83$

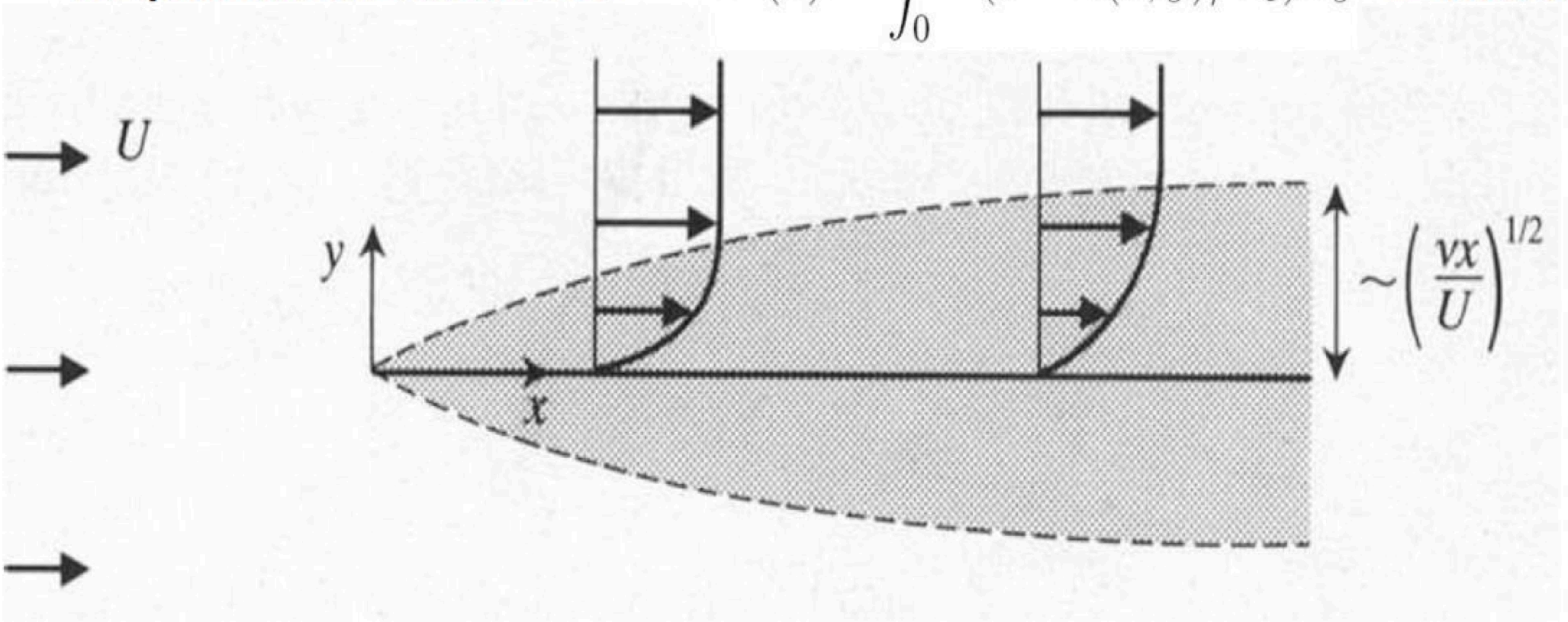
Orr-Sommerfeld equation

$$\overset{=0}{\overline{[U(y)-c]}}[\phi'' - k^2\phi] - U''(y)\,\phi = \frac{1}{i\,k\,Re}\left(\frac{d^2}{dy^2} - k^2\right)^2 \phi$$

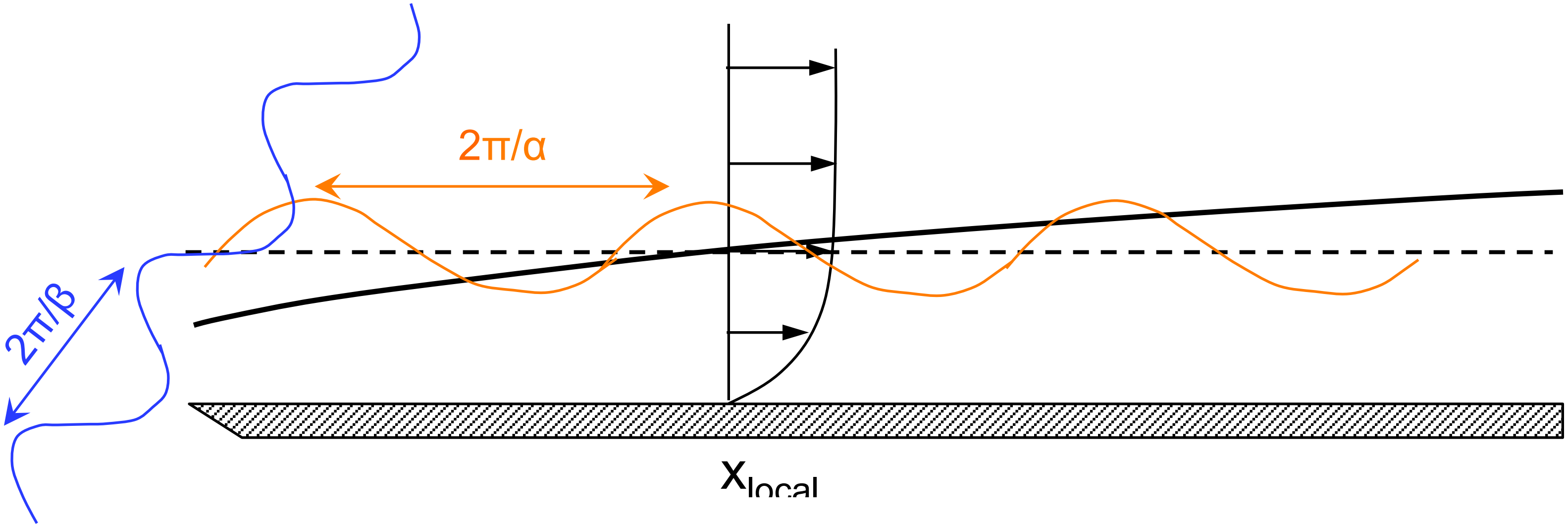
Tollmien-Schlichting waves in the boundary layer

Boundary layer thickness

Displacement thickness $\delta^*(x) = \int_0^\infty (1 - u(x,y)/U_e)\, dy$ **~ 1.73 δ**



Local parallel flow approximation



$$(\mathbf{u}, p) = (\mathbf{u}(y), p(y))\, \mathrm{e}^{\sigma t + i(\alpha x + \beta z)}$$

⇨Orr-Sommerfeld-Squire equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}(y)\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{U}(y) = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$

$$\nabla.\mathbf{u} = 0$$

$$(\mathbf{u},p) = (\mathbf{u}(y),p(y))\,\mathrm{e}^{\sigma t+i(\alpha x+\beta z)}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}(y)\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{U}(y) = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$

$$\nabla.\mathbf{u} = 0$$

⇨Orr-Sommerfeld-Squire equation

$$\left[(-i\omega + i\alpha U)(\mathcal{D}^2 - k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(\mathcal{D}^2 - k^2)^2\right]\tilde{v} = 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{\mathrm{Re}}(\mathcal{D}^2 - k^2)\right]\tilde{\eta} = -i\beta U'\tilde{v}$$

$$v(x,y,z,t) = \tilde{v}(y)\; e^{i(\alpha x+\beta z-\omega t)}$$

$$\eta(x,y,z,t) = \tilde{\eta}(y)\; e^{i(\alpha x+\beta z-\omega t)}$$

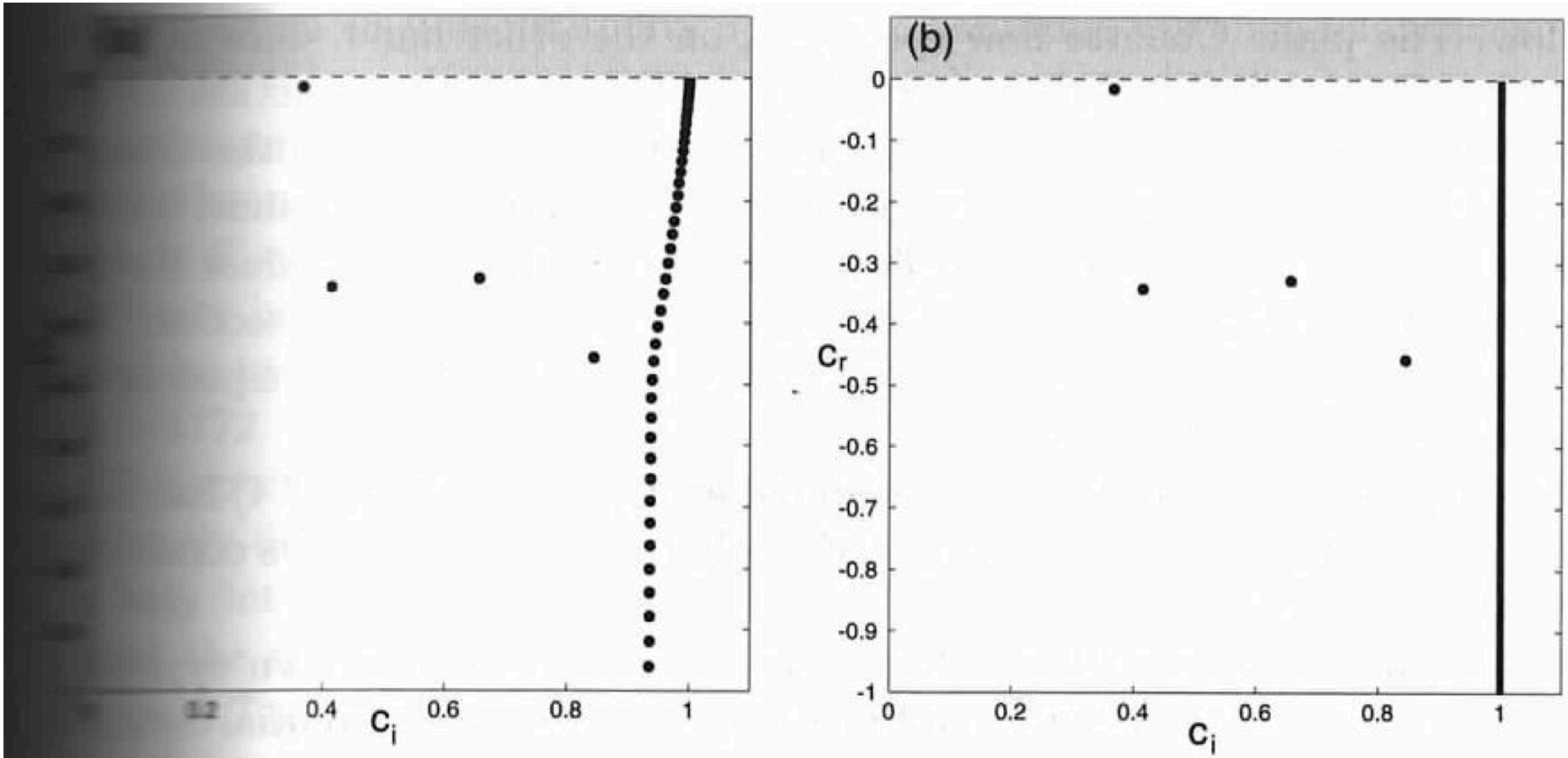
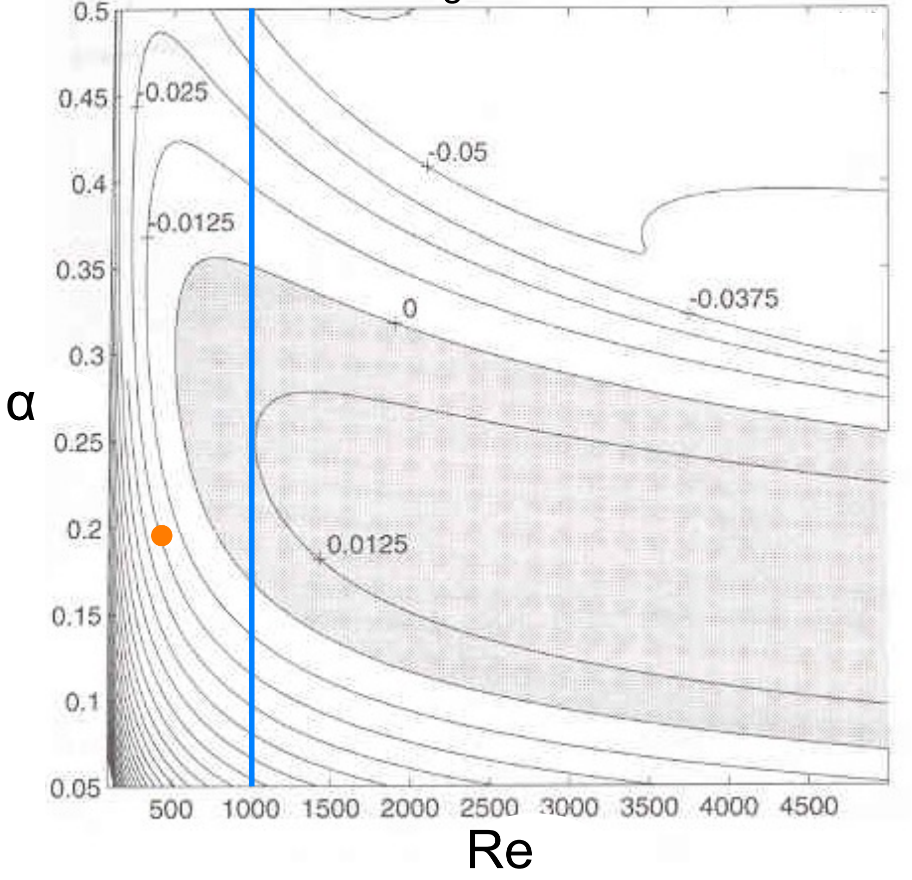
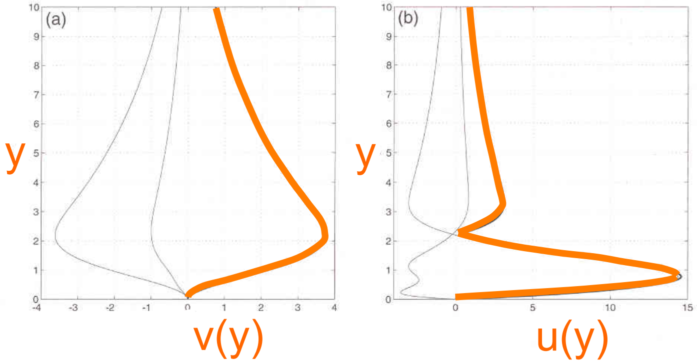


FIGURE 3.4. Spectrum for Blasius boundary layer flow for $\alpha = 0.2, \mathrm{Re} = 500$. (a) Numerically obtained spectrum displaying a discrete representation of the continuous spectrum with a particular choice of discretization parameters. (b) Exact spectrum displaying the discrete and continuous part.

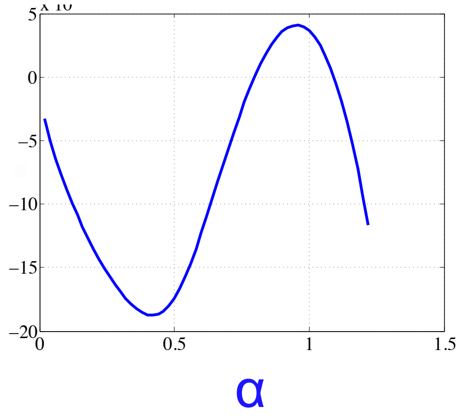
Tollmien Schlichting waves

Schmid and Henningson 2002

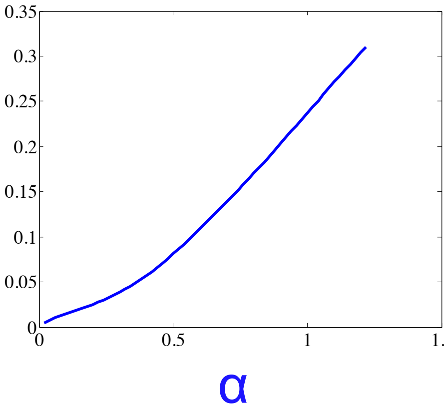




Growth rate



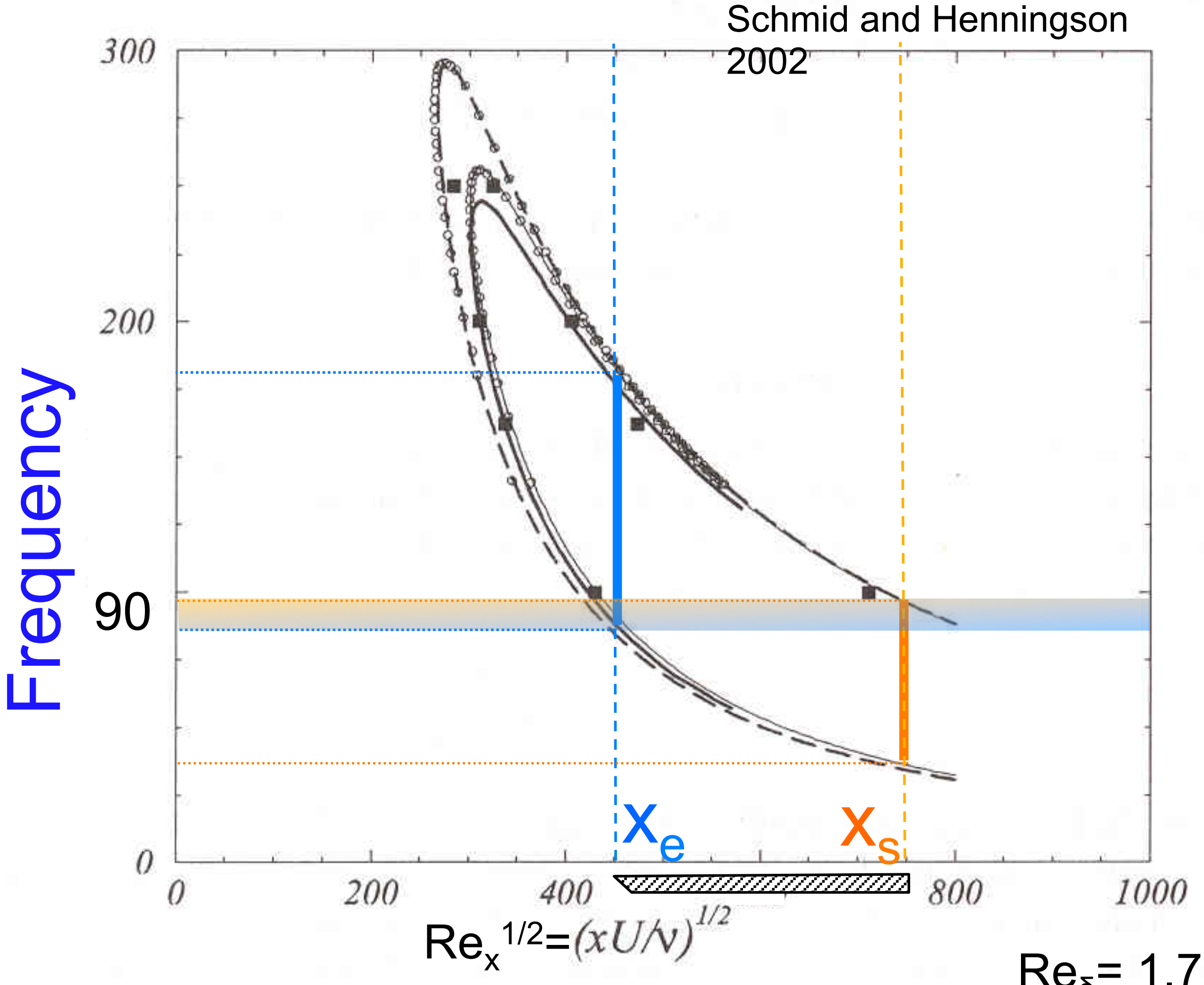
Frequency



α: axial wavenumber

β: transverse wavenumber

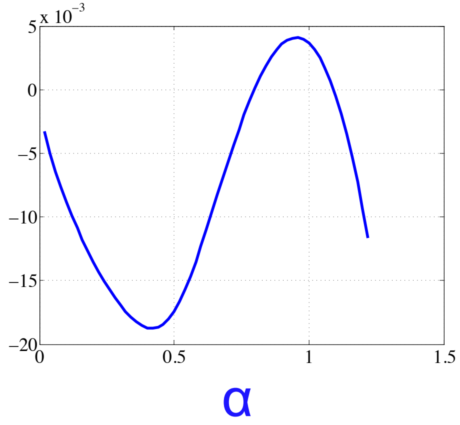
Neutral curve



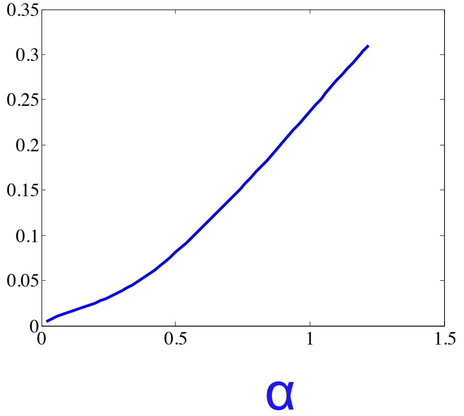
Schmid and Henningson 2002

$Re_\delta = 1.72 Re_x^{1/2}$

Growth rate

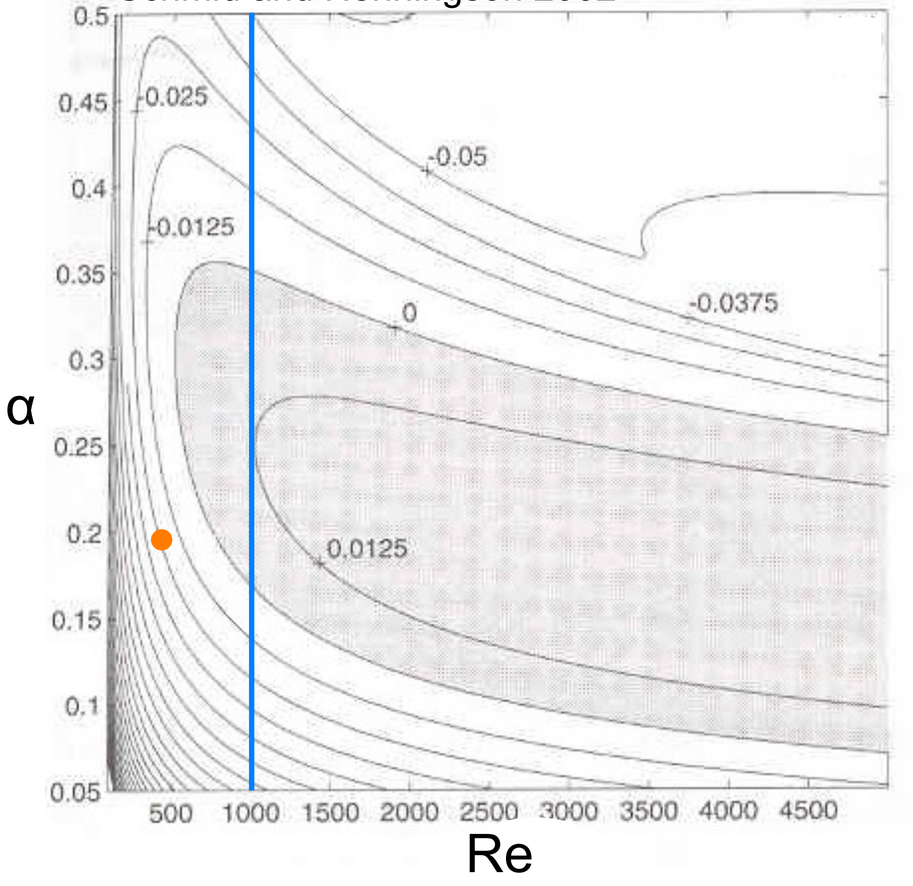


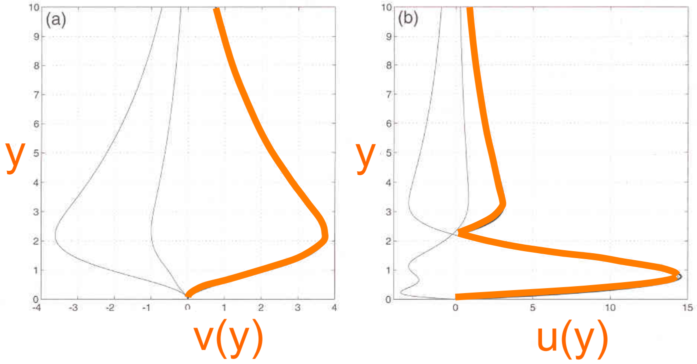
Frequency

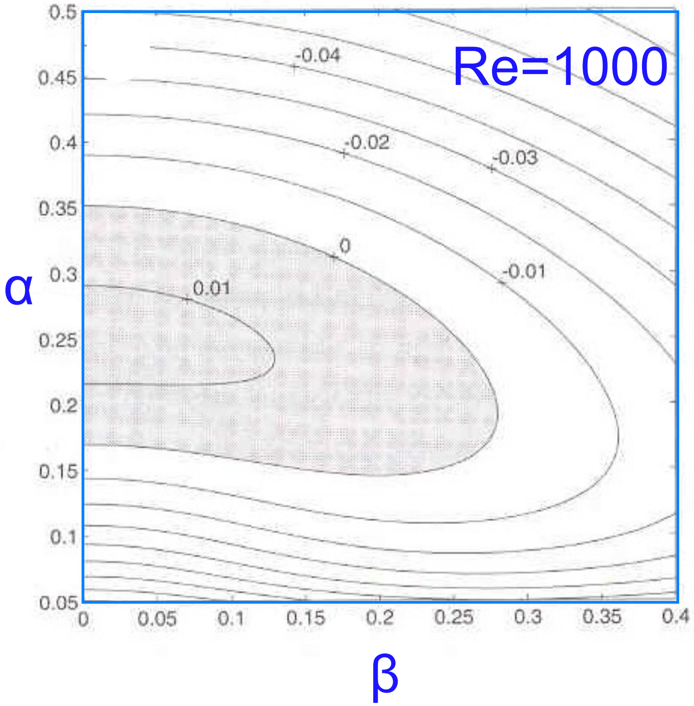


Tollmien Schlichting waves

Schmid and Henningson 2002



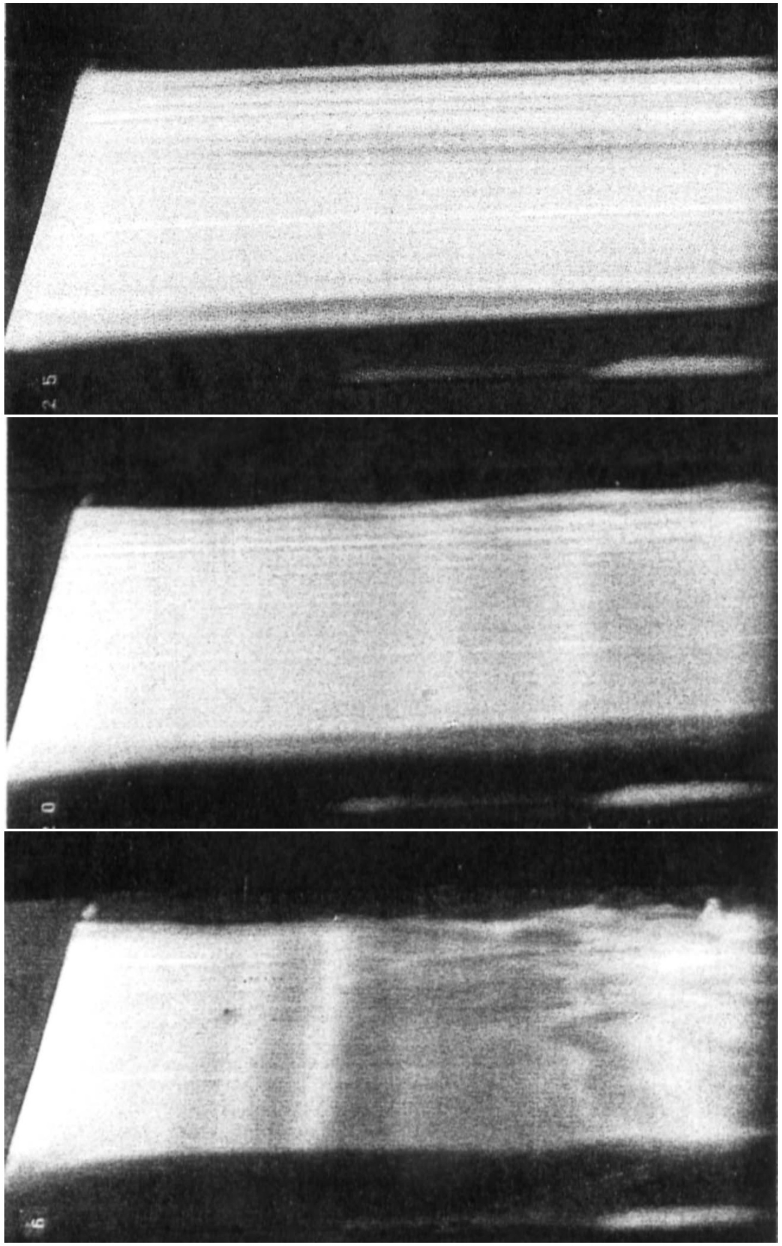




α: axial wavenumber

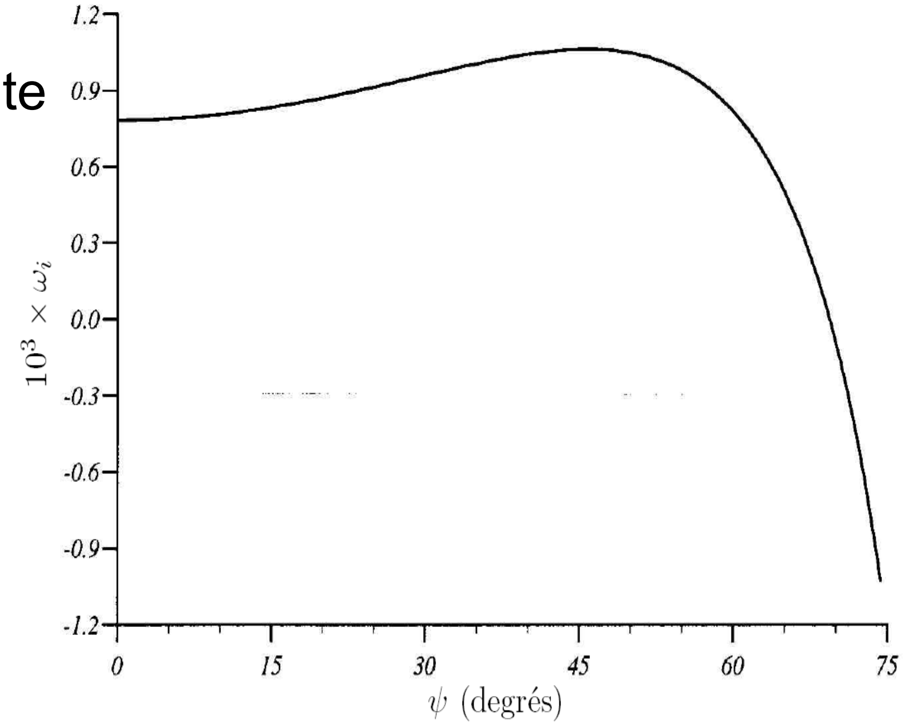
β: transverse wavenumber

Boundary layer at increasing Reynolds number



Boundary layer at Re_δ =1500

Growth-rate



ψ= tan(β/α)

Taux de croissance temporel (s^{-1}) *de l'instabilité d'une couche limite, en fonction de l'angle* ψ *de la perturbation avec la direction de l'écoulement de base.* $R_{\delta 1} = 1500$, $\omega\nu/U_\infty^2 = 0,3 \times 10^{-4}$ *(calcul G. Casalis, ONERA).*

The most unstable perturbation is oblique!

PARALLEL FLOW CONCEPTS

Viscous 3D instabilities

Squire's transformation

$$\tilde{k}^2 = k_x^2 + k_z^2\,, \qquad \tilde{c} = c\,,$$

$$\tilde{k}\tilde{u} = k_x\hat{u} + k_z\hat{w}\,, \qquad \tilde{v} = \hat{v}\,, \qquad \tilde{p}/\tilde{k} = \hat{p}/k_x\,.$$

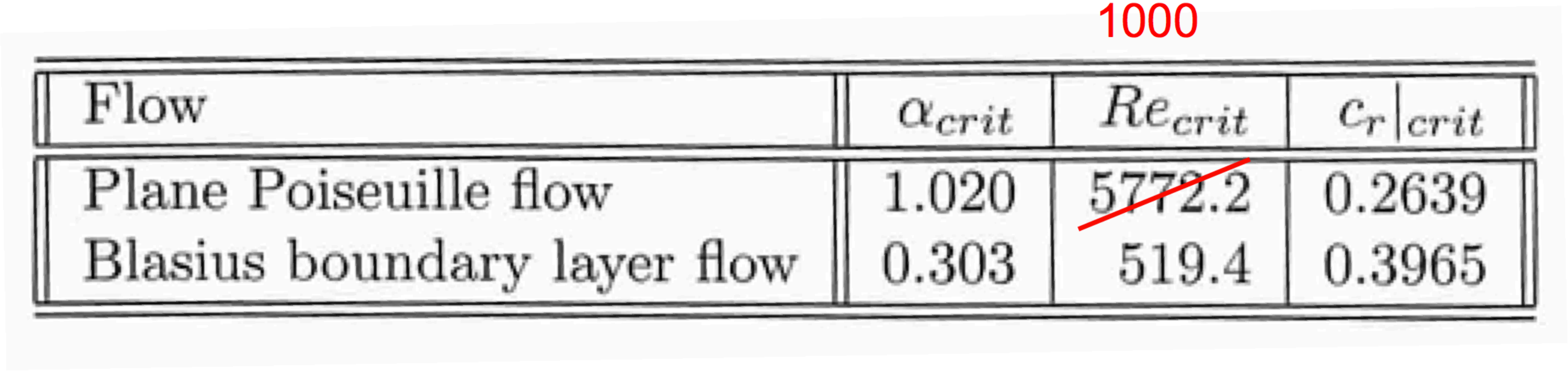
$$\tilde{k}\widetilde{\mathrm{Re}} = k_x\,\mathrm{Re}$$

3D dispersion relation

$$D(\mathbf{k},\omega;\mathrm{Re}) \equiv \tilde{D}\left[(k_x^2+k_z^2)^{1/2}, \frac{(k_x^2+k_z^2)^{1/2}}{k_x}\omega; \frac{k_x}{(k_x^2+k_z^2)^{1/2}}\mathrm{Re}\right] = 0$$

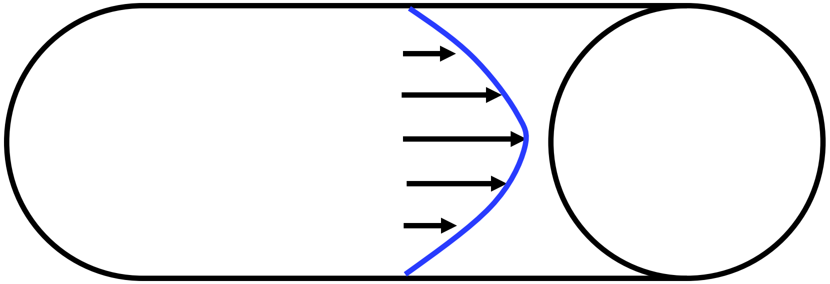
To each oblique mode $(\mathbf{k},\omega)$ of temporal growth rate ω_i, at Reynolds number Re, corresponds a two-dimensional mode $(\tilde{k},\tilde{\omega})$ of larger growth rate $\tilde{\omega}_i = \omega_i\sqrt{k_x^2+k_z^2}/k_x$, at a *lower* Reynolds number $\widetilde{\mathrm{Re}} = \mathrm{Re}\,k_x/\sqrt{k_x^2+k_z^2}$.

Experimental stability threshold!

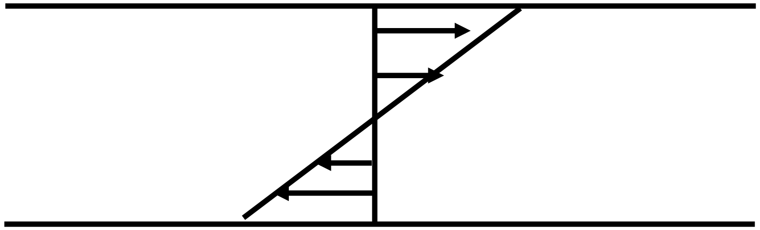


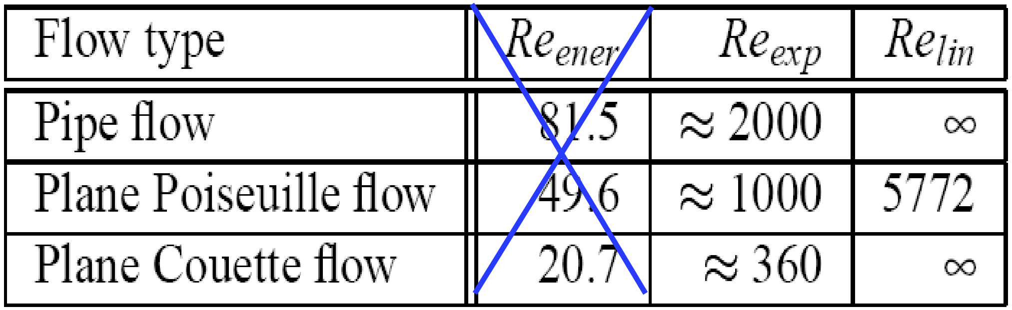
Flow	α_{crit}	Re_{crit}	$c_r\|_{crit}$
Plane Poiseuille flow	1.020	~~5772.2~~ 1000	0.2639
Blasius boundary layer flow	0.303	519.4	0.3965

Hagen Poiseuille



Couette





Flow type	~~Re_{ener}~~	Re_{exp}	Re_{lin}
Pipe flow	~~81.5~~	≈ 2000	∞
Plane Poiseuille flow	~~49.6~~	≈ 1000	5772
Plane Couette flow	~~20.7~~	≈ 360	∞

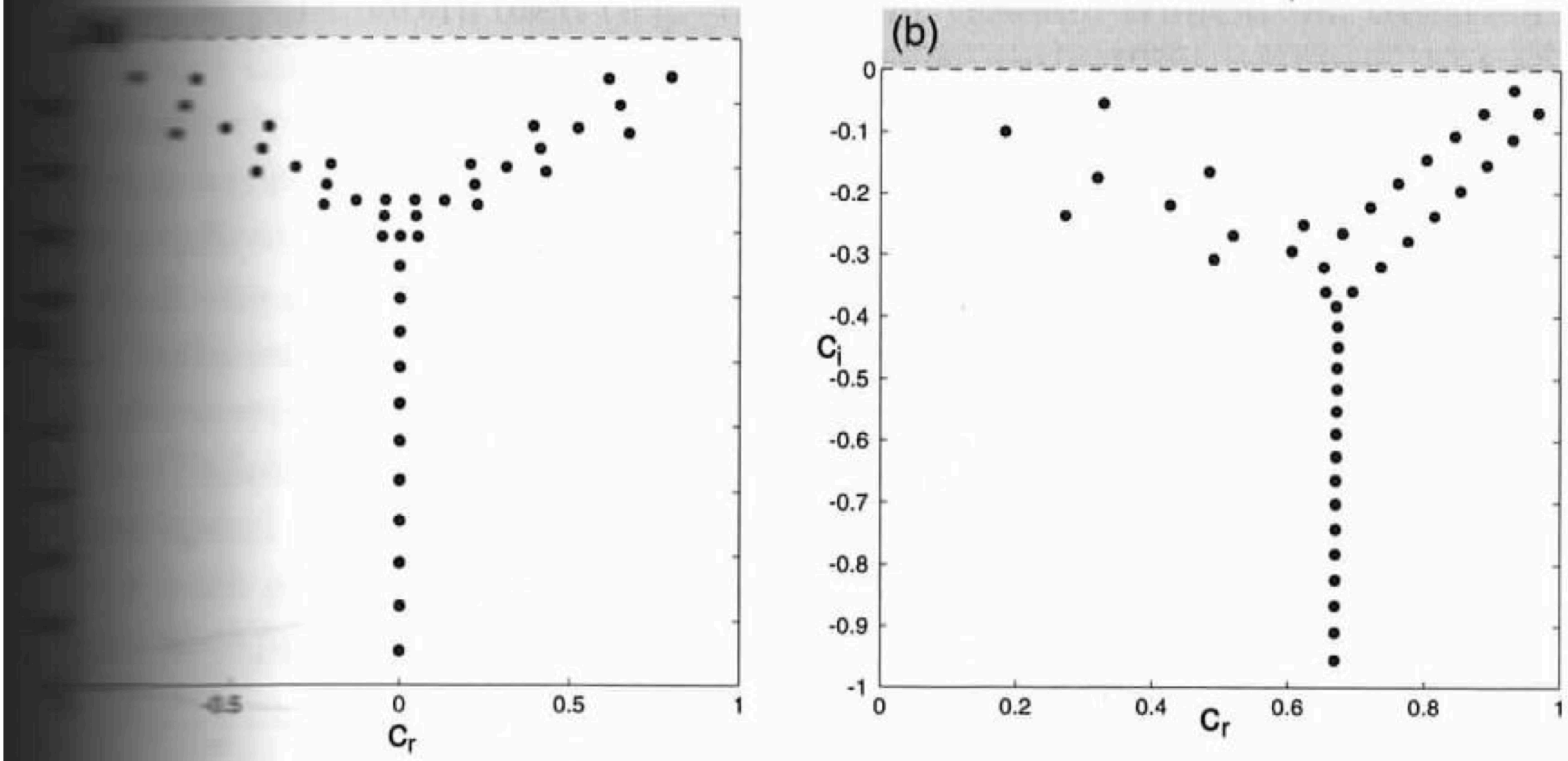
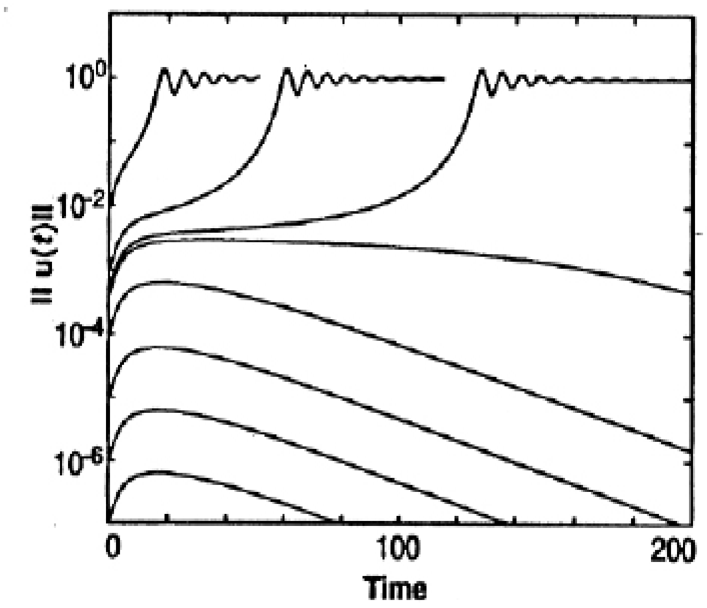


FIGURE 3.3. Spectrum of plane Couette and pipe flow. (a) Plane Couette flow for $\alpha = 1, \beta = 1, \mathrm{Re} = 1000$; (b) Pipe flow for $\alpha = 1, n = 1, \mathrm{Re} = 5000$.

Croissance transitoire et bye pass transition



Stabilité conditionnelle; transition sous-critique
Trefethen 1993