ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
SECTION DE GENIE MECANIQUE

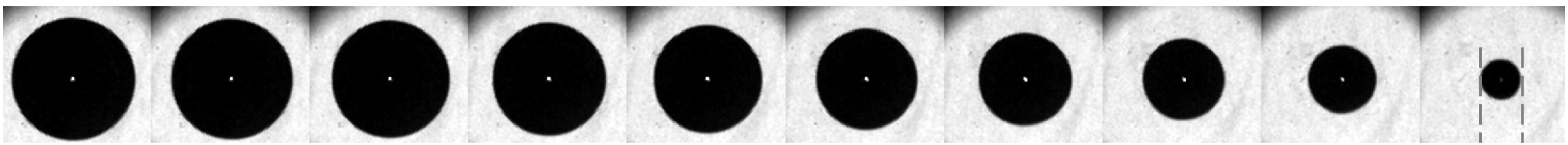
6^{th} & 8^{th} Semester, Fall 2025

CAVITATION AND INTERFACE PHENOMENA

Chapter 2: Stability and Dynamics of a Cavitation Bubble

2.2: Dynamics of a Spherical Bubble

Part 1/4 (incomplete)



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A brief history of Cavitation

1860s – 1870s : Discovery:
High-speed screw propellers on Royal Navy's torpedo-boat destroyers showed sudden loss of thrust and blade erosion.
Typical speed: ~15 to 30 knots,

1895: The term "Cavitation" introduced for the 1st time
Thornycroft & Barnaby introduced the term 'cavitation', Suggested by R. E. Froude (son of William Froude).

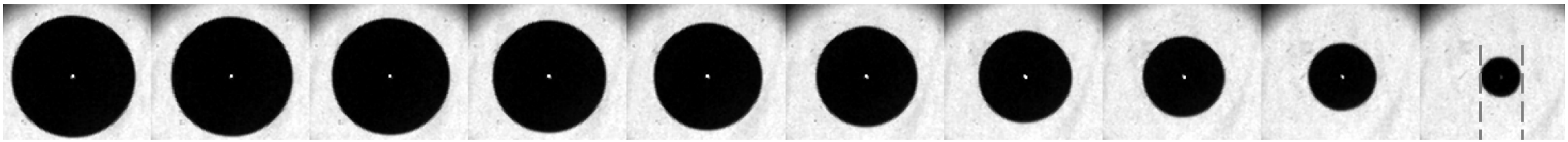
1894 – 1897: First experimental evidence of the phenomenon
Charles Parsons built a test channel and, with stroboscopic lighting, gave the first clear visual evidence of vapor cavities on fast propellers.

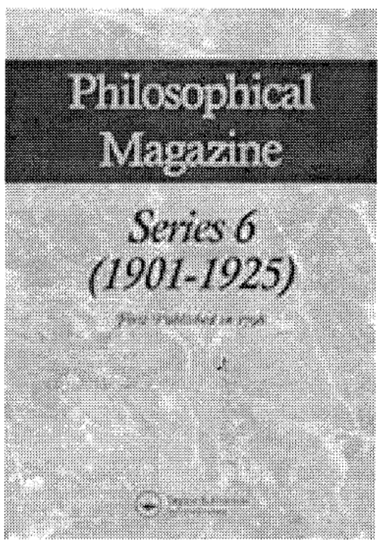
1917 : First Theory
Lord Rayleigh's bubble dynamics theory; foundation of modern cavitation theory (Rayleigh equation).

Dynamics of a Cavitation Bubble

Rayleigh Model

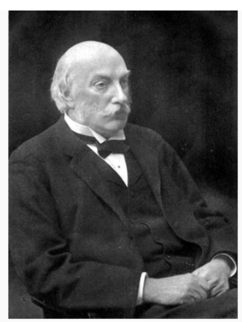
1919: First mathematical model, which describes the motion of an infinite volume of a liquid during the collapse of a low pressure bubble



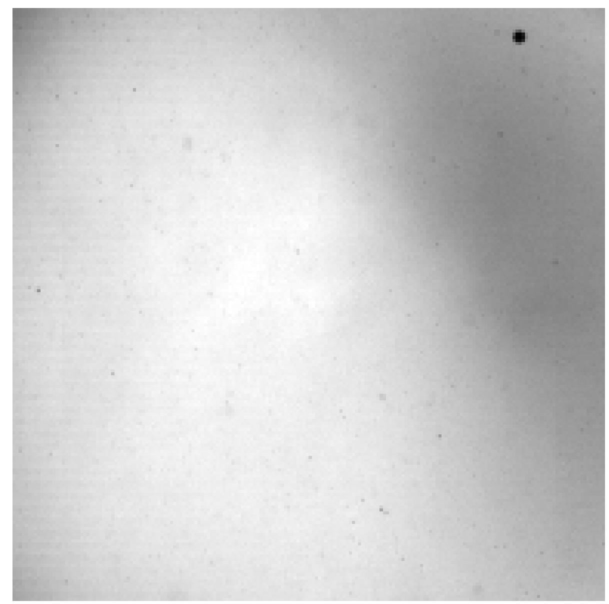


Philosophical Magazine Series 6
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t910323447

VIII. ***On the pressure developed in a liquid during the collapse of a spherical cavity***
Lord Rayleigh



Lord Rayleigh
Nov. 1842 – June 1919



Dynamics of a Cavitation Bubble
Rayleigh Model

Historical Facts:

Published by Lord Rayleigh in 1917:

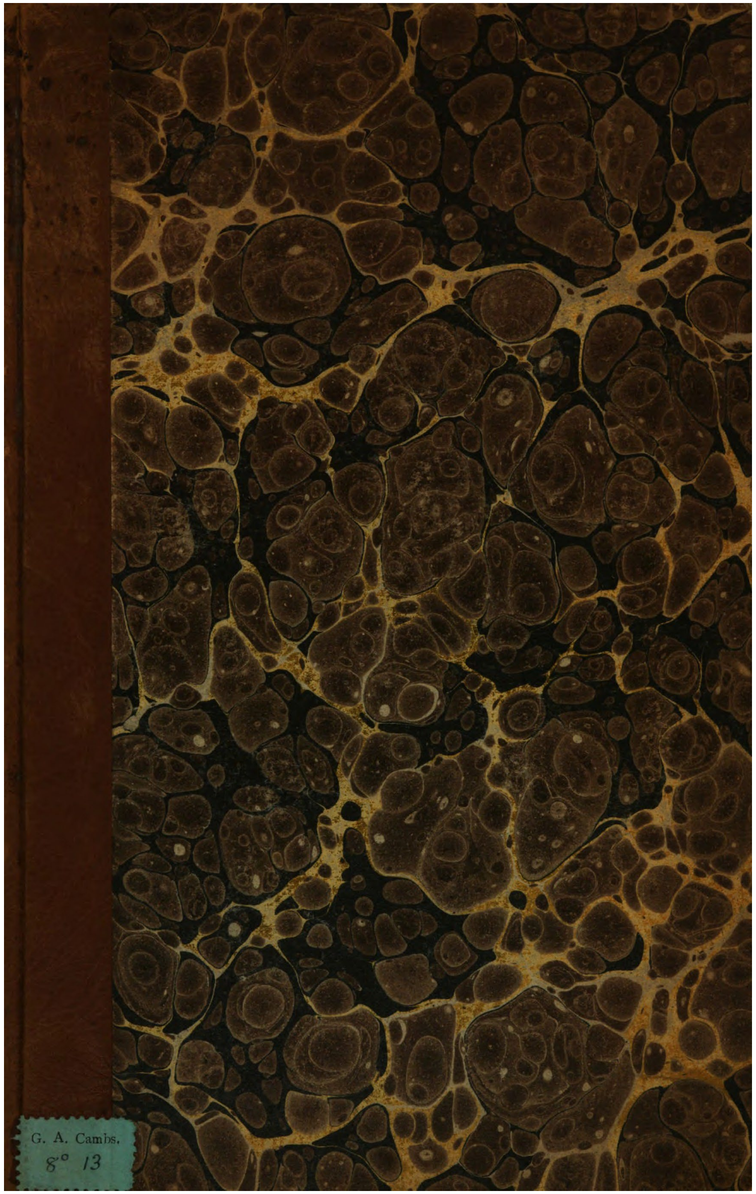
"On the pressure developed in a liquid during the collapse of a spherical bubble",
Philosophical Magazine Series 6, 1917

Rayleigh cited the work previously published by W. H. Besant:
***"A Treatise on Hydrostatics and Hydrodynamics"*, 1859**

W. H. Besant cited the Senate-House Problems, Friday January 8th, 1847

Senate house problems is a distinctive written examination of undergraduate students of the University of Cambridge. It consisted of 16 papers spread over 8 days, totaling 44.5 hours. The total number of questions was 211. The actual marks for the exams were never published, but there is reference to an exam in the 1860s where, out of a total possible mark of 17'000, the best student achieved 7'634, the second 4'123, the lowest around 1'500.

Senate-House Problems, Cambridge, 1847



THE

EXAMINATIONS

FOR THE DEGREE OF

BACHELOR OF ARTS,

CAMBRIDGE,

JANUARY, 1847.

LONDON: GEORGE BELL.

CAMBRIDGE:
J. AND J. J. DEIGHTON: MACMILLAN, BARCLAY AND MACMILLAN J. HALL: E. JOHNSON:
W. P. GRANT: H. WALLIS;

AND ALL BOOKSELLERS.

(1847)

THE SENATE-HOUSE EXAMINATION

FOR

DEGREES IN HONORS,

1847.

EXAMINERS FOR HONORS:

Moderators;

ADAMS, JOHN COUCH, M.A., (B.A. 1843) St. John's college.
STOKES, Rev. GEORGE GABRIEL, M.A., (B.A. 1841) Pembroke college.

Examiners.

MATHISON, Rev. WILLIAM COLLINGS, M.A. (B.A. 1839) Trinity college.
SYKES, JOHN, M.A., (B.A. 1841) Pembroke college.

WEDNESDAY, *January* 6, 1847.

Nine o'clock to half-past Eleven.

1. The opposite sides of a parallelogram are equal to one another, as are also the opposite angles, and the parallelogram itself is bisected by its diagonal.

If the four sides of a quadrilateral figure are equal to one another, the diagonals bisect each other at right angles.

α. The angles in the same segment of a circle are equal to one another.

β. If gold can be beaten out so thin that a grain will form a leaf of 56 square inches, how many of these leaves will make an inch thick, the weight of a cubic foot of gold being 10 cwt. 95 lbs.?

4. Solve the equations,

$$(1)\quad \frac{x}{x-1} = \frac{3}{2} + \frac{x-1}{x},$$

$$(2)\quad 3x - 2y = 3y - 4x = 1.$$

γ. Find the sum of n terms of an arithmetic series.

The sum of n terms of an arithmetic series is $pn + qn^2$ whatever be the value of n; find the m^{th} term.

δ. Find the number of variations which can be formed out of n things taken all together, when p are of one sort, q of another, &c.

There are n of each of the m letters $a, b, c \ldots$; find the whole number of variations which can be made of them without taking more than n letters together.

ϵ. Find the present value of a given half-yearly payment to continue for any number of years, at a given rate per annum, compound interest.

8. Define a logarithm, and prove that

$$\log xy = \log x + \log y, \quad \log x^n = n \log x, \quad \frac{\log_a x}{\log_b x} = \log_a b = \frac{1}{\log_b a}.$$

Given $\log_{10} 2 = \cdot 3010300$, find $\log_{10} \sqrt[3]{(\cdot 0125)}$.

9. Express $\sin A \pm \sin B$ and $\cos A \pm \cos B$ in products involving the sines or cosines of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$.

Find all the solutions of the equation $\cos 3x + \sin 3x = \frac{1}{\sqrt{2}}$.

10. Prove the formula $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$, and apply it to solve a triangle when two sides and the included angle are given.

Point out an objection to the use of this formula in the case in which a and b are nearly equal, and give another method of solution to which this objection does not apply.

ζ. Find the length of the perpendicular let fall from the point (a, b) upon the straight line whose equation referred to rectangular axes is

$$x \cos \alpha + y \sin \alpha = c.$$

η. The tangents at the extremities of any chord of a parabola intersect in the diameter which bisects the chord.

θ. Find the locus of a point the sum of whose distances from two fixed points is constant.

Shew how an ellipse whose semi-axes are 4 and 5 inches may be described mechanically.

14. Express the cosine of an angle of a spherical triangle in terms of the sides.

15. Given the base and vertical angle of an isosceles spherical triangle, find the equal sides.

A cube is turned round one of its diagonals through 180°: find the inclination of the plane of one of the faces to the plane with which that face coincided in its original position.

16. If $f(x)$ is any rational and integral function of x, prove that the highest common factor of $f(x)$ and $f'(x)$ consists of the product of the simple factors which occur more than once in $f(x)$, each raised to a power one less than that with which it occurs in $f(x)$.

Find all the roots of the following equation, which contains equal roots,

$$8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0.$$

One o'clock to Four.

1. Assuming the parallelogram of forces for the direction, prove it for the magnitude of the resultant. If the angle between two equal forces acting on a point be 120°, what is their resultant?

2. Find the relation between the power and the weight on the Inclined Plane, (1) when the power acts parallel to the plane, (2) when it acts horizontally.

3. Find the centre of gravity of a triangle. Find also that of five equal heavy particles placed at five of the angular points of a regular hexagon.

THE SENATE-HOUSE EXAMINATION

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α. Explain the two modes of measuring force in dynamics. What is the ratio of the accelerating forces when equal pressures act on different bodies? If the numerical value of the accelerating force of gravity be g when a second is taken for the unit of time, what is its numerical value when half a second is taken for the unit?

β. A body is projected vertically downwards with a given velocity; shew that the space described in any time is equal to that which would be described in the same time with a uniform velocity equal to half the sum of the velocities at the beginning and end of the time. Also find the velocity of the body in terms of the space described.

γ. Prove that a body projected in any direction inclined to the vertical, and acted on by gravity, will describe a parabola, and shew that the velocity at any point is that which would be acquired in falling from the directrix.

If any number of bodies be projected in different directions from the same point with equal velocities, shew that the foci of the parabolas described will lie in the surface of a sphere.

δ. State and prove Newton, Lemma X. How will the figure and the result be modified, if the force be a uniform one?

ε. If a body move in any orbit about a fixed centre of force, the areas described by lines drawn from the centre to the body lie in one plane, and are proportional to the times of describing them. Prove this, and shew that the reasoning applies whether the force be attractive or repulsive.

ζ. Find the velocity at any point of an ellipse described about a centre of force in the focus. At what point is this velocity equal to that in a circle at the same distance?

10. Find the conditions which must be satisfied when a solid floats at rest in a fluid.

11. Define specific gravity; and find the weight of a cubic inch of glass whose specific gravity is 3.456, the weight of a cubic foot of water being a thousand ounces.

12. Explain how a thermometer is graduated, and shew how to compare the scales of two differently graduated thermometers. What is 20° centigrade in Fahrenheit's scale?

13. Find the deviation of a ray after two successive reflections at plane mirrors inclined to each other, the course of the ray lying in a plane perpendicular to their line of intersection. What must be the first angle of incidence, that at a third reflection the course of the ray may be exactly reversed?

14. Find the relative index of refraction between two media, having given the refractive index between each of them and a third. Does the demonstration hold for all angles of incidence and refraction at the common suface of the two media?

15. Describe the simple astronomical telescope; and shew that its magnifying power is measured, (1) by the ratio of the focal length of the object glass to that of the eye glass, (2) by the ratio of the diameter of the object glass to that of the emergent pencil.

η. Explain the reason why navigators in sailing round the world gain or lose a day in their reckoning, according as they sail eastwards or westwards.

θ. Explain the different modes of measuring time in use among astronomers. Find the mean time at which a known star crosses the meridian.

ι. Find the length of the day at any given place. What is the lowest latitude at which the Sun does not set for 24 hours?

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THURSDAY, *January* 7, 1847.
Nine o'clock to half-past Eleven.

1. Prove the parallelogram of couples, shewing how the directions in which the couples act in their own planes are taken into account.

2. Shew that the surface generated by the revolution of a plane curve about an axis in its plane is equal to the rectangle of which the sides are the length of the curve and the length of the path of its centre of gravity.

How must the enunciation be altered if the curve lie on both sides of the axis?

α. Find the differential equation between u and θ to the path of a body under the action of a central force; and shew how the constants which occur in its complete integral may be determined, when the velocity, distance, and direction of projection are given.

4. When a heavy particle moves on a curve in a vertical plane, find the pressure upon any point of the curve.

If the particle move on the interior of a circle, find the pressure at the lowest point when the velocity is just sufficient to carry the particle entirely round.

β. State Kepler's Laws. What information does each of them afford respecting the force which retains the planets in their orbits?

γ. A hollow cylinder closed at both ends is partly filled with a homogeneous liquid, not acted on by gravity, which revolves uniformly about the axis of the cylinder; find the force which tends to separate the two portions into which the cylinder is divided by an imaginary plane passing through its axis.

7. When a mass of fluid is in equilibrium, prove that if the portion comprised between the free surface and any other surface of equal pressure be removed, the equilibrium of the rest will not be affected, provided the forces acting on the fluid do not depend on the mutual attraction of its parts.

Prove that such a portion can also be removed, if the surfaces of equal pressure are similar and concentric ellipsoids, and the particles of the fluid attract each other according to the law of gravitation.

δ. When a ray of light passes through a prism denser than the surrounding medium, in a plane perpendicular to the edge of the prism, the deviation is from the edge.

If a ray will not pass through when the prism is in the position of minimum deviation, it will not pass through when the prism is in any other position.

ε. When is a single lens said to be equivalent to a system of lenses through which an excentrical pencil passes, the axis of the pencil before incidence being nearly parallel to the axis of the lenses? Find the single lens equivalent to two lenses separated by a given interval, and apply the result to Ramsden's eye-piece when used with an object-glass of great focal length.

ζ. The law of force in an orbit nearly circular being given, find an approximate value of the apsidal angle. Newton, Sect. IX.

11. Shew that in consequence of the eccentricity of the Earth's orbit, the excess of the Moon's true longitude over the mean is greatest about three months after the Earth has passed its aphelion, and greatest negatively about three months after it has passed its perihelion.

η. When the vertical plane in which a transit instrument moves nearly coincides with the meridian, find the deviation by observing the time of transit of two known stars.

What stars are the most eligible for the purpose?

13. Shew how to determine the nadir point of a mural circle by observing the cross wires directly, and by reflection in a trough of mercury.

14. Shew how to determine the latitude and the time by observing the altitudes of two known stars.

One o'clock to Four.

1. Prove that

$$(ax_1^2 + 2bx_1y_1 + cy_1^2)(ax_2^2 + 2bx_2y_2 + cy_2^2) - \{ax_1x_2 + b(x_1y_2 + x_2y_1) + cy_1y_2\}^2$$
$$= (ac - b^2)(x_1y_2 - x_2y_1)^2.$$

2. If all the odd numbers be arranged in order of magnitude, and then separated into groups, so that the first group contains one number, the second two, and so on, prove that the sum of the numbers in any group is equal to the cube of the natural number which marks the order of the group.

3. A straight line and two circles are given; find the point in the straight line from which two tangents drawn to the circles shall be equal.

4. If r be the radius of the circle inscribed in a triangle, and r_a, r_b, r_c the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively, prove that

$$\sqrt{r_b r_c} + \sqrt{r_c r_a} + \sqrt{r_a r_b} = r.$$

5. Find the locus of a point, such that if from it a pair of tangents be drawn to an ellipse, the product of the perpendiculars dropped from the foci upon the line joining the points of contact shall be constant.

Also determine the curve to which the chord of contact is always a tangent.

6. A circular ring without weight, and having a heavy particle attached to a point in its circumference, rolls without sliding on the concave side of a semicircle the plane of which is vertical, and the radius equal to the diameter of the ring. Shew that every position of the ring, consistent with the condition that the particle is initially somewhere in the horizontal diameter of the semicircle, is a position of statical equilibrium.

7. If v, v', v'' be the velocities at three points P, Q, R of the path of a projectile, where the inclinations to the horizon are α, $\alpha - \beta$, $\alpha - 2\beta$, and if t, t' be the times of describing PQ, QR respectively, shew that

$$v''t = vt', \text{ and } \frac{1}{v} + \frac{1}{v''} = \frac{2\cos\beta}{v'}.$$

8. If three heavy particles be projected simultaneously from the same point in any directions and with any velocities, prove that the plane passing through them will always remain parallel to itself.

9. A particle is constrained to move in a circle, and is acted on by a force tending to a fixed point and varying inversely as the distance; prove that the sum of the squares of the velocities of the particle at the extremities of any chord drawn through the centre of force is constant.

10. An elastic string, not acted on by gravity, is made to whirl with a given angular velocity round one end which is fixed; find the length to which the string will be stretched, and explain why the length becomes infinite for a finite angular velocity.

11. A particle is describing an ellipse about a centre of force in the focus; when the particle is at a given point, the absolute force is slightly diminished, find the consequent alterations of the semi-axis major, eccentricity, and position of the apse.

If at the same time the square of the velocity had been diminished in the same proportion as the absolute force, shew that the position and dimensions of the orbit would have been unaffected.

12. Prove that the number of ways in which any number x can be composed of n numbers (not necessarily different from each other), is equal to the number of ways in which x can be composed of n and numbers not exceeding n, the order in which the numbers occur not being considered.

13. When x is any prime number, prove that the integral part of $\left(\cot\frac{\pi}{8}\right)^x - 2$ is divisible by $4x$, and that of $\left(\cot\frac{\pi}{12}\right)^x - 2^{x+1} + 1$ is divisible by $6x$.

Also; when x and y are any odd numbers, prove that the integral part of $(\sqrt{3})^y\left(\cot\frac{\pi}{12}\right)^x$ is divisible by 6 as long as y is less than x; and if x be given find the least value of y which makes the integral part of $(\sqrt{3})^y\left(\cot\frac{\pi}{12}\right)^x$ not divisible by 6.

14. If the two pairs of opposite sides of a quadrilateral inscribed in a Conic Section be produced to meet, and likewise the two pairs of tangents to the curve drawn from the opposite angles of the quadrilateral, prove that the four points of intersection will be in the same straight line.

Also if from any point in the Conic Section perpendiculars be drawn to the sides of the quadrilateral, prove that the product of the perpendiculars on one pair of opposite sides is to the product of the perpendiculars on the other pair of sides in a constant ratio.

15. In any Conic Sections if PQ, PR make equal angles with a fixed chord PK, and QR be joined, prove that QR will pass through a fixed point for all positions of PQ, PR.

Apply this property to prove that if PQ, PR be any two chords of an ellipsoid, in the same plane with, and inclined at the same angle to a fixed chord PK, then the locus of all the possible intersections of PQ, $P'Q'$, &c. will be the line PK and a plane curve.

16. If an ellipse be projected on a plane, prove that the projections of any pair of conjugate diameters of the ellipse will be a pair of conjugate diameters of its projection.

Also, having given the ratio of the axes of the original ellipse, and the ratio of their projections on the given plane, together with the angles which those projections make with a fixed line in that plane, determine the inclination of the plane of the ellipse to the fixed plane, and the angle which their line of intersection makes with the fixed line.

17. At the point in which the surface

$$\left(\frac{x^2}{a^2} - 1\right)^2 + \left(\frac{y^2}{b^2} - 1\right)^2 = \frac{z^2}{c^2} + 1$$

meets the axis of z, an elliptic paraboloid may be found, which has, at its vertex, a complete contact of the third order with the surface.

18. If through any point P, within a spherical triangle ABC, great circles be drawn from the angular points A, B, C, to meet the opposite sides in a, b, c respectively, prove that

$$\frac{\sin Pa \cdot \cos PA}{\sin Aa} + \frac{\sin Pb \cdot \cos PB}{\sin Bb} + \frac{\sin Pc \cdot \cos PC}{\sin Cc} = 1,$$

and thence deduce the corresponding property of a plane triangle.

19. ABC, $A'B'C'$ are two equal spherical triangles, the equal angles being placed in the same order; if the vertices of the equal angles be joined by arcs of great circles AA', BB', CC', prove that the great circles which bisect these arcs at right angles meet in a point, and that AA', BB', CC' subtend equal angles at that point.

20. Supposing the orbits of comets to be equably distributed through space, prove that their mean inclination to the plane of the ecliptic is the angle subtended by an arc equal to the radius.

21. Having given the ratio which the daily arc of retrogradation of a planet when in opposition bears to the Sun's daily motion, determine the distance of the planet from the Sun, supposing the orbits of the Earth and planet to be circular and in the same plane.

If the orbit of the planet be really an ellipse, find at what point of it the planet must be at the time of observation, in order that the distance found on the above hypothesis may coincide with the true distance.

22. A tube of small bore, in the form of a circle, revolves about one of its diameters as a fixed vertical axis, and contains a smooth particle; the angular velocity with which the tube is set in motion is known, and the particle starts from its highest position with a given velocity. Determine the angular velocity of the tube, and that of the particle relatively to the tube, for any given position of the particle, and shew that there are three other positions of the particle in the tube for which the angular velocity of the tube, and one other for which that of the particle is the same as for the given position.

23. If R be the radius of absolute curvature at any point of a curve defined by the intersection of two surfaces $u_1 = 0$, $u_2 = 0$, and r_1, r_2 be the radii of curvature of the sections of $u_1 = 0$, $u_2 = 0$ made by the tangent planes to $u_2 = 0$, $u_1 = 0$ respectively at that point, prove that R, r_1, r_2 will be connected by the relation

$$\frac{1}{R^2} = \frac{1}{r_1^2} - \frac{2\cos\theta}{r_1 r_2} + \frac{1}{r_2^2},$$

θ being the angle between the tangent planes.

FRIDAY, *January* 8, 1847.

Nine o'clock to half-past Eleven.

1. Similar triangles are to each other in the duplicate ratio of their homologous sides.

α. Draw a straight line perpendicular to a plane from a given point above it.

3. Find the positive integral values of x and y, which satisfy the equation $99x + 19y = 1900$; and shew that the number of positive integral solutions of the equation $ax + by = c$ cannot differ by more than unity from the greatest integer contained in $\frac{c}{ab}$.

4. Find the two middle terms of the expansion of $(a + x)^{13}$; and expand $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$ in ascending powers of x, writing down the n^{th} term both when n is odd and when it is even.

β. If the coefficients of an equation be rational, surd roots of the form $a + \sqrt{b}$ enter by pairs, and if $\sqrt{a} + \sqrt{b}$ be one root, where $\sqrt{a}$ and $\sqrt{b}$ do not

B

involve the same surd part, the three other values obtained by changing the signs of $\sqrt{a}$ and $\sqrt{b}$ will also be roots of the equation.

γ. Investigate the exponential expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$. State what is taken for the unit of angular measure in these expressions, and apply them to prove that

$$x\sin\theta + \frac{x^2\sin 2\theta}{1.2} + \frac{x^3\sin 3\theta}{1.2.3} + \ldots = \epsilon^{x\cos\theta}\sin(x\sin\theta).$$

δ. If from the several points of any straight line, the equation to which is given, pairs of tangents be drawn to an ellipse, prove that the corresponding chords of contact will all pass through a fixed point, and determine the co-ordinates of that point.

What does this proposition become when the given line passes through the centre of the ellipse?

ε. Prove that the semi-conjugate axis of a hyperbolic section of a right cone is a mean proportional between the perpendiculars dropped from the vertices of the hyperbola upon the axis of the cone.

ζ. Find the length of the perpendicular dropped from a given point upon a given plane, and the projections of this perpendicular upon the co-ordinate axes.

10. Differentiate the following functions:

$$\log_\epsilon \frac{\sqrt{a^2+x^2}+x}{a}, \quad \frac{\tan^3\theta}{3} - \tan\theta + \theta, \quad \text{and } \left(\frac{x}{n}\right)^{nx}.$$

11. Shew how the differential calculus may be applied to determine the limiting value of a fraction $\frac{f(x)}{\varphi(x)}$ which, for a particular value of x, assumes the form $\frac{0}{0}$. Prove that the same rule applies if the fraction assume the form $\frac{\infty}{\infty}$, and find by means of it the limiting values of $\frac{\sin^3 x}{x - \frac{1}{2}\sin 2x}$ when $x = 0$, and $\frac{\log x}{x}$ when $x = \infty$.

12. Find the expression for the radius of curvature in terms of p and r; and shew that when the angle between the perpendicular and the radius vector is a maximum or a minimum, the radius of curvature $= \frac{r^2}{p}$.

13. Integrate $\frac{1}{x^2 - a^2}$, $\frac{1}{x\sqrt{x^2+a^2}}$, and $e^{ax}\sin mx$; and find the value of $\int_0^\infty x^n e^{-x}dx$, n being a positive integer.

η. Trace the curve $y^2 = x^2 \cdot \frac{a+x}{a-x}$, and shew that the whole area of the curve $= 4a^2$, the area being supposed to be bounded on one side by the asymptote.

One o'clock to Four.

1. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in $1^{\text{h}}.\ 40^{\text{m}}.$; supposing the river to have a current of 2 miles per hour, find the rate at which the crew would row in still water.

2. If a and x both lie between 0 and 1, prove that $\frac{1-a^x}{1-a} > x$.

3. In the figure of Euclid, Book I. Prop. 35, (*Parallelograms on the same base and between the same parallels are equal,*) if two diagonals be drawn to the two parallelograms respectively, one from each extremity of the base, and the intersection of the diagonals be joined with the intersection of the sides (or sides produced) in the figure, prove that the joining line will bisect the base.

4. A piece of paper a mile long is rolled into a solid cylinder; find approximately the diameter of the cylinder, supposing 240 leaves of such paper to have a thickness of one inch.

5. PT, QT are two equal tangents to a parabola, P and Q being the points of contact; if PT, QT be both cut by a third tangent, prove that their alternate segments will be equal.

6. Two concentric ellipses which have their axes in the same directions intersect, and four common tangents are drawn so as to form a rhombus, and the points of intersection of the ellipses are joined so as to form a rectangle; prove that the product of the areas of the rhombus and rectangle is equal to half the continued product of the four axes.

7. If $\phi f(x) = \phi F(x)$ for all values of x from a to b, and if c be a quantity not less than a nor greater than b such that $f(c) = F(c)$, and $f'(c)$, $F'(c)$ have opposite signs, prove that $\phi(y)$ is necessarily a maximum or minimum when $y = f(c)$.

8. An imperfectly elastic ball is projected with a given velocity from a given point in a smooth inclined plane; find the direction of projection in order that the ball may cease to hop just as it returns to the point of projection.

9. A small plane touches a self-luminous paraboloid of revolution at its vertex, and is then moved parallel to itself along the axis produced; prove that the illumination of the plane varies inversely as its distance from the focus.

10. Three plane mirrors A, B, C are parallel to the same straight line; find the position of a luminous point, which is so distant that parallax may be neglected, in order that the rays reflected from A may be parallel to those which are reflected from B and C in succession.

11. Two heavy particles are connected by a light string, and immersed in a fluid whose specific gravity is intermediate between those of the particles, and which revolves uniformly about a fixed axis, and is not acted on by gravity; find the position of equilibrium of the particles relatively to the fluid, and shew that when the equilibrium is possible it is stable.

12. A ray of light passes from air into a transparent prism of either single or double refraction, and emerges after any number of internal reflections; apply the principle of quickest propagation to prove that the emergent ray or rays are inclined to the edges of the prism at the same angle as the incident ray.

13. A system of ellipses is described such that each ellipse touches two rectangular axes, to which its axes are parallel, and that the rectangle under the axes of the ellipses is constant; prove that each ellipse is touched by two rectangular hyperbolas, the rectangle under the transverse axes of which is equal to the rectangle under the axes of any one of the ellipses.

14. There are two systems of curves which cut each other everywhere at right angles; OM, NP are two curves of one system, ON, MP, are two of the other, and the arc $OM = x$, $ON = y$, $NP = \xi$, $MP = \eta$, and ρ, ρ' are the radii of curvature of PN, PM at the point P; prove that

$$\rho \frac{d^2\xi}{dxdy} = \rho' \frac{d^2\eta}{dxdy} = \frac{d\xi}{dx}\frac{d\eta}{dy},$$

the differential coefficients being taken on the supposition that O remains fixed while P alters.

15. The radius vector of any point in the surface of a solid differs from a constant by a small quantity of the order α, and the angle which it makes with the normal is of the same order; prove that if small quantities of the order α^2 are neglected the sphere which has the same volume as the given solid is also that which has the same surface.

16. A uniform flexible and inextensible heavy string AB is laid in the form of a circular arc on a smooth horizontal plane; a given impulsive force being applied at A in the direction of the tangent, find the impulsive tension at any point, and shew that the direction of the initial motion of A makes with the direction of the force an angle whose tangent $= \frac{\varepsilon^{\alpha} - \varepsilon^{-\alpha}}{\varepsilon^{\alpha} + \varepsilon^{-\alpha}}$, α being the angle which AB subtends at the centre.

17. Tangent planes to the surface whose equation, referred to rectangular co-ordinates, is $\frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = 1$ pass through a point P; prove that a sphere can be described through the curve of contact provided P lie in a certain right line passing through the origin.

18. The equations to a system of right lines in space contain two arbitrary parameters; prove that when the roots of a certain quadratic are real and unequal, there are two planes passing through a given line of the system which contain consecutive lines.

19. Find the value of $\int_0^\infty \left\{\varepsilon^{-(a+\alpha\sqrt{-1})x} - \varepsilon^{-(b+\beta\sqrt{-1})x}\right\}\frac{dx}{x}$, where a and b are positive, but α and β positive or negative; and shew that it is wholly real when $\frac{\alpha}{a} = \frac{\beta}{b}$.

20. Supposing the luminiferous ether to gravitate to the Sun, and to be incompressible and at rest, and supposing the velocity of light to vary by a small quantity proportional to the pressure of the ether, shew that the observed times of the eclipses of Jupiter's satellites will be affected by an inequality expressed by $c \log \tan \frac{1}{2}(E + S) \cot \frac{1}{4}E$, where c is a constant, and E, S, are two angles of the triangle ESJ.

21. A heavy particle is attached to a light string which passes over a horizontal cylinder; supposing the particle to perform small finite oscillations in a plane perpendicular to the axis of the cylinder, find approximately the correction to the time of oscillation due to the finite arc of oscillation, and shew that it vanishes when the radius of the cylinder is to the length of string hanging down as $\sqrt{3}$ to 2.

22. The refractive index (μ) of a transparent medium varies continuously from point to point; prove by means of the principle of quickest propagation,

THE SENATE-HOUSE EXAMINATION

or otherwise, that the differential equations to a ray of light within the medium are

$$\frac{ds}{dx}\left(\frac{d\mu}{dx} - \mu\frac{d^2x}{ds^2}\right) = \frac{ds}{dy}\left(\frac{d\mu}{dy} - \mu\frac{d^2y}{ds^2}\right) = \frac{ds}{dz}\left(\frac{d\mu}{dz} - \mu\frac{d^2z}{ds^2}\right),$$

where the differential coefficients of μ are partial.

23. An infinite mass of homogeneous incompressible fluid acted on by no forces is at rest, and a spherical portion of the fluid is suddenly annihilated; find the instantaneous alteration of pressure at any point of the mass, and prove that the cavity will be filled up in the time $\left(\frac{6\rho}{\varpi}\right)^{\frac{1}{2}} a \int_0^1 \frac{\lambda^4 d\lambda}{\sqrt{1-\lambda^6}}$, where a is the initial radius of the sphere, and ϖ the pressure at an infinite distance, which is supposed to remain constant.

24. A narrow hoop is rolled along a rough horizontal plane in such a manner as to move nearly in a vertical plane, but make small oscillations on each side of it; find the time of oscillation, and shew that oscillations of this sort are not possible unless the velocity of the centre of the hoop be greater than that due to one third of the radius.

SATURDAY, *January* 9, 1847.
Nine o'clock to half-past Eleven.

1. AB is a common chord of the segments ACB, $ADEB$ of two circles, and through C, any point in ACB, are drawn the straight lines ACE, BCD; prove that the arc DE is invariable.

2. If CP, CD be any conjugate diameters of an ellipse $APBDA'$, and BP, BD be joined, and also AD, $A'P$, these latter intersecting in O, shew that $BDOP$ is a parallelogram, and that its greatest area is $ab(\sqrt{2}-1)$.

3. Find the limiting value of $x\epsilon^{-x^2}\int_0^x \epsilon^{x^2}\,dx$ when $x = \infty$.

4. A regular octahedron is inscribed in a cube, so that the corners of the octahedron are in the centres of the faces of the cube; prove that the volume of the cube is six times that of the octahedron.

5. A bucket partly filled with water is attached to a weight by a string which passes over a fixed pully; supposing the water to revolve with a given angular velocity as the bucket is ascending or descending, find the form of the free surface.

6. A body acted on by a central force P is moving in a medium whose resistance = c (velocity); prove that

$$\frac{d^2r}{dt^2} + P - \frac{h^2}{r^3}\epsilon^{-2ct} + c\frac{dr}{dt} = 0,$$

where h is an arbitrary constant.

7. A plane drawn through a given point is illuminated by two self-luminous spheres; find the position of the plane when the illumination at the given point is a maximum.

8. OAA_1, is a spherical triangle, right-angled at A_1; the arc A_1A_2 of a great circle is drawn perpendicular to OA, A_2A_3 is drawn perpendicular to OA_1, and so on; prove that A_nA_{n+1} vanishes when n becomes infinite, and find the value of $\cos AA_1 . \cos A_1A_2 . \cos A_2A_3 \ldots$ *to infinity*.

9. Having given the times by a sidereal clock at which three known stars are observed to have the same zenith distance, the absolute zenith distance

being unknown, find the latitude of the place of observation and the error of the clock.

10. A circular disc is suspended by a fine wire attached to the centre, and immersed horizontally in a fluid; the wire being suddenly turned through a given angle, determine the motion of the disc, supposing each element of the surface acted on by a friction varying as the velocity, and shew that the successive arcs described from rest to rest are in geometric progression. Shew also that if the friction exceed a certain quantity the disc will not come to rest at all.

11. A sphere touches each of two right lines which are inclined to each other at a right angle, but do not intersect; prove that the locus of its centre is a hyperbolic paraboloid.

12. A slender rod suspended horizontally by two equal parallel strings attached to two points equidistant from its ends oscillates round a vertical line; find the time of a small oscillation.

If in the position of equilibrium the strings are inclined at equal angles to the vertical, shew that the time of oscillation is the same as it would be if the strings were parallel, of a length equal to the projection of either of them on a vertical line, and at a distance equal to a mean proportional between their distances at the points of suspension and attachment respectively.

13. A given quantity of incompressible fluid is contained in an elastic spherical envelope, and just fills the space inclosed without stretching the envelope; if the particles of the fluid be acted on by a repulsive force varying inversely as the square of the distance from the centre, find the absolute force when the space originally occupied by the fluid is left a vacuum.

14. A circle always touches the axis of z at the origin, and passes through a fixed straight line in the plane of xy; find the equation to the surface generated. Shew that the origin is a singular point, and that in its immediate neighbourhood the surface may be conceived to be generated by a circle having its plane parallel to that of xy, and its radius proportional to z^2.

15. P is a point in the base of a tetrahedron $VABC$, of which V is the vertex, so taken that the volume of the parallelepiped constructed on VP as diagonal, and having three of its edges coincident with VA, VB, VC, is a maximum; supposing the base of the tetrahedron to turn in all directions round a given point, while the edges adjacent to V remain fixed in position, find the locus of P.

16. A sphere touches an elliptic paraboloid at the vertex, and has its diameter a mean proportional between the parameters of the principal sections of the paraboloid; supposing the curve of intersection of the sphere and paraboloid to be projected on the tangent plane at the vertex, find the area of the curve of projection.

17. A tube of small bore, in the form of a logarithmic spiral, revolves with a uniform angular velocity about an axis passing through its pole and perpendicular to its plane, which is horizontal, and contains a particle which moves freely in it; supposing the initial velocity of the particle relatively to the tube to be equal to the velocity of the point of the tube in contact with the particle, shew that the path of the particle is another logarithmic spiral.

18. Shew that there cannot be any curve such that if tangents PT, QT be drawn at any points P and Q to meet in T, the angle which PT subtends at a fixed point S may always bear a constant ratio to that subtended by QT, except when that ratio is one of equality, and prove that in this case the curve is a conic section having S for its focus.

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Bubble Dynamics

Cavity collapse limited by liquid inertia

Origin: Senate-House Problems, Friday, Jan. 8^{th} 1847. By G. C. Stokes

70 years before the Rayleigh publication !!

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Challenge:

Solve this 23^{rd} question (and only this) using the knowledge of the 19^{th} century

TO BE CONTINUED …