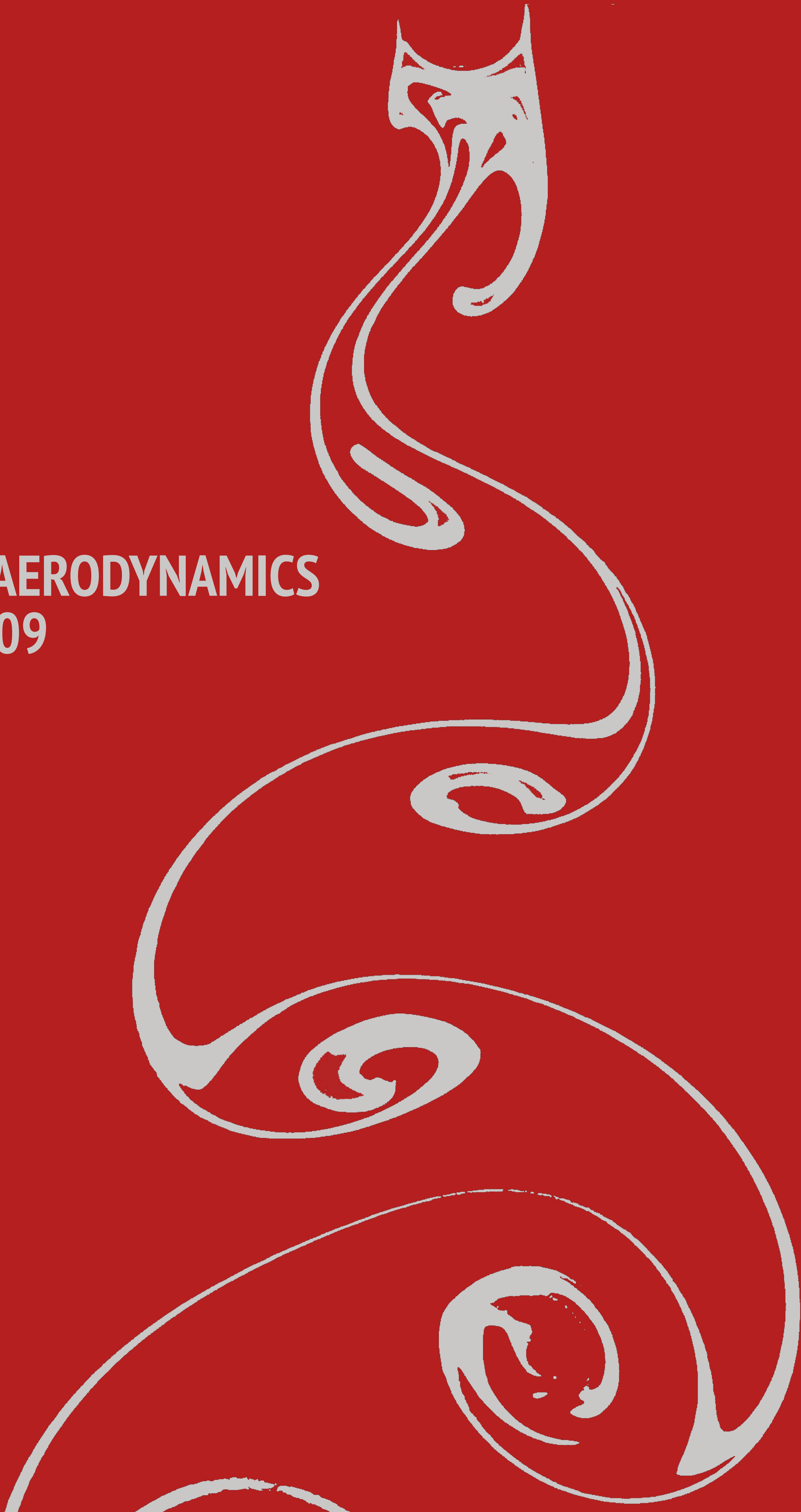
ME-445 AERODYNAMICS
Exercise 09
Week 8



Formula sheet

Cylindrical coordinates

$$\nabla\vec{u} = \left(\frac{\partial v_r}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0\right)$$

$$\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla\times\vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

Potential flow

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

Uniform parallel flow $\quad w = \phi + i\psi = U_\infty e^{-i\alpha}\, z$

Potential vortex in z_0 $\quad w = -\frac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $\quad w = \frac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $\quad w = \frac{\mu}{2\pi(z - z_0)}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{\mathrm{d}y_c}{\mathrm{d}x} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{x}{c} = \frac{(1-\cos\theta)}{2}$$

we know:

$$k = 2U_\infty\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\cos n\theta \mathrm{d}\theta$$

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{m,1/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_i(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_i < 0$

Prandtl's lifting-line theory

$$U_\infty\alpha_i(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{(\mathrm{d}\Gamma/\mathrm{d}y)}{y - y_0}\mathrm{d}y$$

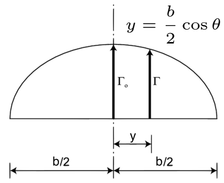
$$\alpha(y_0) = \alpha_{\text{eff}}(y_0) + \alpha_i(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_i = \frac{C_L}{\pi \mathrm{AR}}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}$$



General loading $\Gamma(\theta) = 2bU_\infty\sum_{n=1}^{\infty} A_n \sin n\theta$

$$w(\theta) = U_\infty\sum_{n=1}^{\infty} n A_n \frac{\sin n\theta}{\sin\theta}$$

$$C_L = \pi A_1 \,\mathrm{AR}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}(1+\delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\,(A_n/A_1)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}$ boundary layer growth

$C_f = \frac{1.328}{\sqrt{Re_x}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\frac{\delta}{x} = \frac{0.37}{Re_x^{1/5}}$ boundary layer growth

$C_f = \frac{0.074}{Re_x^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity	$\nu = 1 \times 10^{-6}\,\mathrm{m^2/s}$
density	$\rho = 1000\,\mathrm{kg/m^3}$

air

kinematic viscosity	$\nu = 1.5 \times 10^{-5}\,\mathrm{m^2/s}$
density	$\rho = 1.2\,\mathrm{kg/m^3}$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$
$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$
$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. The circulation around the wing at any point y is denoted Γ. If the circulation has a parabolic form:

$$\Gamma = \Gamma_0 \left[1 - \left(\frac{2y}{b} \right)^2 \right]$$

(a) What is the downward induced velocity behind the wing with the span b?

Solution:

$$\Gamma = \Gamma_0 \left[1 - \left(\frac{2y}{b} \right)^2 \right] = \Gamma_0 - \Gamma_0 \frac{4y^2}{b^2}$$

$$\frac{d\Gamma}{dy} = -\frac{8\Gamma_0 y}{b^2}$$

$$w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y - y_0} dy = \frac{2\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{y - y_0} dy$$

At mid-span:

$$w(y_0 = 0) = \frac{2\Gamma_0}{\pi b}$$

(b) Compare w at mid-span with that obtained when the same lift is distributed elliptically.

Solution: Downwash at mid-span:

$$w_q(y_0 = 0) = \frac{2\Gamma_0}{\pi b}$$

For the elliptical distribution:

$$w_e(y_0 = 0) = \frac{\Gamma_0}{2b}$$

Lift force:

$$L_q = \rho \mathrm{U}_\infty \int_{-b/2}^{b/2} \Gamma_0 \left(1 - \frac{4y^2}{b^2} \right) dy = \frac{2}{3} \rho \mathrm{U}_\infty \Gamma_0 b$$

For the elliptical distribution:

$$L_e = \frac{\pi}{4} \rho \mathrm{U}_\infty \Gamma_0 b$$

For the same lift:

$$\frac{\pi}{4} \Gamma_{0,e} = \frac{2}{3} \Gamma_{0,q}$$

$$\frac{w_{0,q}}{w_{0,e}} = \frac{2\Gamma_{0,q}}{\pi b} \frac{2b}{\Gamma_{0,e}} = \frac{4}{\pi} \frac{w_{0,q}}{w_{0,e}} = \frac{3}{2}$$

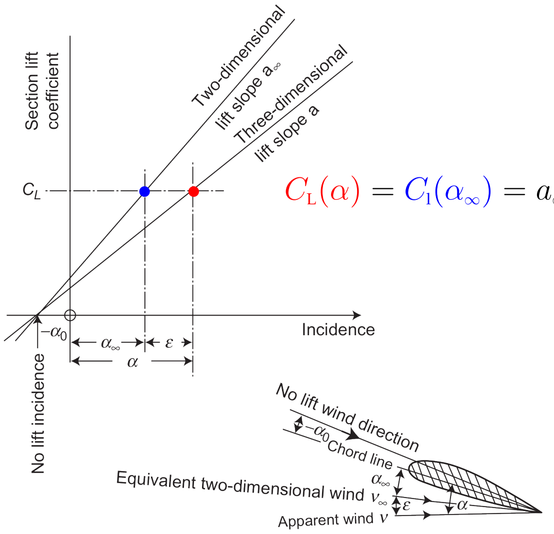
$\rightarrow$ The downwash of the elliptic wing is 2/3 of the downwash of the quadratic wing.

2. Consider a rectangular wing with an aspect ratio $\mathrm{AR} = 6$, an induced drag factor $\delta_{\mathrm{AR=6}} = 0.055$, and a zero-lift angle of attack $\alpha_0 = -2°$. At an angle of attack of $3.4°$, the induced drag coefficient for this wing is $C_{\mathrm{D,i,AR=6}} = 0.01$. Calculate the induced drag coefficient for another rectangular wing with the same airfoil section at the same angle of attack, but with an aspect ratio $\mathrm{AR} = 10$ with $\delta_{\mathrm{AR=10}} = 0.105$.

   (a) Show that the lift coefficient slope for the 3D wing can be expressed as:

$$a = \frac{\mathrm{d}C_{\mathrm{L}}}{\mathrm{d}\alpha} = \frac{a_\infty}{1 + \dfrac{a_\infty}{\pi AR}(1+\delta)}$$

   with a_∞ the lift coefficient slope for the 2D airfoil



$$\color{red}{C_{\mathrm{L}}(\alpha)} = \color{blue}{C_{\mathrm{l}}(\alpha_\infty)} = a_\infty(\alpha_\infty - \alpha_0)$$

   Solution: From the lift coefficient distribution

$$C_{\mathrm{L}}(\alpha) = C_{\mathrm{l}}(\alpha_\infty) = a_\infty(\alpha_\infty - \alpha_0)$$

$$C_{\mathrm{L}} = a_\infty(\alpha - \alpha_{\mathrm{i}} - \alpha_0)$$

   with

$$\alpha_{\mathrm{i}} = \frac{C_{\mathrm{D,i}}}{C_{\mathrm{L}}} = \frac{C_{\mathrm{L}}}{\pi \mathrm{AR}}(1+\delta)$$

$$C_{\mathrm{L}} = a_\infty\alpha - a_\infty\frac{C_{\mathrm{L}}}{\pi\mathrm{AR}}(1+\delta) - a_\infty\alpha_0$$

$$C_{\mathrm{L}}\left(1 + \frac{a_\infty}{\pi\mathrm{AR}}(1+\delta)\right) = a_\infty\alpha - a_\infty\alpha_0$$

$$\Rightarrow \quad a = \frac{\mathrm{d}C_{\mathrm{L}}}{\mathrm{d}\alpha} = \frac{\partial C_{\mathrm{L}}}{\partial\alpha} = \frac{a_\infty}{1 + \dfrac{a_\infty}{\pi AR}(1+\delta)}$$

   (b) Calculate the induced drag coefficient for another rectangular wing with the same airfoil section at the same angle of attack, but with an aspect ratio $\mathrm{AR} = 10$ with $\delta_{\mathrm{AR=10}} = 0.105$.

Solution: For $AR = 6$:

$$C_L^2 = \frac{\pi \text{AR} C_{D,i}}{1+\delta} = \frac{\pi \times 6 \times 0.01}{1.055} = 0.1787$$
$$C_L = 0.423$$
$$\frac{dC_L}{d\alpha} = \frac{\Delta C_L}{\Delta \alpha} = \frac{0.423}{3.4+2} = 0.078/° = 4.485/\text{rad}$$
$$\frac{dC_L}{d\alpha} = a = \frac{a_\infty}{1 + \frac{a_\infty}{\pi \text{AR}}(1+\delta)}$$
$$a_\infty = \frac{a_{\text{AR=6}}}{1 - \frac{a_{\text{AR=6}}}{\pi \text{AR}}(1+\delta_{\text{AR=6}})} = \frac{4.485}{1 - \frac{4.485}{\pi \times 6} \times 1.055} = 5.99/\text{rad}$$

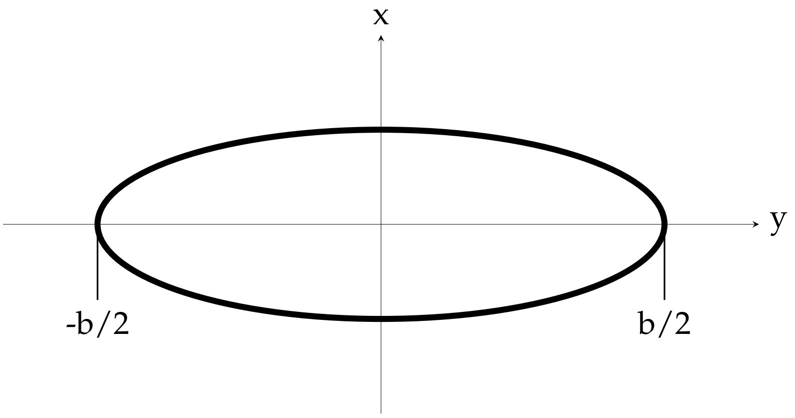
The second wing with $AR = 10$ has the same 2D airfoil profiles, thus the lift coefficient slopes are as follows:

$$a_{\text{AR=10}} = \frac{a_\infty}{1 + \frac{a_\infty}{\pi \text{AR}}(1+\delta_{\text{AR=10}})} = \frac{5.99/\text{rad}}{1 + \frac{5.99/\text{rad}}{\pi \times 10} \times 1.105} = 4.95/\text{rad} = 0.0864/°$$
$$C_L = a_{\text{AR=10}}(\alpha - \alpha_0) = 0.0864 \times (3.4+2) = 0.467$$
$$C_{D,i} = \frac{C_L^2}{\pi \text{AR}}(1+\delta) = \frac{0.467^2}{\pi \times 10} \times 1.105 = 0.0077$$

3. Consider an untwisted, unswept wing which has an elliptical lift distribution $\Gamma_{\text{elliptical}}$ under normal flight conditions. The wing is equipped with a system to reduce torsional moments when encountering gusts. When approaching a storm front, rudder and flap motions alter the span-wise circulations distribution to $\Gamma_{\text{altered}}(\theta) = \Gamma_0 \left[\sin\theta - \frac{1}{4}\sin 3\theta\right]$ with Γ_0 the same constant as in the expression for $\Gamma_{\text{elliptical}}$.

(a) Draw a sketch of the most probable planform area of this wing seen from above.

Solution: Elliptical planform no sweep



(b) Determine the span-wise lift distribution under normal conditions $L_{\text{elliptical}}(\theta)$.

Solution:

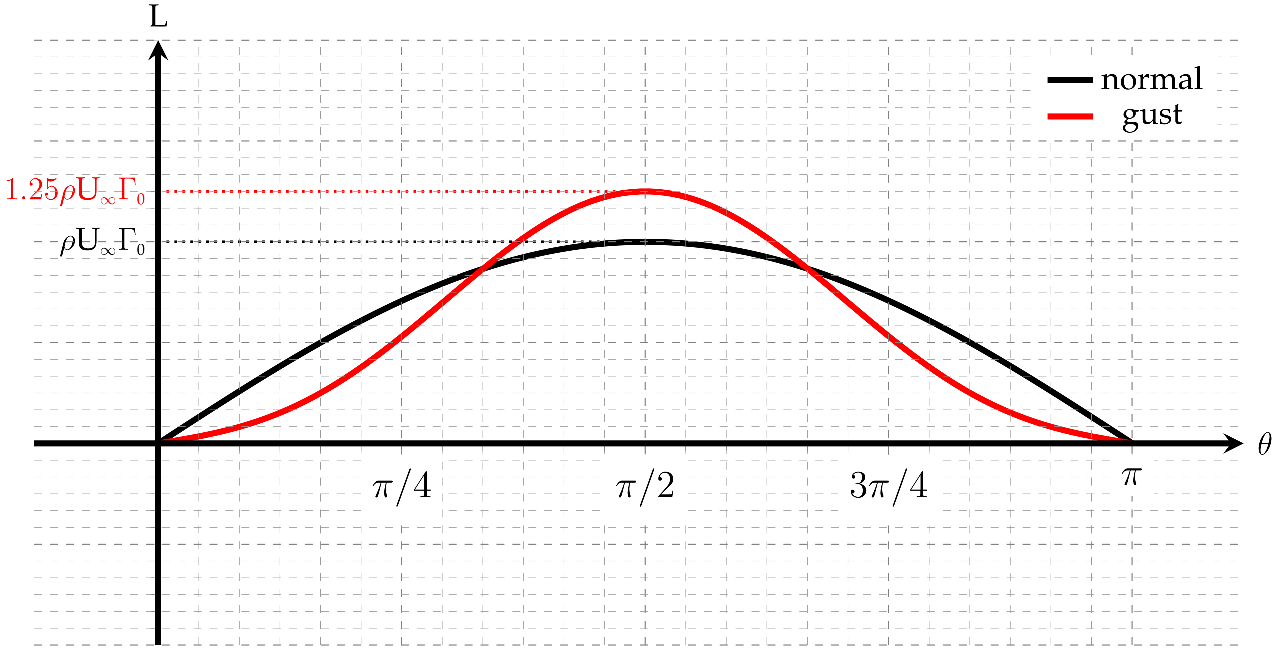
$$\begin{aligned} L_{\text{normal}}(\theta) &= \rho \mathrm{U}_\infty \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2} \\ &= \rho \mathrm{U}_\infty \Gamma_0 \sin\theta \end{aligned}$$

(c) Determine the span-wise lift distribution under altered gusty conditions $L_{\text{altered}}(\theta)$.

Solution:

$$L_{\text{altered}}(\theta) = \rho \mathrm{U}_\infty \Gamma_0 \left[\sin\theta - \frac{1}{4}\sin 3\theta\right]$$

(d) Draw both lift distributions in function of the span-wise location and indicate explicitly the extrema values.



(e) Determine for both flight conditions the total lift coefficient C_L

Solution: $$\int_{-b/2}^{b/2} \Gamma(y)dy = \frac{b}{2}\int_0^{\theta} \Gamma(\theta)\sin\theta d\theta$$

$$C_L = \frac{L}{1/2\rho U_\infty^2 S} = \frac{\rho U_\infty}{1/2\rho U_\infty^2 S}\int_{-b/2}^{b/2}\Gamma(y)dy = \frac{b}{U_\infty S}\int_0^{\theta}\Gamma(\theta)\sin\theta d\theta$$

Elliptical distribution:

$$C_L = \frac{b}{U_\infty S}\int_0^{\theta}\Gamma_0 \sin^2\theta d\theta$$

$$= \frac{b\,\pi\,\Gamma_0}{S\,2\,U_\infty}$$

Altered distribution:

$$C_L = \frac{b}{U_\infty S}\int_0^{\theta}\Gamma_0\sin^2\theta d\theta + \underbrace{\frac{b}{U_\infty S}\int_0^{\theta} -\frac{\Gamma_0}{4}\sin 3\theta \sin\theta d\theta}_{=0}$$

$$= \frac{b\,\pi\,\Gamma_0}{S\,2\,U_\infty}$$

Solution:

(f) Determine the extra propulsion force that is required to remain the same flight altitude when the lift distribution is altered from $L_{\text{elliptical}}(\theta)$ to $L_{\text{altered}}(\theta)$.

(Hint: use $C_{\text{D,i}} = \dfrac{C_{\text{L}}^2}{\pi \text{AR}}(1+\delta)$ with $\delta = \sum_{n=2}^{\infty} n\,(A_{\text{n}}/A_{1})^2$)

Solution:

General loading: $\Gamma(\theta) = 2b\text{U}_\infty \sum_{n=1}^{\infty} A_{\text{n}} \sin n\theta$ and $\delta = \sum_{n=2}^{\infty} n\,(A_{\text{n}}/A_{1})^2$

Elliptical distribution:

$$\Gamma(\theta) = \Gamma_0 \sin\theta = 2b\text{U}_\infty A_1 \sin\theta \Rightarrow A_1 = \frac{\Gamma_0}{2b\text{U}_\infty}$$

$$\delta = 0$$

Altered distribution:

$$\Gamma(\theta) = \Gamma_0 \left[\sin\theta - \frac{1}{4}\sin 3\theta\right] = 2b\text{U}_\infty A_1 \sin\theta + = 2b\text{U}_\infty A_3 \sin 3\theta$$

$$\Rightarrow A_1 = \frac{\Gamma_0}{2b\text{U}_\infty} \quad , \qquad A_3 = -\frac{\Gamma_0}{8b\text{U}_\infty}$$

$$\delta = 3\left(\frac{A_3}{A_1}\right)^2 = 3\left(\frac{2}{-8}\right)^2 = \frac{3}{16}$$

Extra propulsion necessary to compensate the increased drag

$$\begin{aligned}\Delta C_{\text{D,i}} &= \delta_{\text{altered}} \frac{C_{\text{L}}^2}{\pi AR} \\ &= \frac{3}{16}\frac{C_{\text{L}}^2}{\pi AR} = \frac{3}{16}\frac{b^2\pi^2\Gamma_0^2}{S^2\, 4\text{U}_\infty^2}\frac{S}{\pi b^2} = \frac{3}{128}\frac{\pi\Gamma_0^2}{1/2\text{U}_\infty^2\, S} \\ \Rightarrow \Delta D &= \frac{3}{128}\rho\pi\Gamma_0^2\end{aligned}$$

(g) Can you think of a reason why it would help to change the lift distribution in the proposed way in terms of gust load alleviation?

Solution: Higher load distribution near the centre reduces the bending moments near the tips.