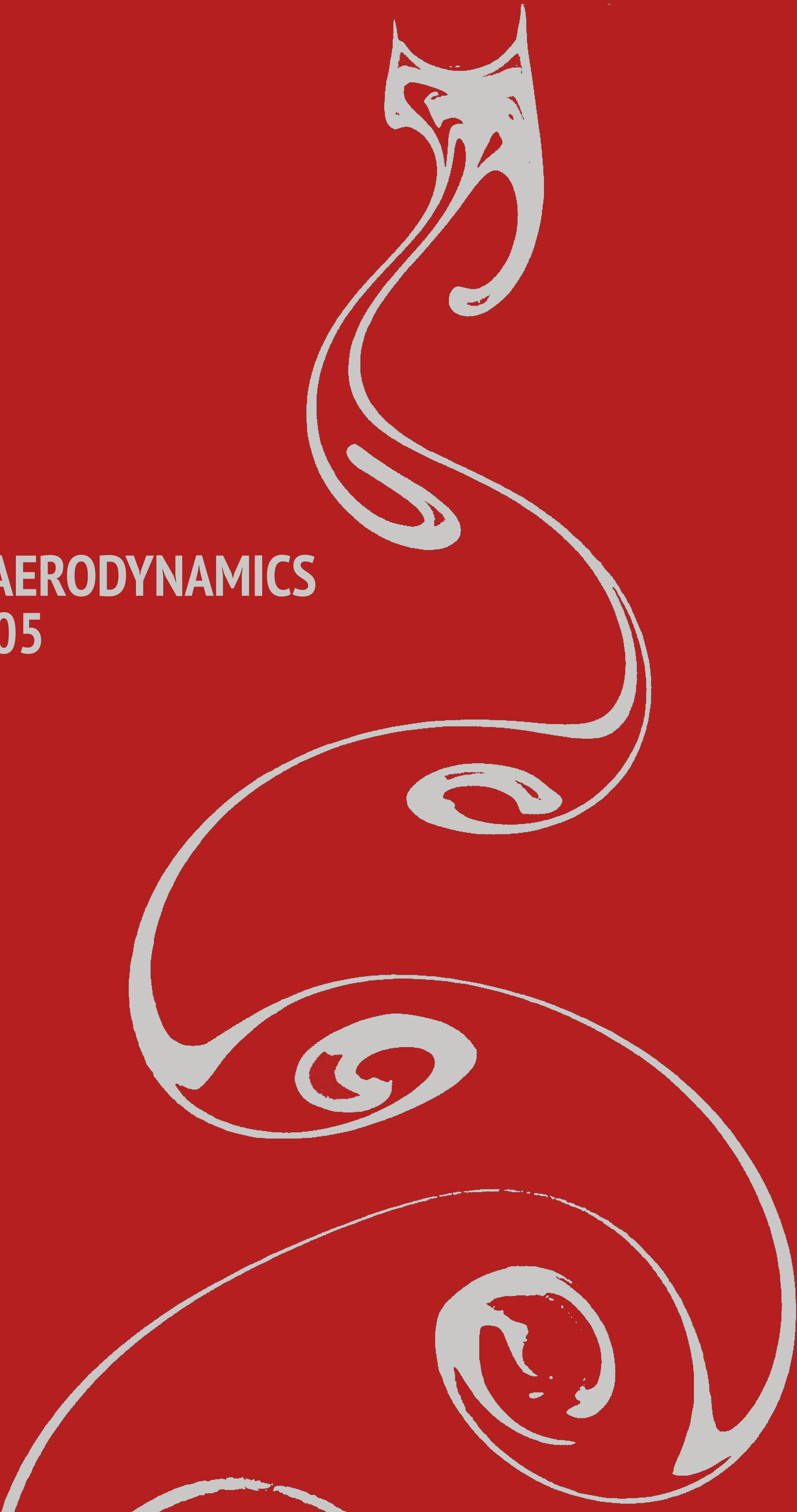
EPFL

ME-445 AERODYNAMICS
Exercise 05
Week 4



Formula sheet

Cylindrical coordinates

$$\nabla \vec{u} = \left(\frac{\partial v_r}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0 \right)$$

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right] \right)$$

Potential flow

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $w = \phi + i\psi = \mathrm{U}_\infty e^{-i\alpha} z$

Potential vortex in z_0 $w = -\dfrac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $w = \dfrac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $w = \dfrac{\mu}{2\pi(z - z_0)}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{\mathrm{d}y_c}{\mathrm{d}x} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{x}{c} = \frac{(1 - \cos\theta)}{2}$$

we know:

$$k = 2\mathrm{U}_\infty \left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\cos n\theta \mathrm{d}\theta$$

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{m,1/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_i(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_i < 0$

Prandtl's lifting-line theory

$$\mathrm{U}_\infty \alpha_i(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(\mathrm{d}\Gamma/\mathrm{d}y)}{y - y_0}\mathrm{d}y$$

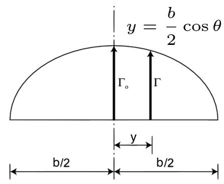
$$\alpha(y_0) = \alpha_{\text{eff}}(y_0) + \alpha_i(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_i = \frac{C_L}{\pi \mathrm{AR}}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}$$



General loading $\Gamma(\theta) = 2b\mathrm{U}_\infty \sum_{n=1}^{\infty} A_n \sin n\theta$

$$w(\theta) = \mathrm{U}_\infty \sum_{n=1}^{\infty} nA_n \frac{\sin n\theta}{\sin\theta}$$

$$C_L = \pi A_1 \mathrm{AR}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}(1 + \delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\,(A_n/A_1)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\dfrac{\delta}{x} = \dfrac{5}{\sqrt{Re_x}}$ boundary layer growth

$C_f = \dfrac{1.328}{\sqrt{Re_x}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\dfrac{\delta}{x} = \dfrac{0.37}{Re_x^{1/5}}$ boundary layer growth

$C_f = \dfrac{0.074}{Re_x^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity $\nu = 1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$

density $\rho = 1000\,\mathrm{kg\,m^{-3}}$

air

kinematic viscosity $\nu = 1.5 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$

density $\rho = 1.2\,\mathrm{kg\,m^{-3}}$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$
$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$
$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. The velocity components of a two-dimensional inviscid incompressible flow are given by

$$u = 2y + \frac{y}{\sqrt{x^2 + y^2}}$$

$$v = -2x - \frac{x}{\sqrt{x^2 + y^2}}$$

(a) Find the stream function ψ that satisfies the boundary condition ψ (0,0) = 0 in cartesian and polar coordinates.

Solution: From the definition of the stream function

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

By integration we obtain

$$\psi = y^2 + \sqrt{x^2 + y^2} + f(x) + C_1$$
$$\psi = x^2 + \sqrt{x^2 + y^2} + g(y) + C_2$$

where C_1 and C_2 are constants of integration, and f and g are unknown functions of x and y respectively. By comparing the two results for ψ, we get

$$\psi = x^2 + y^2 + \sqrt{x^2 + y^2} + C$$

where we apply the boundary condition at (0, 0) to get

$$\psi = x^2 + y^2 + \sqrt{x^2 + y^2}$$

In polar coordinates, $x^2 + y^2 = r^2$, so this can be rewritten as:

$$\psi = r^2 + r$$

(b) Is this flow irrotational? Hint:

$$\nabla \times \vec{U} = \left(0, 0, \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)\right)$$

Solution: In cartesian coordinates, the equation for the z component of the vorticity is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega_z$$

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= 2 + \frac{1}{\sqrt{x^2+y^2}} - \frac{1}{2}\frac{2y^2}{(\sqrt{x^2+y^2})^3} \\
\frac{\partial v}{\partial x} &= -2 - \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{2}\frac{2x^2}{(\sqrt{x^2+y^2})^3} \\
\Rightarrow \omega_z &= -2 - \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{2}\frac{2x^2}{(\sqrt{x^2+y^2})^3} - 2 - \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{2}\frac{2y^2}{(\sqrt{x^2+y^2})^3} \\
&= -4 - \frac{2}{\sqrt{x^2+y^2}} + \frac{x^2+y^2}{(\sqrt{x^2+y^2})^3} \\
&= -(4 + \frac{1}{\sqrt{x^2+y^2}})
\end{aligned}
$$

In polar coordinates,

$$\omega_z = \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

where

$$v_\theta = -\frac{\partial \psi}{\partial r} = -2r - 1$$

and

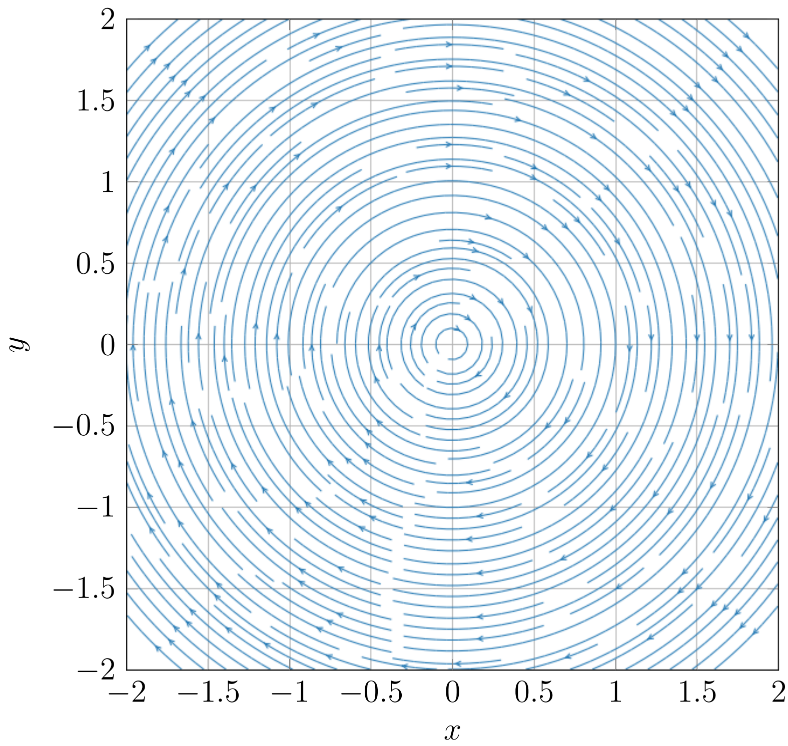
$$v_r = \frac{1}{r}\left(\frac{\partial \psi}{\partial \theta}\right) = 0$$

From this,

$$\omega_z = \frac{1}{r}\left(\frac{\partial r(-2r-1)}{\partial r}\right) = -\left(4 + \frac{1}{r}\right)$$

Alas, no, the flow is not irrotational.

(c) Sketch the streamlines.



Solution:

(d) What is the circulation Γ in the contour given by $\psi = 1$?

Solution: Let a be the radius of the contour given by $\psi = 1$. The circulation Γ is defined as

$$\Gamma = \oint_C \vec{V} \cdot \hat{t}\, ds = \iint_A \vec{\omega_z} r\, dr\, d\theta = -\int_0^{2\pi}\int_0^a \left(4 + \frac{1}{r}\right) r\, dr\, d\theta = 2\pi(-2a^2 - a)$$

We can determine the value of a by plugging in 1 for ψ and a for r in the polar equation from 1(a) and solving for a. We obtain the following equation:

$$\psi = 1 = a^2 + a.$$

Applying the quadratic formula we obtain two solutions:

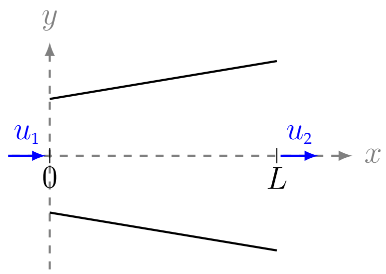
$$a = \frac{-1 \pm \sqrt{5}}{2}$$

Since the value of the radius must be positive, we take the positive solution, or

$$a = \frac{-1 + \sqrt{5}}{2} = 0.6180$$

which we can plug into our formula for Γ to obtain Γ = -8.683 m^3/s, which by convention means clockwise circulation.

2. Consider the steady 2D potential flow in a diverging channel. The velocity field is given by $\vec{U} = (u, v)$; the x-component of $\vec{U}$ is given by $u = \alpha x + \beta$, with α and β constant. The velocity at $x = 0$ is equal to u_1 and the x-component of the velocity at $x =$ L is u_2.



(a) What are the assumptions of potential flow?

Solution: A potential flow is incompressible ($\nabla \cdot \vec{U} = 0$) and irrotational ($\nabla \times \vec{U} = 0$).

(b) Use the fact that the flow is incompressible to derive an expression for the y-component of the velocity field given that $v(y = 0) = 0$.

Solution:
If the flow is incompressible: $\nabla \cdot \vec{U}$

$$\begin{aligned} \Rightarrow \frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial x} \\ &= -\frac{\partial}{\partial x}(\alpha x + \beta) \\ &= -\alpha \end{aligned}$$

Integration yields: $v = -\alpha y + C$ where C is the integration constant.
Using the boundary condition $v(y = 0) = 0$ we see that $C = 0$ and

$$v = -\alpha y$$

(c) Is this flow irrotational?

Solution: The flow is irrotational if

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

or

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow \frac{\partial(\alpha x + \beta)}{\partial y} = 0 = \frac{\partial(-\alpha y)}{\partial x}$$

Yes! The flow is irrotational!

(d) Derive the expression for the complex potential $w(z)$, with $z = x + iy$ and the boundary condition $w(0) = 0$.

Solution:

$$\frac{dw}{dz} = u - iv = \alpha x + \beta + i\alpha y = \alpha z + \beta$$

Integrating the equation above we obtain the complex potential that satisfies the above boundary condition:

$$w(z) = \frac{1}{2}\alpha z^2 + \beta z$$

(e) A solid circular cylinder with radius R is mounted on the central axis of the diverging channel. Assume that the diameter of the cylinder is small compared to the local width of the channel. Determine the complex potential $w(z)$ of the diverging flow in which this cylinder is placed.
Hint: Use the Milne-Thomson circle theorem.

Solution: According to the Milne-Thomson circle theorem a new stream function for a fluid flow when a cylinder is placed into that flow is given by

$$g(z) = w(z) + \overline{w(\frac{R^2}{\overline{z}})} = \frac{1}{2}\alpha z^2 + \beta z + \frac{1}{2}\alpha\frac{R^4}{z^2} + \beta\frac{R^2}{z}$$

(f) (i) What is the stream function ψ for the flow over the circular cylinder?

Solution: The stream function for the flow over a circular cylinder is the imaginary part of the complex potential, or:

$$\psi(z) = \text{Im}(g(z))$$
$$\psi(z) = \frac{1}{2}\alpha r^2 \sin 2\theta + \beta r \sin\theta - \frac{1}{2}\alpha\frac{R^4}{r^2}\sin 2\theta - \beta\frac{R^2}{r}\sin\theta$$

Since $u(x = 0) = u_1$, $\beta = u_1$, and so:

$$\psi(z) = \frac{1}{2}\alpha r^2 \sin 2\theta + u_1 r \sin\theta - \frac{1}{2}\alpha\frac{R^4}{r^2}\sin 2\theta - u_1\frac{R^2}{r}\sin\theta$$

(ii) Find the velocity on the surface of the cylinder given that

$$v_\theta = -\frac{\partial\psi}{\partial r}.$$

Solution:

$$v_\theta = -\frac{\partial\psi}{\partial r} = -\left(\alpha r \sin 2\theta + u_1 \sin\theta + \alpha\frac{R^4}{r^3}\sin 2\theta + u_1\frac{R^2}{r^2}\sin\theta\right)$$

The velocity distribution on the surface of the cylinder is given by $v_r = 0$ and $v_\theta = -2u_1 \sin\theta$, where θ is a polar angle.

(iii) Find the lift force exerted on the cylinder. Assume that $\text{U}_\infty = u(L/2)$.

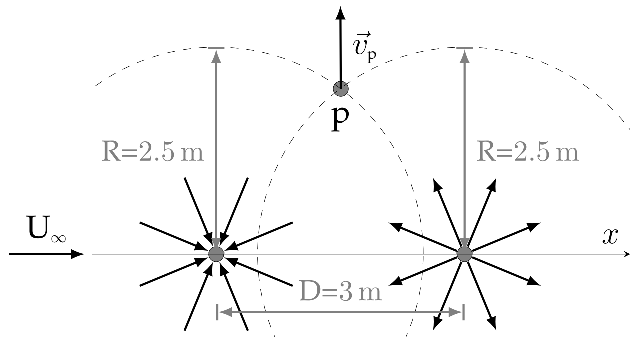
Solution: Using the boundary conditions at $x = 0$ and $x = \text{L}$,

$$u(x = 0) = u_1 = \beta$$
$$u(x = L) = u_2 = \alpha L + u_1 \Rightarrow \alpha = \frac{u_2 - u_1}{L}$$
$$\text{U}_\infty = u(L/2) = \alpha\frac{L}{2} + \beta = \frac{u_2 - u_1}{2} + u_1 = \frac{u_1 + u_2}{2}$$
$$c_p = 1 - \frac{v^2}{\text{U}_\infty^2} = 1 - \frac{4u_1^2 \sin^2\theta}{(u_2 + u_1)^2}$$
$$c_l = -\int_0^{2\pi} c_\text{p} \sin\theta d\theta = 0 \Rightarrow \text{lift} = 0$$

(iv) In what direction should the cylinder be translated in order for a non-zero resultant force to be exerted on it?

Solution: Currently no lift is generated over the cylinder because the incoming flow is symmetric. However, translating it such that there is flow of greater velocity over one side of the cylinder (either the top or the bottom) will result in a net resultant force pushing up or down on the cylinder. For this, the cylinder can be shifted up *or* down with respect to the centerline (x-axis) of the diverging channel.

3. A sink of strength $20\,\mathrm{m^2\,s^{-1}}$ is located $3\,\mathrm{m}$ upstream of a source of $40\,\mathrm{m^2\,s^{-1}}$ in a horizontal uniform irrotational flow that goes from left to right. At a point p located $2.5\,\mathrm{m}$ from both the source and the sink. Find the velocity at point p and the velocity of the uniform flow U_∞ that satisfy the condition that the resulting local velocity at p is vertical.



Solution:

The complex potentials for a sink at z_{sink}, a source at z_{source}, and a free stream flow are:

$$
\begin{aligned}
w_{\text{sink}}(z) &= \frac{Q_{\text{sink}}}{2\pi}\ln(z - z_{\text{sink}}) \\
w_{\text{source}}(z) &= \frac{Q_{\text{source}}}{2\pi}\ln(z - z_{\text{source}}) \\
w_{\text{flow}}(z) &= \mathrm{U}_\infty z
\end{aligned}
$$

$$
\begin{aligned}
w(z) &= w_{\text{sink}}(z) + w_{\text{source}}(z) + w_{\text{flow}}(z) \\
&= \frac{Q_{\text{sink}}}{2\pi}\ln(z - z_{\text{sink}}) + \frac{Q_{\text{source}}}{2\pi}\ln(z - z_{\text{source}}) + \mathrm{U}_\infty z
\end{aligned}
$$

We can treat the point p as lying on the y-axis, so that the radial distance to the sink, z_{sink}, is -D/2 and the radial distance to the source, z_{source}, is D/2, given that the distance between the source and the sink is D. Since the source has a strength that is double in magnitude but opposite in sign of the sink, we can write Q_{source} in terms of Q_{sink} and plug it into the above formula, such that:

$$
w(z) = -\frac{Q_{\text{sink}}}{2\pi}\ln(z + D/2) + \frac{2Q_{\text{sink}}}{2\pi}\ln(z - D/2) + \mathrm{U}_\infty z
$$

$$
\begin{aligned}
u - iv &= \frac{dw}{dz} \\
&= -\frac{Q_{\text{sink}}}{2\pi}\frac{1}{z + D/2} + \frac{2Q_{\text{sink}}}{2\pi}\frac{1}{z - D/2} + \mathrm{U}_\infty \\
&= \frac{Q_{\text{sink}}}{2\pi}\left(-\frac{1}{z + D/2} + \frac{2}{z - D/2}\right) + \mathrm{U}_\infty \\
&= \frac{Q_{\text{sink}}}{2\pi}\left(\frac{z + 3D/2}{z^2 - D^2/4}\right) + \mathrm{U}_\infty
\end{aligned}
$$

Since p resides on the y-axis, the x and y-coordinates of P can be written as:

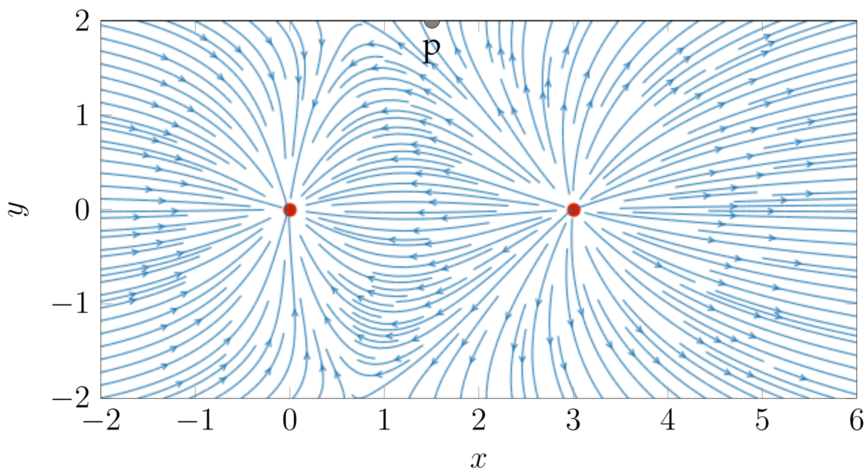
$$
\begin{aligned}
&x_\text{p} = 0 \\
&y_\text{p} = \sqrt{R^2 - D^2/4} \\
&\quad \Rightarrow z_\text{p} = i\sqrt{R^2 - D^2/4} = (\sqrt{R^2 - D^2/4})\, e^{\pi/2}
\end{aligned}
$$

where D is the distance between the source and the sink and R the distance from p to the source and to the sink.

Since point p only experiences a vertical velocity, we can describe the velocity field as:

$$
\begin{aligned}
\cancelto{0}{u(p)} - iv(p) &= \frac{Q_\text{sink}}{2\pi}\left(\frac{i\sqrt{R^2 - D^2/4} + 3D/2}{-(R^2 - D^2/4) - D^2/4}\right) + \text{U}_\infty \\
&= \frac{Q_\text{sink}}{2\pi}\left(\frac{i\sqrt{R^2 - D^2/4} + 3D/2}{-R^2}\right) + \text{U}_\infty \\
&= \underbrace{\text{U}_\infty - \frac{3Q_\text{sink}D}{4\pi R^2}}_{u=0} - i\underbrace{\frac{Q_\text{sink}}{2\pi R^2}(\sqrt{R^2 - D^2/4})}_{v}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \quad \text{U}_\infty = \frac{3Q_\text{sink}D}{4\pi R^2} = 2.29\,\text{m}\,\text{s}^{-1} \\
&\Rightarrow \quad v(p) = \frac{Q_\text{sink}}{2\pi R^2}(\sqrt{R^2 - D^2/4}) = 1.02\,\text{m}\,\text{s}^{-1}
\end{aligned}
$$



4. A two-dimensional source is placed in a uniform flow of $U_\infty = 2\,\mathrm{m\,s^{-1}}$ from left to right along the x-direction. The volume flow rate coming from the source is $4\,\mathrm{m^2\,s^{-1}}$.

(a) Find the location of the stagnation point.

Solution: The streamlines of the combination of a freestream and a source flow are:

$$\psi = \psi_{\text{stream}} + \psi_{\text{source}} = \frac{Q_{\text{source}}}{2\pi}\theta + U_\infty r\sin\theta = \frac{Q_{\text{source}}}{2\pi}\arctan\frac{y - y_{\text{source}}}{x - x_{\text{source}}} + U_\infty y$$

To find the location of stagnation point, we solve the following equations: $u = 0$, $v = 0$

$$u = \frac{\partial\psi}{\partial y} = 0 = \frac{Q_{\text{source}}}{2\pi}\frac{x}{x^2+y^2} + U_\infty$$

$$v = -\frac{\partial\psi}{\partial x} = 0 = \frac{Q_{\text{source}}}{2\pi}\frac{y}{x^2+y^2} \Rightarrow y = 0$$

$$u = 0 = \frac{Q_{\text{source}}}{2\pi}\frac{1}{x_0} + U \Rightarrow \boxed{x_0 = -\frac{Q_{\text{source}}}{2\pi U_\infty}}$$

$$x_0 = -\frac{4\,\mathrm{m^2\,s^{-1}}}{2\pi\cdot 2\,\mathrm{m\,s^{-1}}} = -0.32\,\mathrm{m}$$

The stagnation point is upstream of the source ($x_0 < 0$). The streamline containing the stagnation point is called the dividing streamline. It separates the fluid coming from the freestream and the fluid radiating from the source flow.

(b) Sketch the body shape passing through the stagnation point. Find the maximum width of the body.

Solution: At $x = x_0$, we have $\theta = \pi$ and $r = |x - x_0| = \frac{Q_{\text{source}}}{2\pi U_\infty} = 0.32\,\mathrm{m}$. Substituting these values in the expression for ψ, we get the value of ψ at the stagnation point to be

$$\psi_{\text{stagnation}} = \frac{Q_{\text{source}}}{2\pi}\pi + U_\infty\frac{Q_{\text{source}}}{2\pi U_\infty}\sin\pi = \frac{Q_{\text{source}}}{2}$$

An equation to the streamline passing through the stagnation point is obtained as follows,

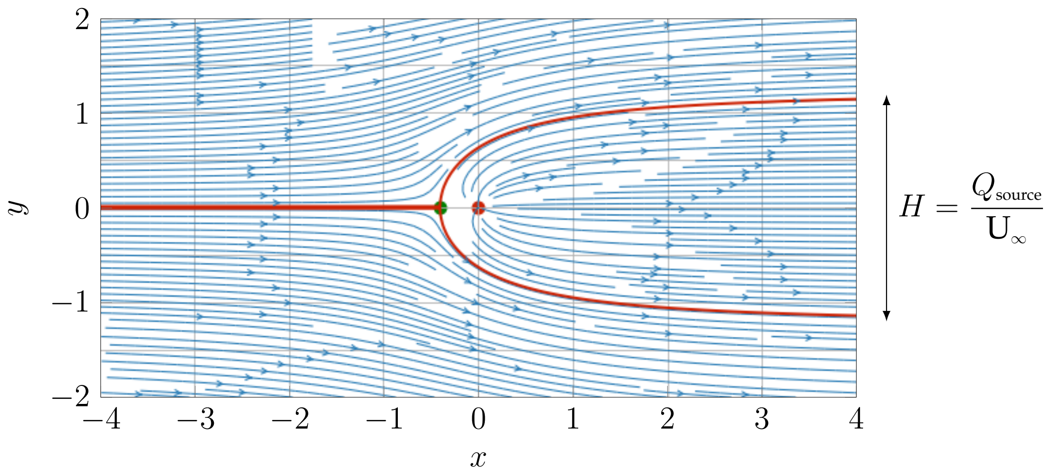
$$\psi_{\text{stagnation}} = \frac{Q_{\text{source}}}{2} = \frac{Q_{\text{source}}}{2\pi}\theta + U_\infty r\sin\theta$$

$$\Rightarrow U_\infty r\sin\theta = \frac{Q_{\text{source}}}{2\pi}(\pi - \theta)$$

Hence

$$\boxed{y = r\sin\theta = \frac{Q_{\text{source}}}{2\pi U_\infty}(\pi - \theta)}$$

The streamlines for this flow are sketched in the figure.



Limits of θ for this body are 0 and 2π. At these values we have $y = \pm\frac{Q_{\text{source}}}{2U_\infty}$. The width of the body is $H = \frac{Q_{\text{source}}}{U_\infty} = \frac{4\,\text{m}^2\,\text{s}^{-1}}{2\,\text{m}\,\text{s}^{-1}} = 2\,\text{m}$.

(c) Find the maximum and minimum pressure coefficients on the body.

Solution: The velocity components for this flow are given by

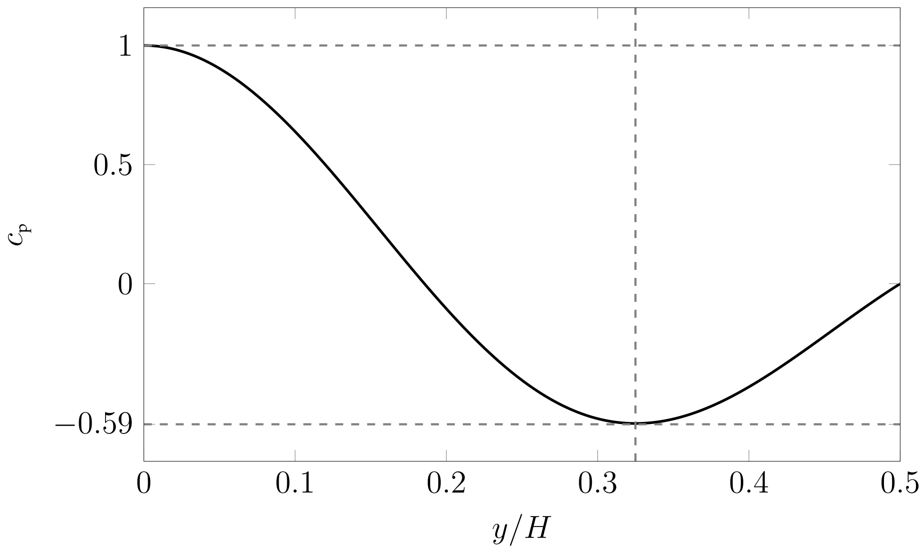
$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{Q_{\text{source}}}{r2\pi} + U_\infty \cos\theta$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty \sin\theta$$

$$\begin{aligned} v^2 = v_r^2 + v_\theta^2 &= U_\infty^2 \cos^2\theta + 2U_\infty \cos\theta \frac{Q_{\text{source}}}{2\pi r} + \frac{Q_{\text{source}}^2}{4\pi^2 r^2} + U_\infty^2 \sin^2\theta \\ &= U_\infty^2 + 2\frac{Q_{\text{source}}}{2\pi r U_\infty} U_\infty^2 \cos\theta + \left(\frac{Q_{\text{source}}}{2\pi r U_\infty}\right)^2 U_\infty^2 \\ &= U_\infty^2 \left(1 + \frac{H}{\pi r}\cos\theta + \left(\frac{H}{2\pi r}\right)^2\right) \end{aligned}$$

$$\begin{aligned} c_p &= 1 - \frac{v^2}{U_\infty^2} \\ &= 1 - \left(1 + \frac{H}{\pi r}\cos\theta + \left(\frac{H}{2\pi r}\right)^2\right) \\ &= -\frac{H}{\pi r}\cos\theta - \left(\frac{H}{2\pi r}\right)^2 \end{aligned}$$

The maximum $c_p = 1$ is found when the velocity is zero, i.e. in the stagnation point. To find the minimum c_p we can plot c_p versus y/H along the body:


1
0.5
0
−0.59
c_p
0
0.1
0.2
0.3
0.4
0.5
y/H