ME-445 AERODYNAMICS

03 - Potential flow theory

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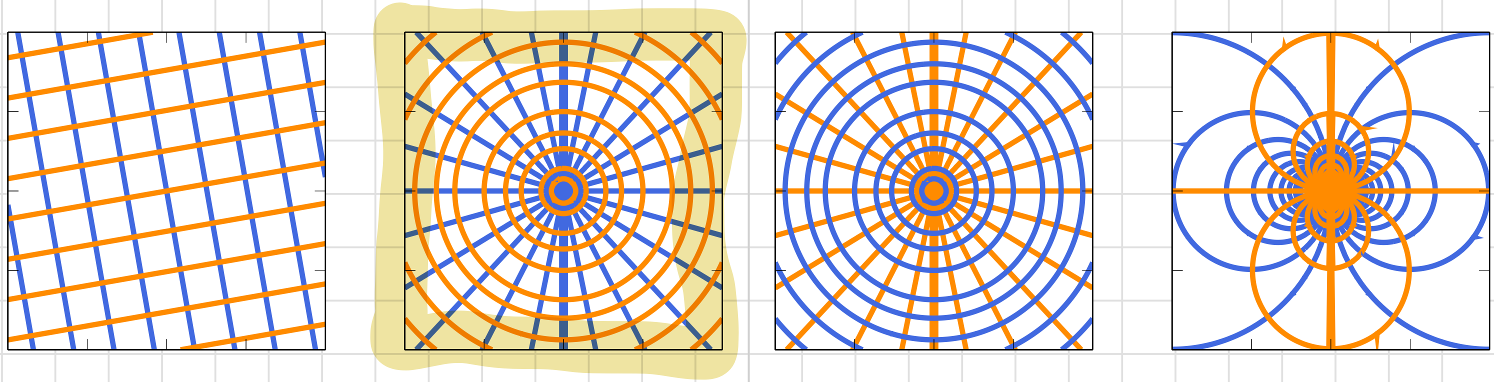
Elementary potential flows

streamlines
equipotential lines

Summary

	w	ϕ	ψ
a. Uniform parallel flow	$U_\infty e^{-i\alpha} z$	$U_\infty (x \cos\alpha + y \sin\alpha)$	$U_\infty (y \cos\alpha - x \sin\alpha)$
b. Potential vortex	$-\dfrac{i\gamma}{2\pi} \ln z$	$\dfrac{\gamma}{2\pi}\theta$	$-\dfrac{\gamma}{2\pi} \ln r$
c. Point source or sink	$\dfrac{Q}{2\pi} \ln z$	$\dfrac{Q}{2\pi} \ln r$	$\dfrac{Q}{2\pi}\theta$
d. Source-sink doublet	$\dfrac{\mu}{2\pi z} e^{i\alpha}$	$\dfrac{\mu}{2\pi r} \cos(\theta - \alpha)$	$-\dfrac{\mu}{2\pi r} \sin(\theta - \alpha)$



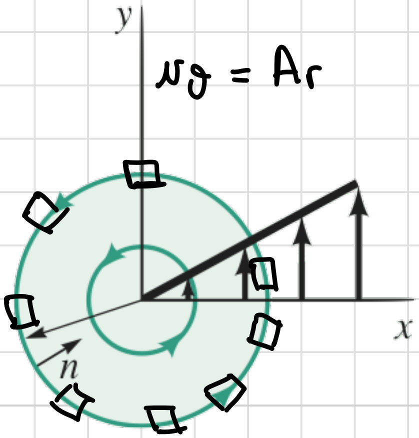
Solid body rotation vs potential or point vortex

$$\boxed{\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}$$

$$\nabla\times\vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \cancelto{0}{\frac{\partial v_r}{\partial \theta}}\right]\right)$$

$v_r = 0$

$v_\theta = Ar$

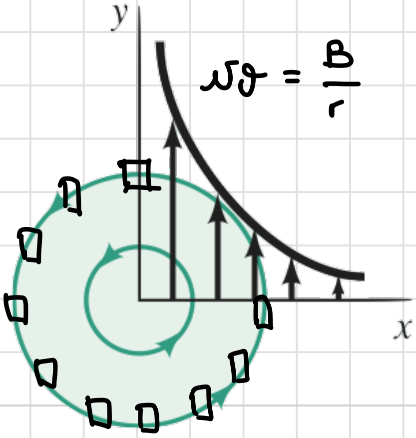


incompressible? $\nabla\cdot\vec{u} = 0$ ✓

irrotational? $\nabla\times\vec{u} \neq 0$

$$\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(Ar^2\right) = \frac{1}{r}2Ar = \underline{\underline{2A}}$$

$v_\theta = \frac{B}{r}$



$\nabla\cdot\vec{u} = 0$ ✓

$\nabla\times\vec{u} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = \frac{1}{r}\frac{\partial}{\partial r}(B) = 0 \quad ✓$$

irrotational + incompr. => potential flow

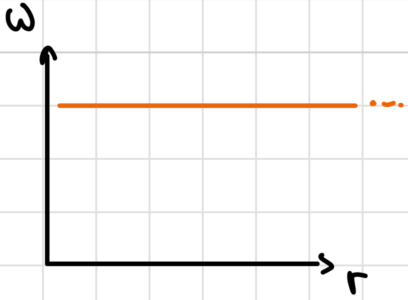
Solid body rotation vs potential or point vortex

$$\nabla\cdot\vec{u}=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

$$\nabla\times\vec{u}=\left(0,0,\frac{1}{r}\left[\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_r}{\partial\theta}\right]\right)$$

Circulation $\Gamma \equiv \iint_S \vec{\omega}\, d\vec{S} = \iint_S (\nabla\times\vec{u})\, dS = \oint_C \vec{u}\, d\vec{\ell}$

$$\Gamma_{sb} = \iint_S \vec{\omega}\, dS = \iint_{S=\odot R} 2A\, dS = 2A\pi R^2$$



not vortex

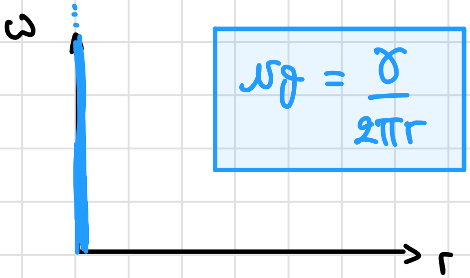
no concentrated distribution of ω

$$\Gamma_{pv} = \oint \vec{u}\, d\vec{\ell} = \int_0^{2\pi} v_\theta\, r\, d\theta$$

$$= \int_0^{2\pi} \frac{B}{r}\, r\, d\theta = 2\pi B \neq 0$$

$$\gamma \overset{?}{=} \iint \vec{\omega}\, d\vec{A}$$

$$\omega = \gamma\,\delta(x)$$



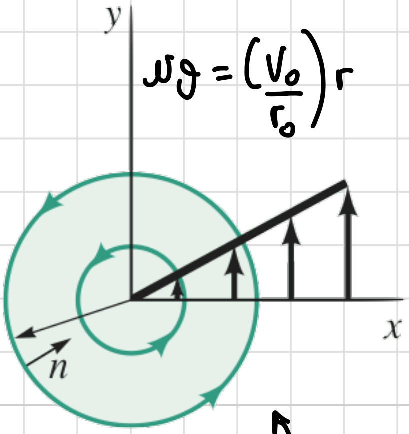
$$\boxed{v_\theta = \frac{\gamma}{2\pi r}}$$

Solid body rotation vs potential or point vortex

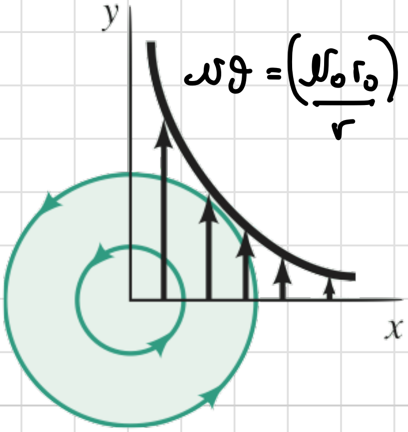
$$\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla\times\vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

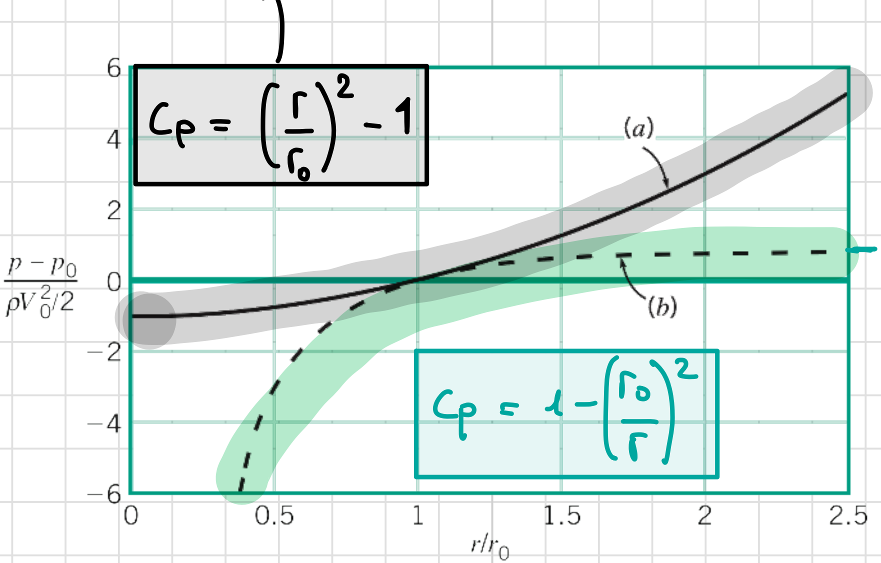
$$\frac{\partial p}{\partial r} = \rho\frac{v_\theta^2}{r}$$



$$v_\theta = \left(\frac{V_0}{r_0}\right) r$$



$$v_\theta = \left(\frac{V_0 r_0}{r}\right)$$



$$C_p = \left(\frac{r}{r_0}\right)^2 - 1$$

$$C_p = 1 - \left(\frac{r_0}{r}\right)^2$$

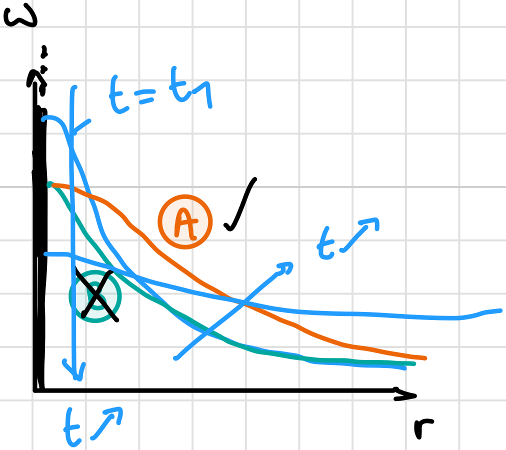
$1 \quad (p = p_\infty)$

Point vortices vs real vortices

Differences:
- singularity in ω
- p, $v_\theta \to \infty$ in centre

potential flow

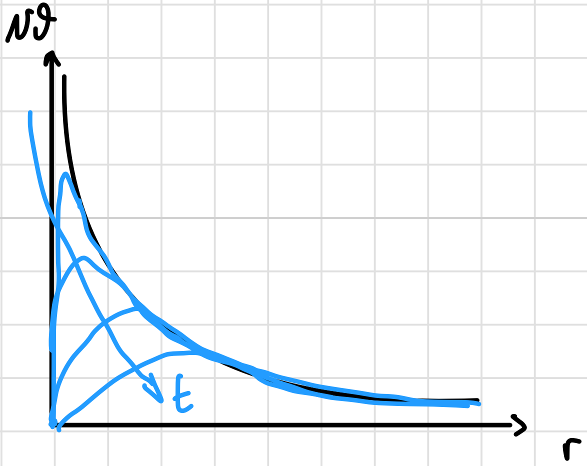
~~$\nu = 0$~~



$\Gamma = cte$

in time: ① $\omega(r=0)$ decreases

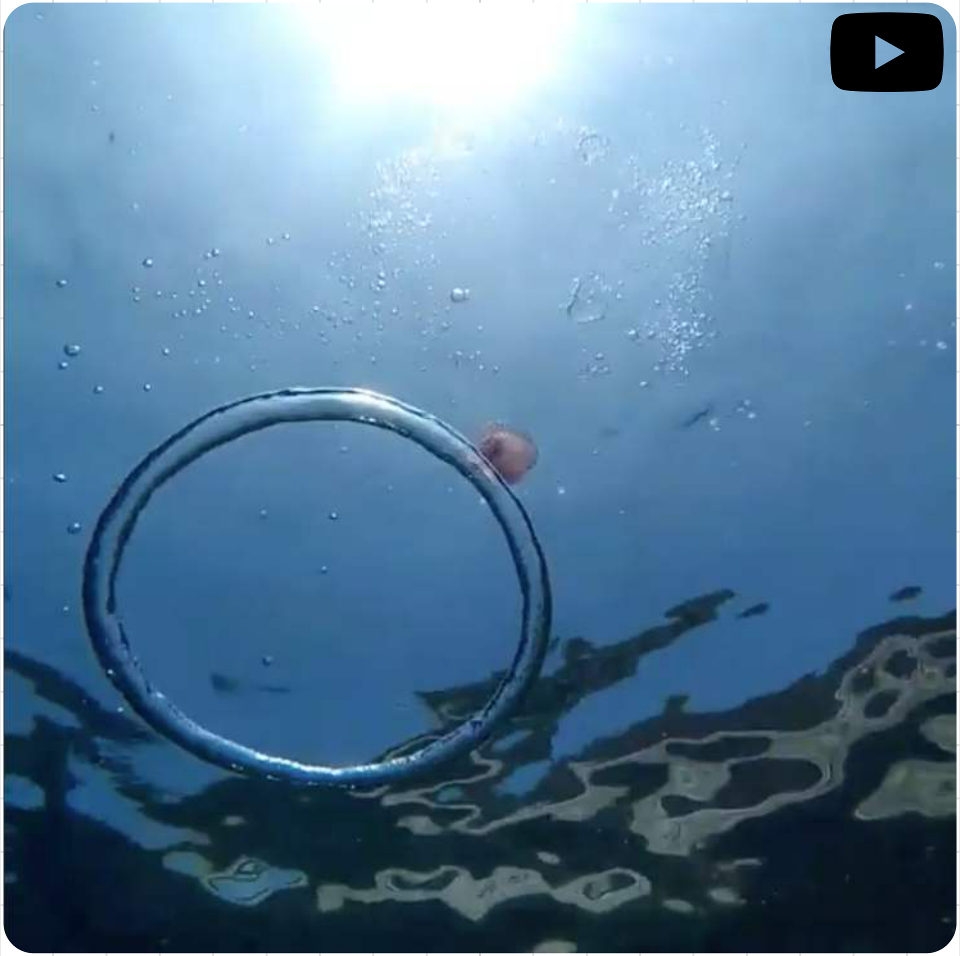
② radius of vortex increases



① v_θ max decreases

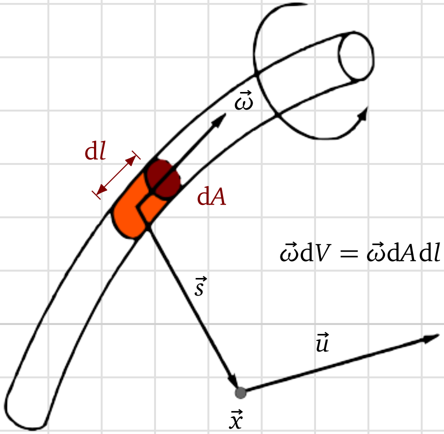
② r @ v_θ max increases

Induced velocity - Biot-Savart



Biot-Savart

determines the velocity field associated with given vorticity field

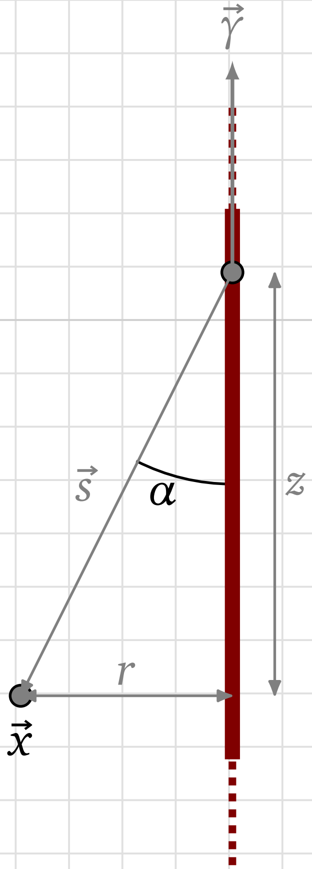


$$\vec{\omega}\mathrm{d}V = \vec{\omega}\mathrm{d}A\mathrm{d}l = \vec{\gamma}\mathrm{d}l$$

$$\boxed{\vec{u}(\vec{x}) = \frac{1}{4\pi}\iiint\limits_V \frac{\vec{\omega}\times\vec{s}}{s^3}\mathrm{d}V = \frac{1}{4\pi}\oint\limits_L \frac{\vec{\gamma}\times\vec{s}}{s^3}\mathrm{d}l}$$

Biot-Savart

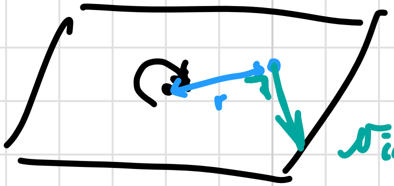
Induced velocity by point vortices



$$\underset{\bar{r}}{u(\vec{x})} = \frac{\gamma r}{4\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{(r^2+z^2)^{3/2}} = \frac{\gamma}{2\pi r}$$

semi-infinite vortex line:

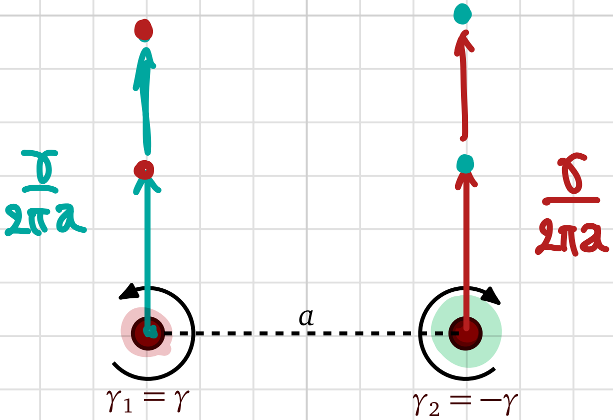
$$u(\vec{x}) = \frac{\gamma r}{4\pi}\int_{0}^{\infty}\frac{\mathrm{d}z}{(r^2+z^2)^{3/2}} = \frac{\gamma}{4\pi r}$$



$$v_{ind} = \frac{\gamma}{2\pi r}$$

Interaction of point vortices

What would be the trajectory of this vortex pair?



Interaction of point vortices

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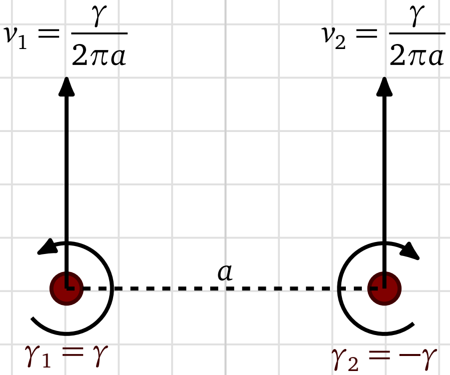

Interaction of point vortices

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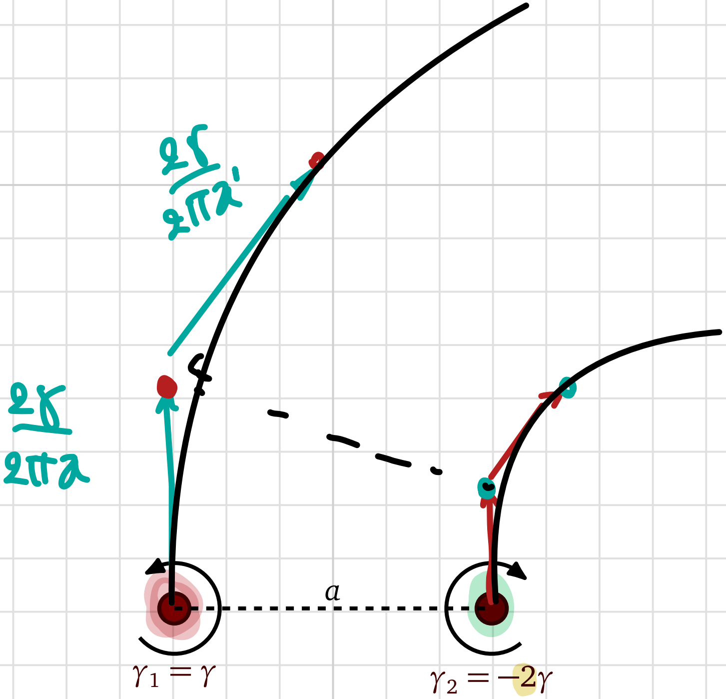


Interaction of point vortices with walls

What would happen to this vortex?

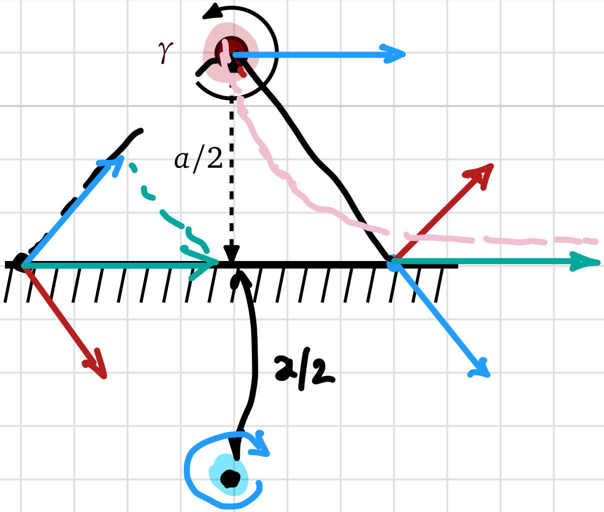
(A) it splits in 2 parts

(B) it disintegrates

(C) it moves to the left

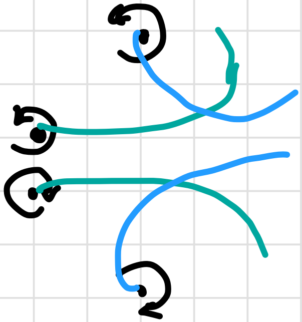
(D) it moves to the right

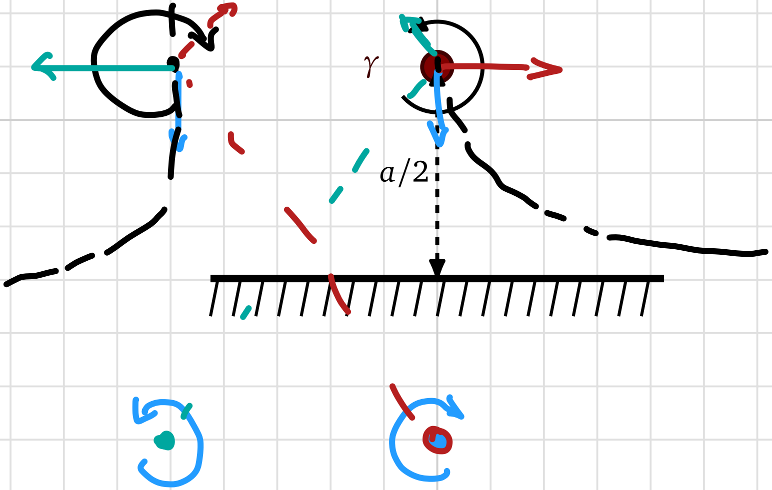
(E) it bounces back up

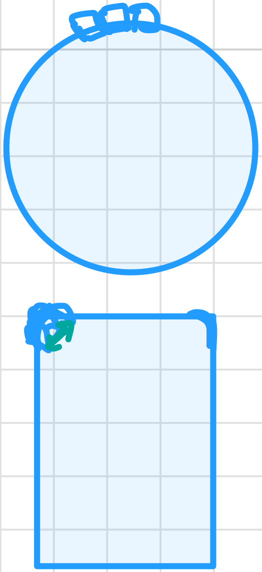


Interaction of point vortices with walls

What would happen to this vortex?

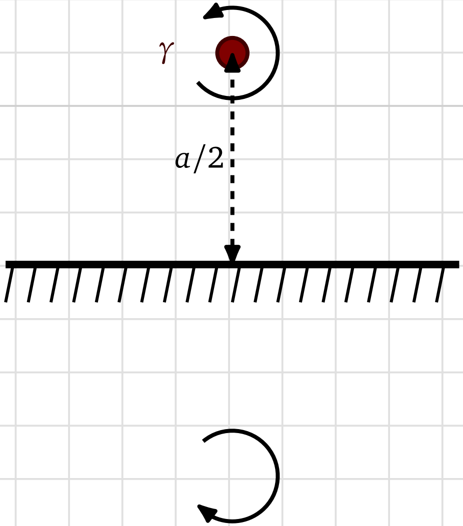






Interaction of point vortices with walls

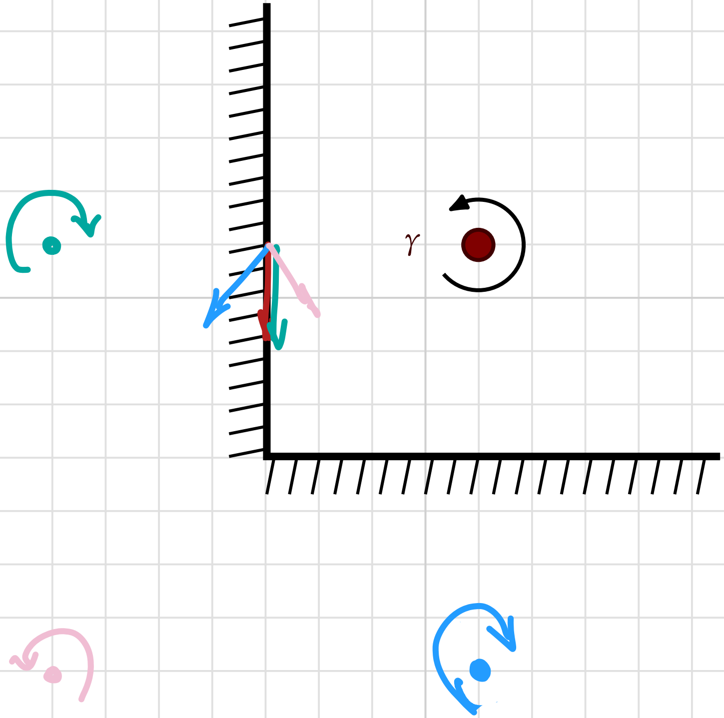
What about the pressure at the wall?



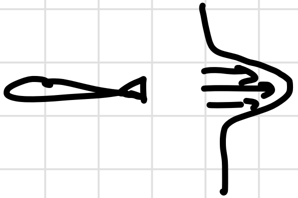
① add the induced velocities due to different vortices @ wall

② use Bernoulli to calculate p from u^2 at the wall.

Interaction of point vortices with walls



Vortex wakes



One of the images shows the wake behind a cylinder, the other one shows the wake behind a fish. Which image belongs to the fish?

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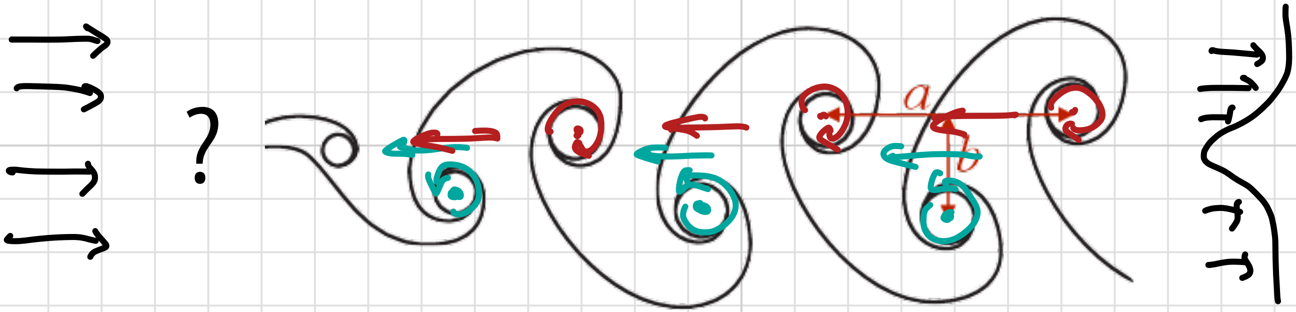
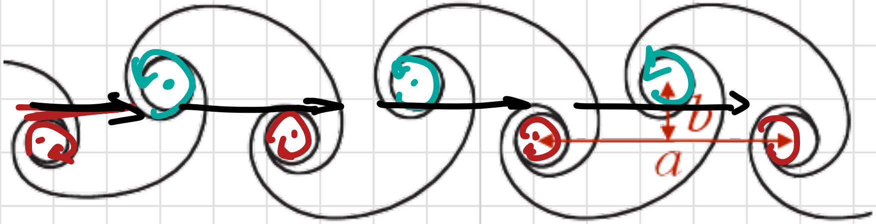


?



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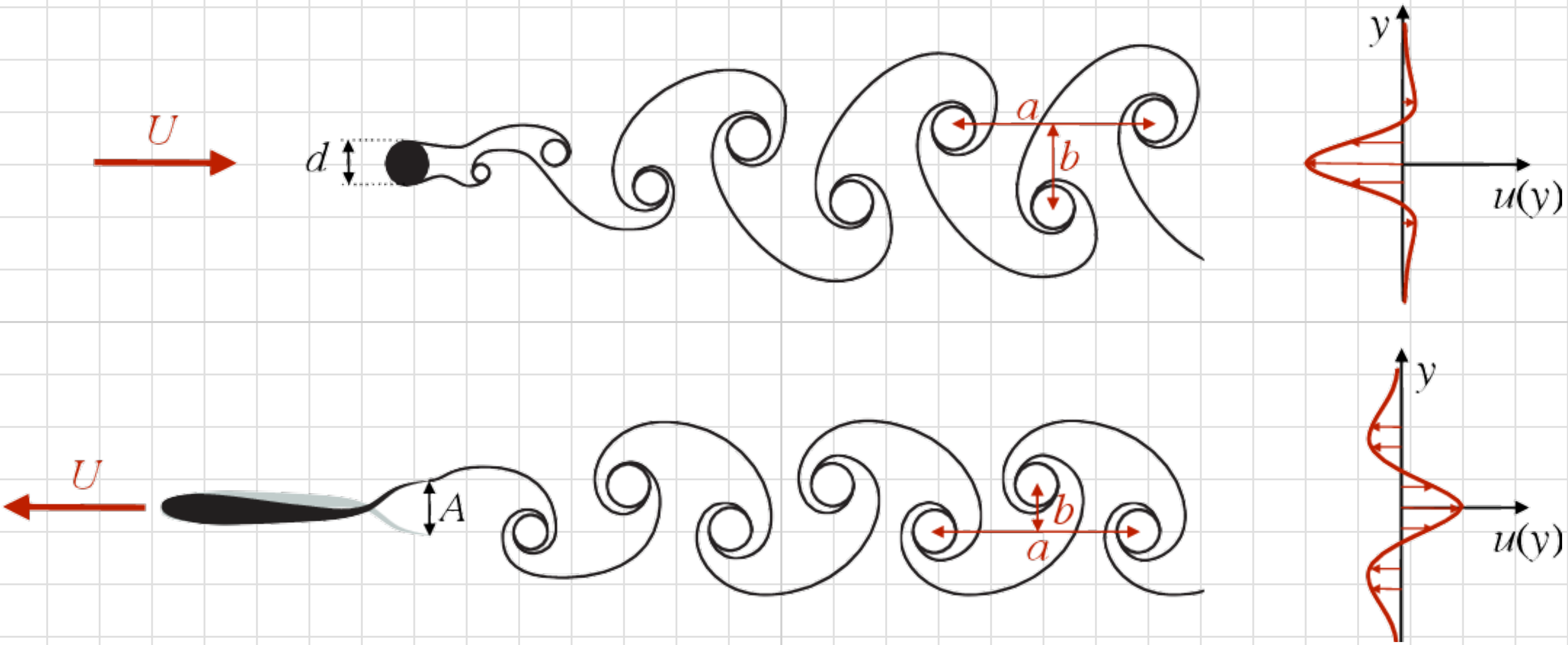
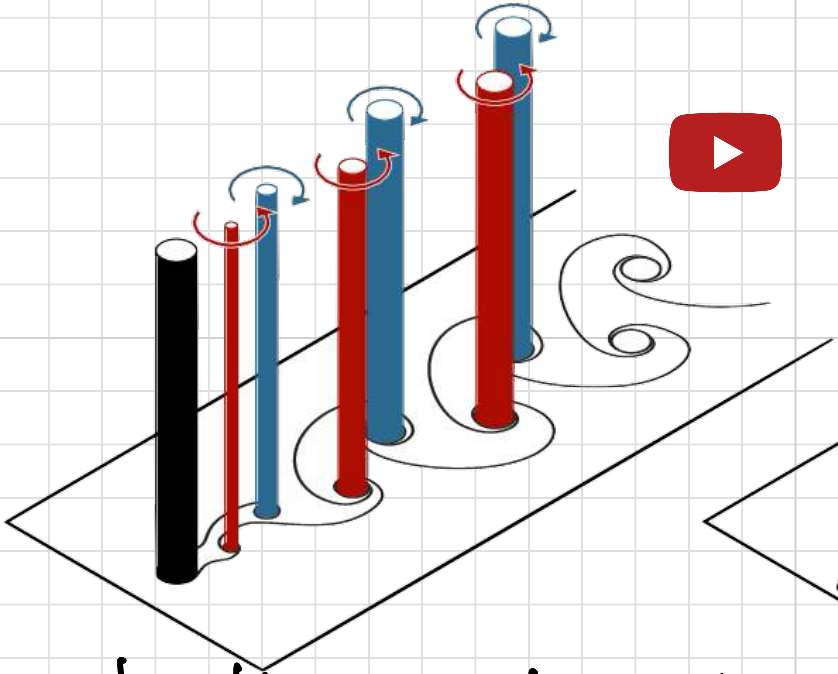
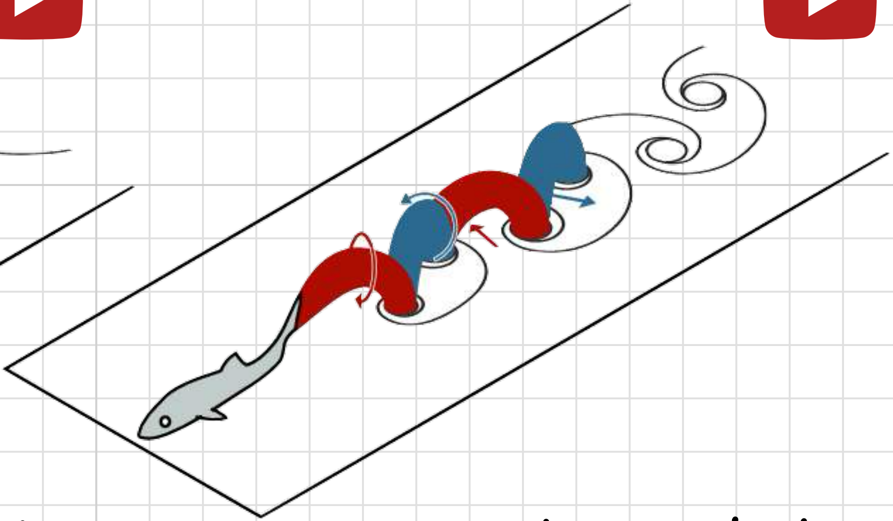


Vortex wakes



Von Karman vortex street



inverse von Karman street

Bénard - Von Karman

Animals and vortices