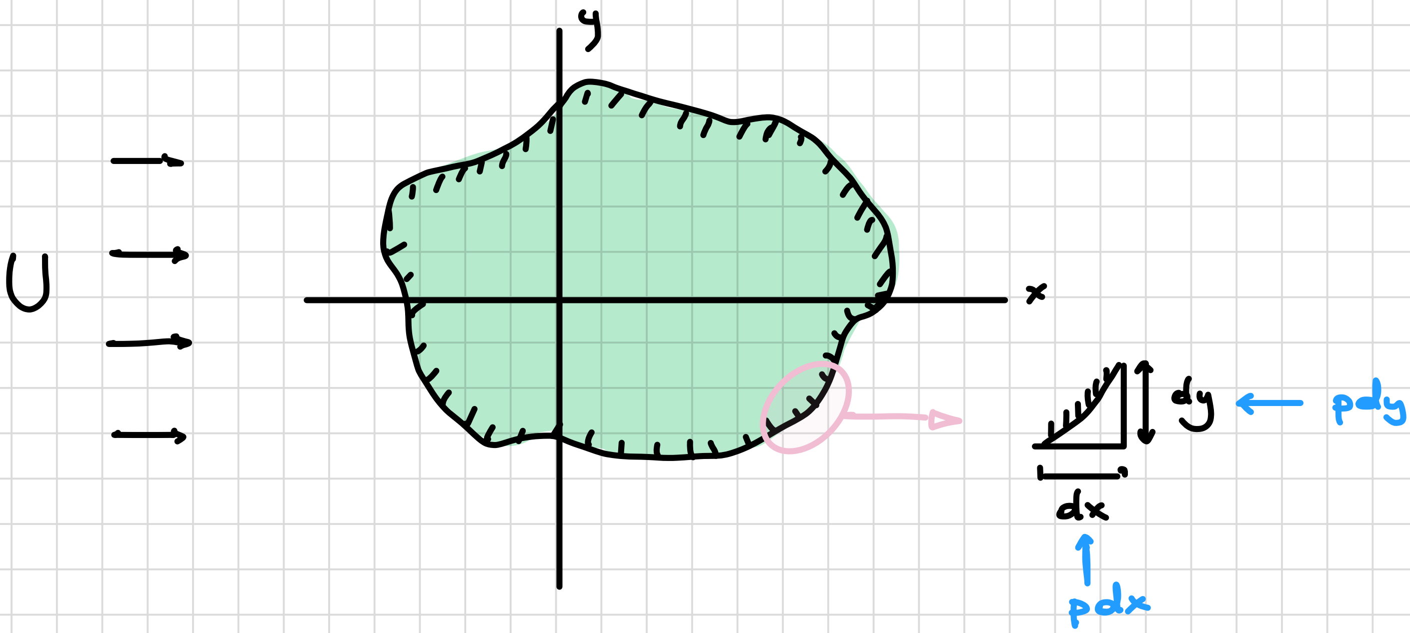
Forces on a 2D body in a potential flow



$$dF_x = -p\,dy \quad ; \quad dF_y = p\,dx$$

$$dM = x\,dF_y - y\,dF_x = p(x\,dx + y\,dy)$$

$$d(F_x - iF_y) = -p\,dy - ip\,dx = -ip(dx - i\,dy) = -ip\,d\bar{z}$$

$$dM = p(x\,dx + y\,dy) = \mathrm{Re}\{(x+iy)(dx - i\,dy)\,p\} = \mathrm{Re}\{p\,z\,d\bar{z}\}$$

$$p + \frac{1}{2}\rho U^2 = cte \longrightarrow p = -\frac{1}{2}\rho U^2 = -\frac{1}{2}\rho \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}$$

$$d(F_x - iF_y) = -ip\,d\bar{z} = i\frac{1}{2}\rho \frac{dw}{dz}\frac{d\bar{w}}{\cancel{d\bar{z}}}\cancel{d\bar{z}} = i\frac{1}{2}\rho \frac{dw}{dz}\,d\bar{w}$$

$$dM = \mathrm{Re}\left\{-\frac{1}{2}\rho \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\,d\bar{z}\,z\right\} = \mathrm{Re}\left\{-\frac{1}{2}\rho z \frac{dw}{dz}\,d\bar{w}\right\}$$

Integration along body surface

pot flow $\underline{=}$ streamline

$\Rightarrow d\psi = 0$

$$dw = d\varphi + i\,\underbrace{d\psi}_{d\psi=0} = d\varphi = d\bar{w}$$

$$\frac{dw}{dz}\,d\bar{w} = \frac{dw}{dz}\frac{d\bar{w}}{dz}\,dz = \frac{dw}{dz}\frac{dw}{dz}\,dz = \left(\frac{dw}{dz}\right)^2 dz$$

$$\Rightarrow \quad \boxed{F_x - iF_y = i\frac{1}{2}\rho \oint \left(\frac{dw}{dz}\right)^2 dz}$$

$$\boxed{M = \mathrm{Re}\left\{-\frac{1}{2}\rho \oint z\left(\frac{dw}{dz}\right)^2 dz\right\}}$$

BLASIUS THEOREMS

Example

Cylinder with circulation:

$$w(z) = \frac{\mu U}{2\pi z U} + Uz - \frac{i\Gamma}{2\pi}\ln z \quad \longrightarrow \quad a = \sqrt{\frac{\mu}{2\pi U}}$$

$$= \frac{a^2}{z}U + Uz - \frac{i\Gamma}{2\pi}\ln z$$

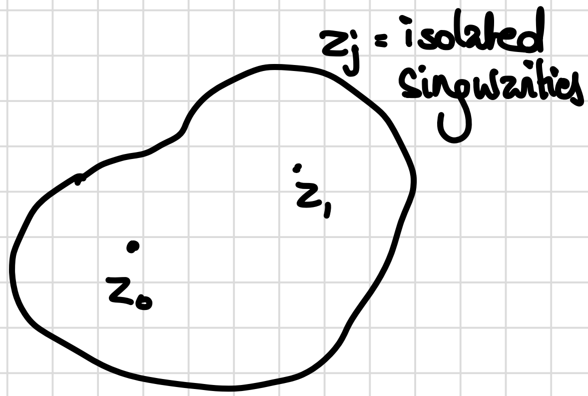
$$\frac{dw}{dz} = -\frac{a^2}{z^2}U + U - \frac{i\Gamma}{2\pi z} = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}$$

$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right) - \frac{\Gamma^2}{4\pi^2 z^2} - 2i\frac{\Gamma}{2\pi z}U\left(1 - \frac{a^2}{z^2}\right)$$

$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1-\frac{2a^2}{z^2}+\frac{a^4}{z^4}\right) - \frac{\Gamma^2}{4\pi^2 z^2} - i\,\frac{\Gamma U}{\pi}\left(\frac{1}{z}-\frac{a^2}{z^3}\right)$$

↳ Laurent series

$$F_x - iF_y = i\,\frac{1}{2}\rho \oint \left(\frac{dw}{dz}\right)^2 dz$$



$$\oint \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \sum_{z_j} \text{residues}$$
$$= 2\pi i \left(-\frac{i\Gamma U}{\pi}\right)$$

Cauchy's residue theorem.

$$\rightarrow F_x - iF_y = i\,\frac{1}{2}\rho\left(2\pi\,\frac{\Gamma U}{\pi}\right) = i\rho\Gamma U$$

$\Rightarrow$

$$\boxed{\begin{array}{l} F_x = \text{drag force} = 0 \ \checkmark \\ F_y = \text{lift force} = -\rho\Gamma U \ \checkmark \end{array}}$$

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \underbrace{\frac{a_{-1}}{(z-z_0)}}_{\text{residue}} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

(Laurent series) $\quad z_0$ = singularity

$$\boxed{M} = \text{Re}\left\{-\frac{1}{2}\rho \oint z\left(\frac{dw}{dz}\right)^2 dz\right\}$$

$$= \text{Re}\left\{-\frac{1}{2}\rho\left(2\pi i\left(-2a^2U^2 - \frac{\Gamma^2}{4\pi^2}\right)\right)\right\} \ \boxed{= 0 \ \checkmark}$$