Exercises – Serie 3 – Added mass

Exercise 1

Consider a sphere of radius $R = 5\ cm$ made of steel (ρ_{steel} = $8000\ kg/m^3$). The sphere is supported by a spring with constant $K = 200\ N/m$ and excited by a sinusoidal force of amplitude A which forces the sphere to oscillate around its equilibrium position at $x = 0$. The sphere is then placed in a water tank and the same force is applied to it.

a) Find the ratio of the natural frequencies of both systems.

b) We replace the ball with one made of POM plastic (ρ_{POM} = $1410\ kg/m^3$). Recompute the ratio of the natural frequencies.

(Note: We ignore the viscous damping, this approximation is acceptable for small amplitude oscillations in which inertial terms (accelerations) dominate.

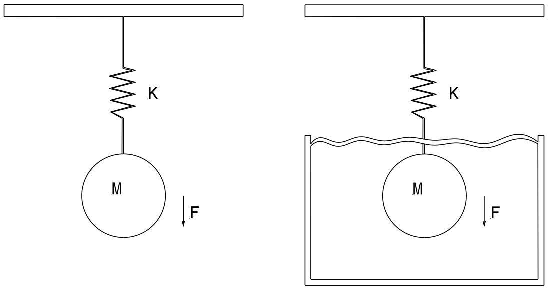


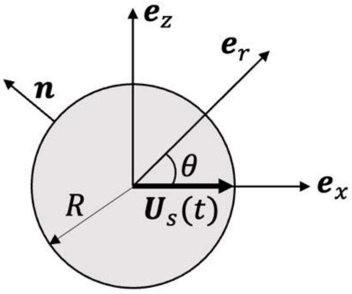
Figure 1: Spring-mass system in the two configurations.

Exercise 2

We want to evaluate the force exerted by a potential flow on a rigid sphere in two cases. The gravitational acceleration is neglected.

1. Unsteady motion of a sphere in a quiescent fluid

Consider a sphere moving with velocity $\boldsymbol{U}_s(t) = U_s(t)\boldsymbol{e}_x$ in a quiescent fluid. The fluid domain is unbounded.



In this case, the velocity potential is given by:

$$\phi = U_s(t)\frac{R^3}{2r^2}\cos\theta$$

a) Give the expression of the velocity field $\boldsymbol{u}(r,\theta,t)$
Potential flow is defined as an incompressible ($\nabla \cdot \boldsymbol{u} = 0$), irrotational ($\nabla \times \boldsymbol{u} = \boldsymbol{0}$) and inviscid ($\mu = 0$) flow.

b) Give the expression of the pressure field $p(r,\theta,t)$ using the unsteady Bernoulli equation $\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 = C(t)$.

c) Find the force $F_x(t) = \boldsymbol{F}(t) \cdot \boldsymbol{e}_x$ exerted by the flow on the sphere

d) Deduce the expression of the added mass m_a?

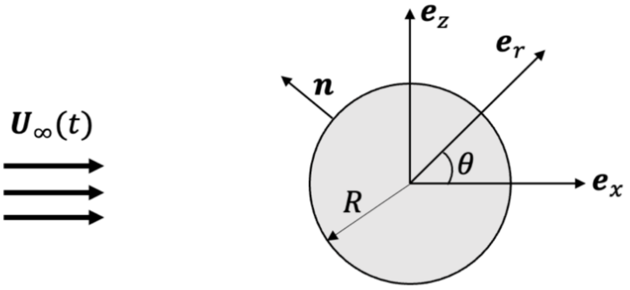
Hints:

The gradient operator in spherical coordinates is given by $\nabla\phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\right)$

The surface element on a sphere of radius R in spherical coordinates is $dS = R^2 \sin\theta \, d\theta d\varphi$

2. Unsteady flow past a fixed sphere

Consider a fixed sphere in an unsteady flow of free-stream velocity $\boldsymbol{U}_\infty(t) = U_\infty(t)\boldsymbol{e}_x$. The fluid domain is unbounded.



In this case, the velocity potential is given by:

$$\phi = U_\infty(t)\left(r + \frac{R^3}{2r^2}\right)\cos\theta$$

a) Give the expression of the velocity field $\boldsymbol{u}(r,\theta,t)$.

b) Give the expression of the pressure field $p(r,\theta,t)$.

c) Find the force $F_x(t) = \boldsymbol{F}(t)\cdot\boldsymbol{e}_x$ exerted by the flow on the sphere.
Write the force F_x as a function of the added mass m_a found in the previous case?

d) If we set $U_\infty(t) = U_s(t)$, is the force amplitude higher, lower, or the same in comparison with the previous case?

e) Now, what is the force if the free-stream velocity is constant $U_\infty = cste$?

Exercise 3

If a 3D body has a characteristic length in one direction that is considerably longer than its length in the other two directions, the *slender body* approximation can be used to formulate the added mass associated with its motion. The idea behind this is to consider the body as a longitudinal stack of thin sections, each having an easily computed added mass, and to integrate the effects of those sections along the length of the body to find the total added mass (c.f. Figure 2 and Figure 3).

By using this approximation, it is possible to estimate the added mass tensor for a 6 degrees of freedom body, m_{ij} , with i, j = $1, 2, 3, 4, 5, 6$, moving in a fluid as follow:

$$\vec{U}(t) = (U_1, U_2, U_3, \omega_1, \omega_2, \omega_3) = (U_1, U_2, U_3, U_4, U_5, U_6)$$

A good way to think about those components m_{ij} is to consider them as the mass associated with a force/moment in the i^{th} direction due to a unit acceleration in the j^{th} direction. Note, however, that the added mass coefficients related to the x_1 axis (c.f. Figure 2) cannot be obtained with the slender body approach. Additionally, the tensor is symmetric: $m_{ij} = m_{ji}$. Given the symmetries of the 3D body depicted on Figure 2, the added mass tensor is given by:

$$\boldsymbol{m} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix} = \begin{pmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{66} \end{pmatrix}$$

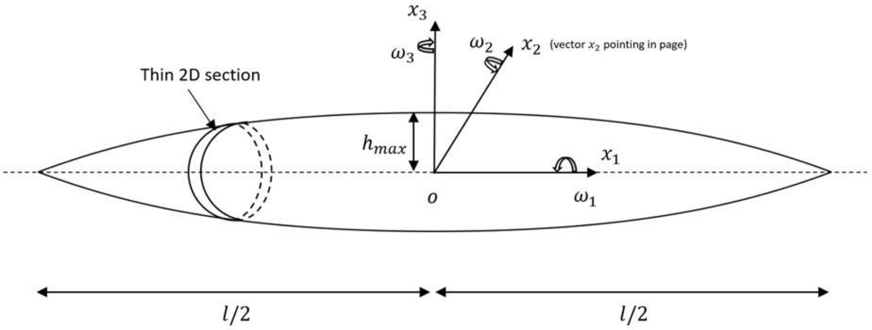
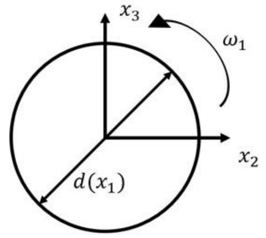


Figure 2: Slender-body approximation. 3D body made of circular 2D sections.



$$a_{22} = a_{33} = \rho\pi\left(\frac{d(x_1)}{2}\right)^2$$

$$a_{44} = 0$$

Figure 3: Added mass coefficients of a 2D circular section.

a) Assuming a solid with the shape $d(x_1) = h_{max}\left(1 - \frac{x_1^2}{l^2/4}\right)$. Calculate the different added masses as a function of l and h_{max}.

$$m_{22} = \int_{-\frac{l}{2}}^{\frac{l}{2}} a_{22} dx_1 \quad m_{33} = \int_{-\frac{l}{2}}^{\frac{l}{2}} a_{33} dx_1 \quad m_{44} = \int_{-\frac{l}{2}}^{\frac{l}{2}} a_{44} dx_1$$

$$m_{55} = \int_{-\frac{l}{2}}^{\frac{l}{2}} x_1^2 a_{33} dx_1 \; m_{66} = \int_{-\frac{l}{2}}^{\frac{l}{2}} x_1^2 a_{22} dx_1$$

The effects of added mass on a body can be represented by forces and moments acting on it. In the case of a body evolving in an unbounded and inviscid fluid, they can be evaluated as follow:

$$F_j = -\dot{U}_i m_{ij} - \epsilon_{jkl} U_i \omega_k m_{li}$$

$$M_j = -\dot{U}_i m_{j+3,i} - \epsilon_{jkl} U_i \omega_k m_{l+3,i} - \epsilon_{jkl} U_k U_i m_{li}$$

where $i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$ and ϵ is the Levi-Cavita symbol with:

$$\epsilon_{jkl} = \begin{cases} 0 & if\ any\ j, k, l\ are\ equal \\ 1 & if\ j, k, l\ are\ in\ cyclic\ order \\ -1 & if\ j, k, l\ are\ in\ anti-cyclic\ order \end{cases}$$

Consider the 3D object moving in the fluid as follow:

$$\vec{U} = (U_1, U_2, U_3, U_4, U_5, U_6) = (U\cos(\alpha), 0, -U\sin(\alpha), 0,0,0)$$

$$\dot{\vec{U}} = \left(\dot{U}\, cos(\alpha), 0, -\dot{U}\, sin(\alpha), 0,0,0\right)$$

b) Calculate the forces and the moments in terms of m_{ij}.

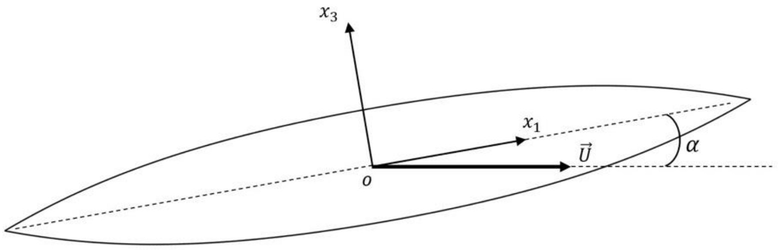


Figure 4: UAV moving in an inviscid and unbounded fluid.

c) Given $l = 1\,m$, $h_{max} = 0.1\,m$, $U = 5\,m/s$, $\dot{U} = 1\,m/s^2$ and $\alpha = 5°$, compute the various forces and moments derived in (b). The added mass coefficient m_{11} can be approximated in this specific case as follow:

$$m_{11} \approx 0.057 \frac{4}{3} \pi \rho \frac{l}{2} h_{max}^2$$

d) What becomes the moment M_2 if we consider a sphere instead of the body depicted on Figure 2?