AEROELASTICITY AND FLUID-STRUCTURE INTERACTION

Chapter 2:

Coupled formulation of fluid and solid motions

Fluid dynamics framework

Dimensional Analysis for Flow Characterization

- ***In fluid mechanics, it is very common to express physical laws with the help of non-dimensional quantities only***

$$f(x_1, x_2, \dots, x_N) = 0$$

where $x_1, \dots, x_N$ ***are dimensional quantities:*** $[x_i] = L^{\alpha_i} M^{\beta_i} T^{\gamma_i}$

(L: ***length,*** M: ***mass and*** T: ***time)***

The function f may be expressed in non-dimensional form F, such as:

$$F(X_1, X_2, \dots, X_P) = 0$$

where $X_1, \dots, X_P$ ***are dimensionless quantities:*** $[X] = L^0 M^0 T^0$

- ***Advantages:***
  - ***Universality of physical laws (e.g. scale independent)***
  - ***Useful to classify the large variety of complex flows***

Fluid dynamics framework

Dimensional Analysis for Flow Characterization

- ***The Buckingham π Theorem:***

Any physically meaningful equation involving N physical variables

$$f(x_1, x_2, \dots, x_N) = 0 \qquad [x_i] = L^{\alpha_i} M^{\beta_i} T^{\gamma_i}$$

may be written with a set of P dimensionless parameters, constructed from the original variables:

$$F(X_1, X_2, \dots, X_P) = 0 \qquad [X_i] = L^0 M^0 T^0$$

$$[X_i] = \frac{Force}{Force} = \frac{Time}{Time} = \dots$$

with :

$$P = N - R \quad and \quad R = rank \begin{bmatrix} \alpha_1 & \dots & \alpha_N \\ \beta_1 & \dots & \beta_N \\ \gamma_1 & \dots & \gamma_N \end{bmatrix}$$

Dimensional Analysis for Flow Characterization

- ***Rank of a matrix:***
  In linear algebra, the rank of a matrix A is the dimension of the vector space generated (or spanned) by its columns (or rows). This corresponds to the maximal number of linearly independent columns (or rows) of A.

- ***Examples:***

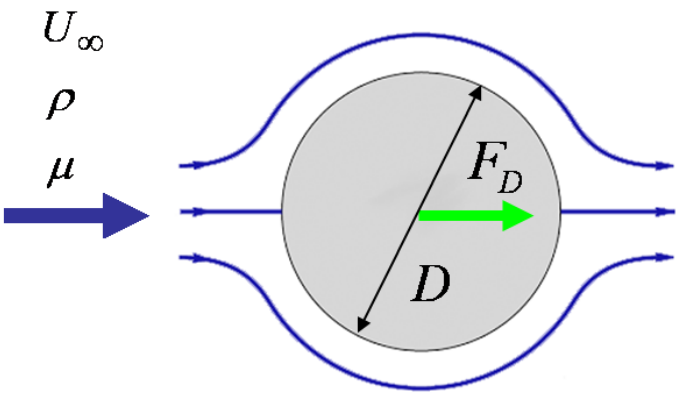
$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \quad rank(A) = 2$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix} \quad rank(A) = rank({}^{T}A) = 1$$

Fluid dynamics framework

Dimensional Analysis for Flow Characterization

- ***Example: Drag on a sphere***



$$f(F_D, U_\infty, \rho, \mu, D) = 0$$

5 dimensional variables

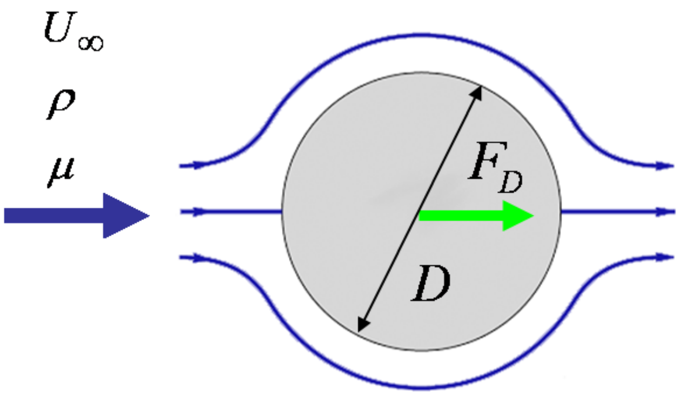
$$R = rank \begin{pmatrix} L \\ M \\ T \end{pmatrix} \begin{bmatrix} 1 & 1 & -3 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ -2 & -1 & 0 & -1 & 0 \end{bmatrix} = 3$$

→ ***P = N – R = 5 – 3 = 2 Only 2 dimensionless variables are required***

Fluid dynamics framework

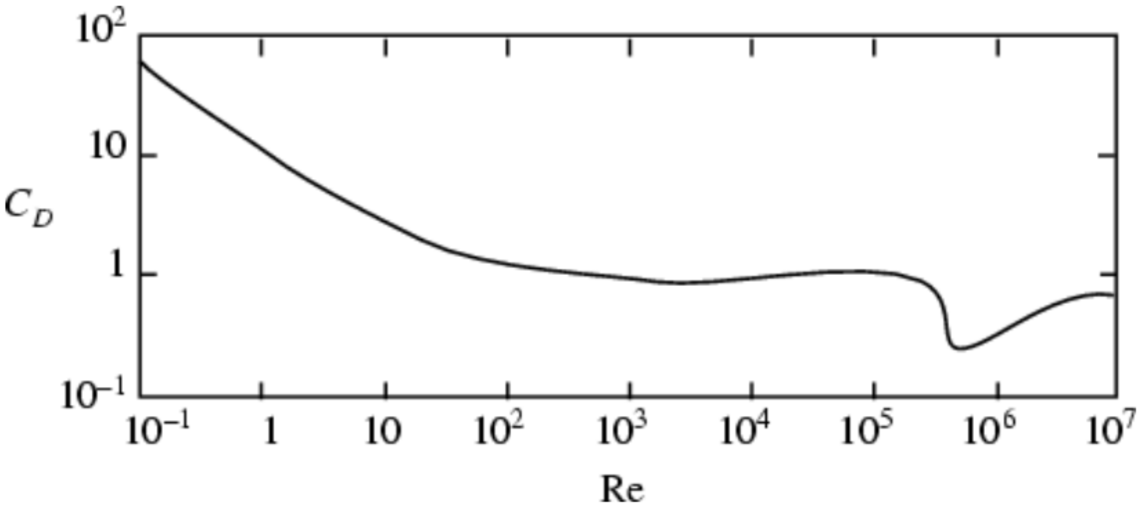
Dimensional Analysis for Flow Characterization

- ***Example: Drag on a sphere***
  - ***dimensionless equation:***



$$\boldsymbol{F\left(\frac{F_D}{\rho U_\infty^2 D^2}, \frac{\rho U_\infty D}{\mu}\right) = 0}$$

$$\boldsymbol{F(C_D, Re) = 0}$$



C_D : Drag Coefficient

R_e : Reynolds Number

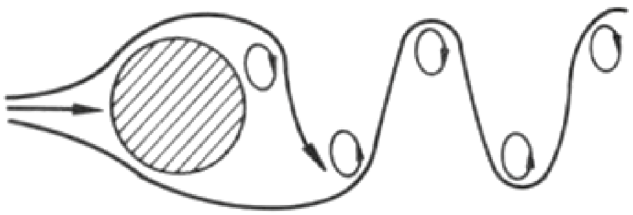
Fluid dynamics framework

Dimensional Analysis for Flow Characterization

- ***Example:***
  Vortex shedding from a cylinder (Karman vortices)
  - ***Shedding Frequency f_s ?***
  - ***Governing equation:***

$$\boldsymbol{f(f_s, U_\infty, \rho, \mu, D) = 0}$$

5 dimensional variables

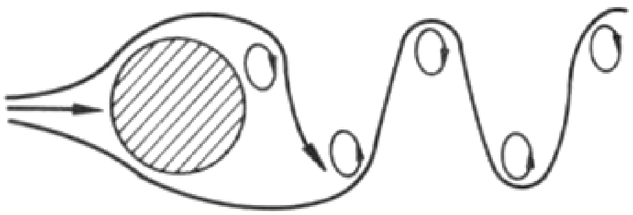


$$\boldsymbol{R = rank\begin{pmatrix} L \\ M \\ T \end{pmatrix}\begin{bmatrix} 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 \end{bmatrix} = 3}$$

→ ***P = N – R = 5 – 3 = 2 Only 2 dimensionless variables are required***

Dimensional Analysis for Flow Characterization

- ***Example:***
  Vortex shedding from a cylinder (Karman vortices)
  - ***Shedding Frequency f_s ?***
  - ***Dimensionless equation:***



$$F\left(\frac{f_s D}{U_\infty}, \frac{\rho U_\infty D}{\mu}\right) = 0$$

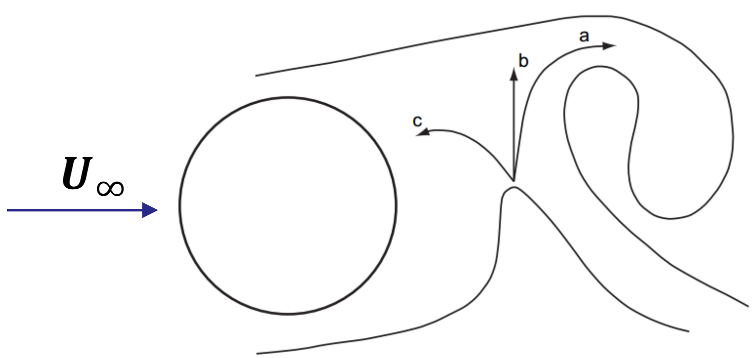
St : Strouhal Number ***R_e : Reynolds Number***

Dimensional Analysis for Flow Characterization

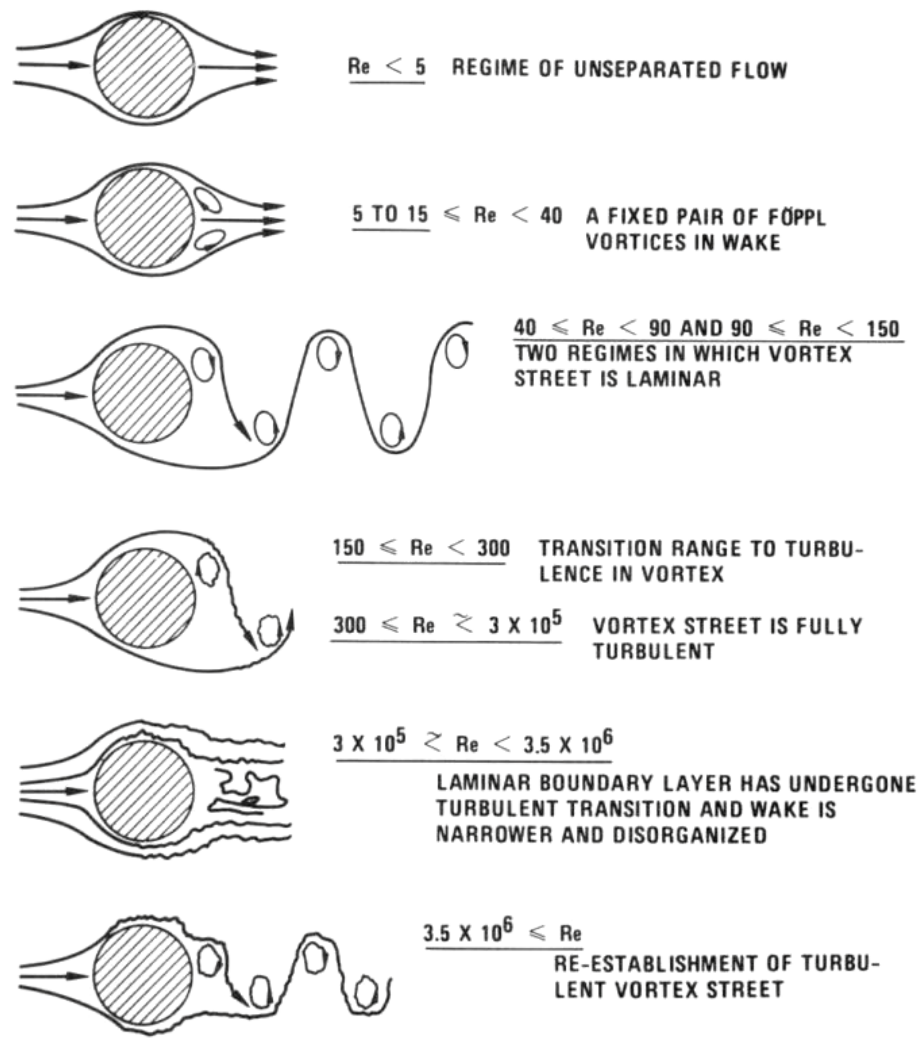
- ***Example:***
  Vortex shedding from a cylinder

- ***Experimental Observations:***

$$St = \frac{f_s D}{U_\infty} \qquad Re = \frac{\rho U_\infty D}{\mu}$$



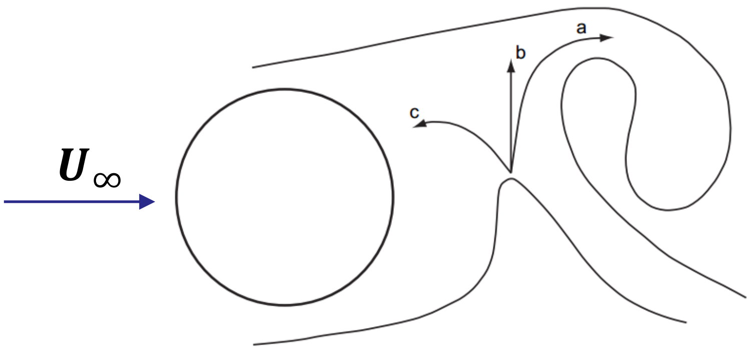
Vortex-formation model, Gerrard (1966)



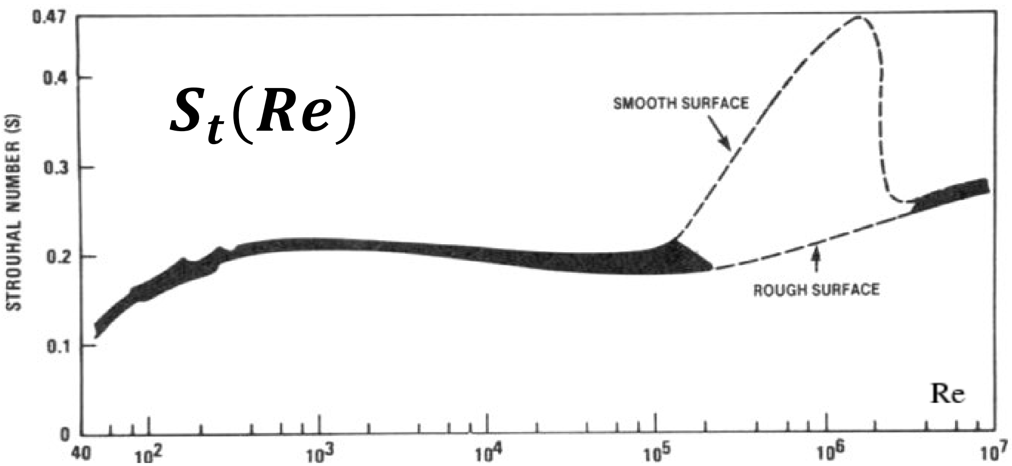
Dimensional Analysis for Flow Characterization

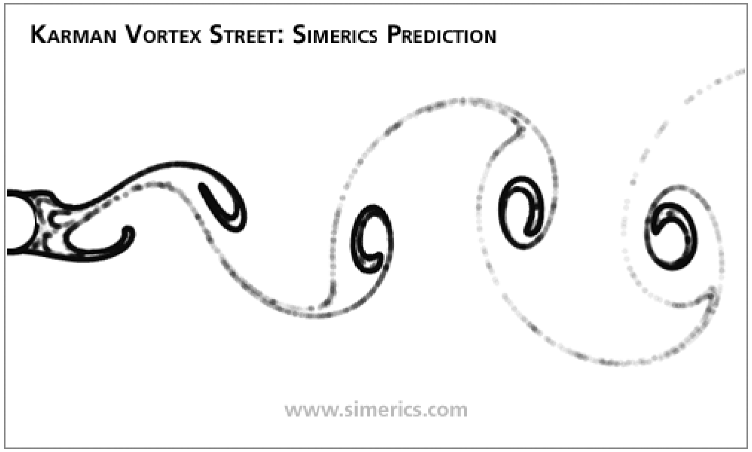
- ***Example:***
  Vortex shedding from a cylinder

- ***Experimental Observations:***



Vortex-formation model, Gerrard (1966)

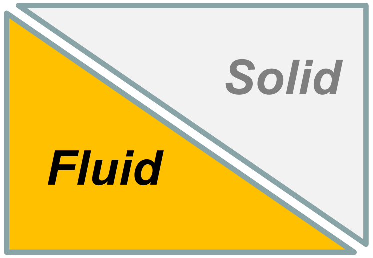




Fluid-Solid Interaction: General Formulation

Uncoupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***The fluid and the solid are treated separately***
    - ***Relevant variables for the fluid motion:***



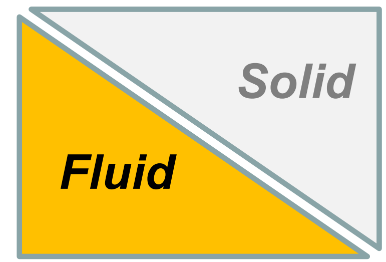
U	Velocity field
x	Coordinates
t	Time
μ	Viscosity
ρ	Density
g	Gravity
U_0	Reference Velocity
L	Reference Length

Fluid-Solid Interaction: General Formulation

Uncoupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***The fluid and the solid are treated separately***
    - ***Dimensional form of the equation of fluid motion :***

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{U}, \boldsymbol{\mu}, \boldsymbol{L}, \boldsymbol{g}, \boldsymbol{\rho}, \boldsymbol{U_0}) = \mathbf{0}$$



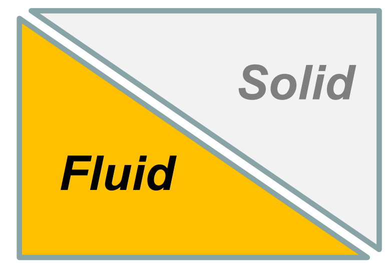
    - ***Dimensionless form of the equation of motion:***

$$\boldsymbol{R} = \boldsymbol{rank}\begin{pmatrix} \boldsymbol{L} \\ \boldsymbol{M} \\ \boldsymbol{T} \end{pmatrix}\begin{bmatrix} 1 & 0 & 1 & -1 & 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -2 & 0 & -1 \end{bmatrix} = 3$$

→ *Problem dimension = 8-3=5*

Uncoupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***The fluid and the solid are treated separately***
    - ***Dimensionless form of the equation of fluid motion:***



$$F\left(\frac{x}{L},\frac{U}{U_0},\frac{t}{L/U_0},\frac{\rho U_0 L}{\mu},\frac{U_0}{\sqrt{gL}}\right)=0$$

$\rho U_0 L/\mu$ *: Reynolds Number (Ratio of inertia to the viscous forces)*

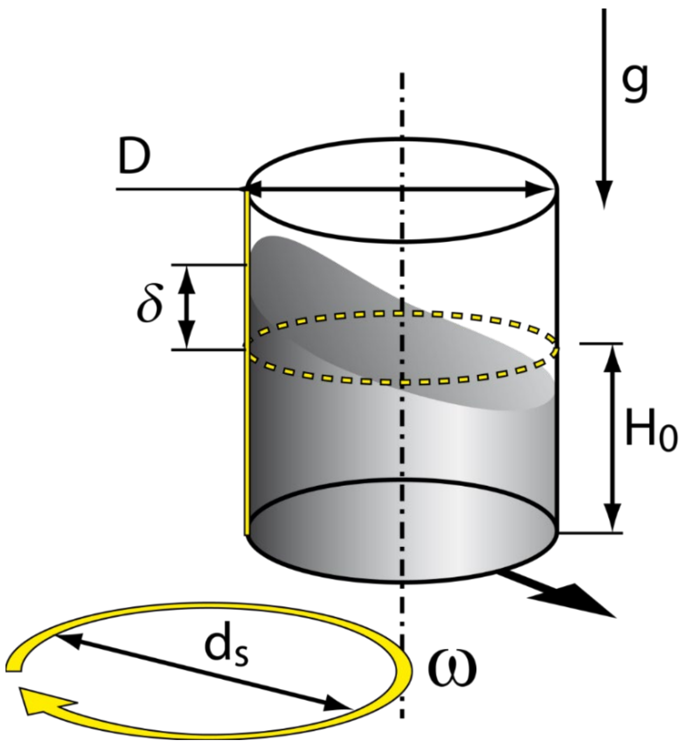
$\frac{U_0 t}{L}=\frac{t}{T_{Fluid}}$ *:Time t is normalized by the time required for a particle moving at the reference velocity U_0 to travel the reference distance L*

$U_0/\sqrt{gL}$ *: Froude Number (Ratio of inertia to the gravity forces)*

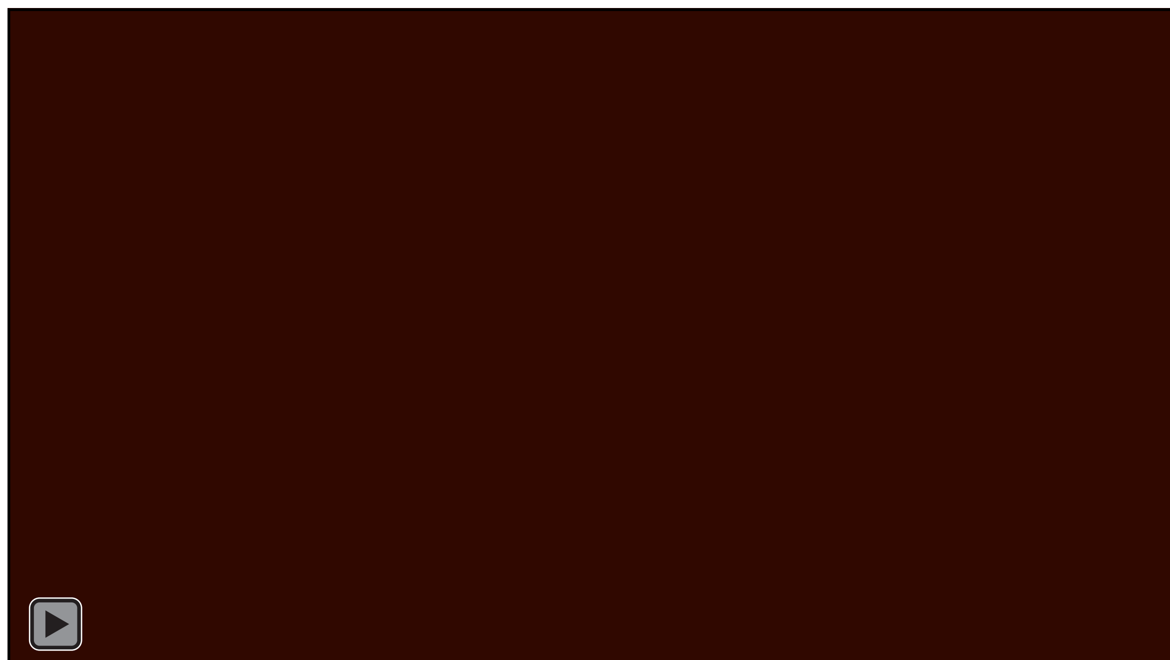
Fluid-Solid Interaction: General Formulation

Illustration of the role of Froude Number

- ***Sloshing phenomena in an orbitally shaken reservoir:***
  - ***The gentle motion enhances mixing and oxygenation (Used in cell cultivation)***
  - ***The sloshing is mainly governed by the Froude number***



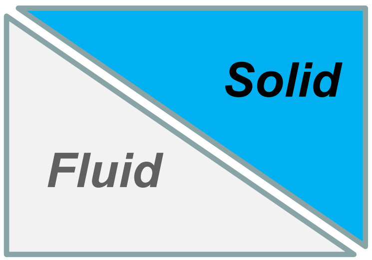
Link to video on Youtube



Fluid-Solid Interaction: General Formulation

Uncoupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***The fluid and the solid are treated separately***
    - ***Relevant variables for the solid motion:***



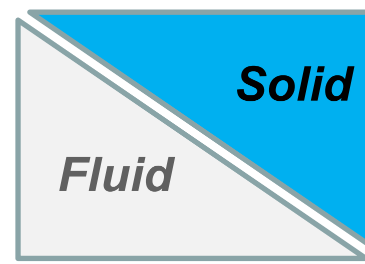
ξ	Displacement field
x	Coordinates
t	Time
E	Stiffness
L	Reference Length
g	Gravity
ρ_s	Density
ξ_0	Reference Displacement

Fluid-Solid Interaction: General Formulation

Uncoupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***The fluid and the solid are treated separately***
    - ***Dimensional form of the equation of solid motion:***

$$f(x, t, \xi, E, L, g, \rho_s, \xi_0) = 0$$



- ***Dimensionless form of the equation of solid motion:***

$$R = rank\begin{pmatrix} L \\ M \\ T \end{pmatrix}\begin{bmatrix} 1 & 0 & 1 & -1 & 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & -2 & 0 & 0 \end{bmatrix} = 3$$

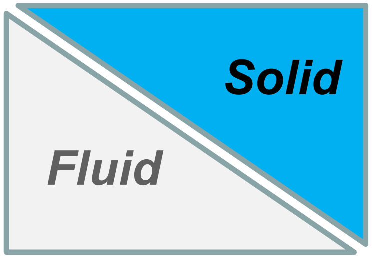
→ *Problem dimension* $= \ 8 - 3 = 5$

Fluid-Solid Interaction: General Formulation

Uncoupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***The fluid and the solid are treated separately***
    - ***Dimensionless form of the equation of solid motion:***

$$F\left(\frac{x}{L},\frac{\xi}{L},\frac{t\sqrt{E/\rho_s}}{L},\frac{\xi_0}{L},\frac{\rho_s g L}{E}\right)=0$$



$\frac{\xi_0}{L}$ *: Displacement number (Displacement scaled by the reference length)*

$\frac{\rho_s g L}{E}$ *: Elastogravity number (ratio of gravity force over elastic force)*

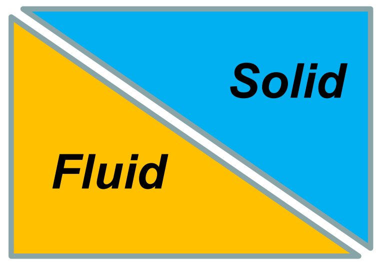
$$\frac{t\sqrt{E/\rho_s}}{L}=\frac{t}{T_{solid}}\ ;T_{solid}=\frac{L}{\sqrt{E/\rho_s}}=\frac{L}{c}$$

: Time for elastic waves to travel the reference length L

Fluid-Solid Interaction: General Formulation

Coupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
    - ***Fluid motion depends on both <u>fluid and solid</u> variables:***

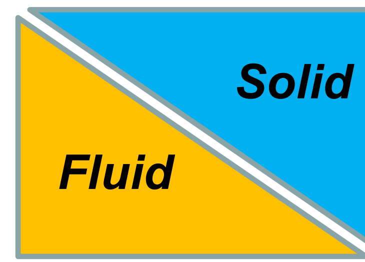


Coordinates	$\underline{x}$	$\underline{x}$	Coordinates
Time	t	t	Time
Velocity field	$\underline{U}$	$\underline{\xi}$	Displacement field
Viscosity	μ	E	Stiffness
Size	L	L	Size
Gravity	g	g	Gravity
Density	ρ	ρ_S	Density
Velocity Data	U_0	ξ_0	Displacement Data

Fluid-Solid Interaction: General Formulation

Coupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***
  - ***Fluid motion depends on both <u>fluid and solid</u> variables:***



$$g(U, x, t, \mu, \rho, U_0, L, g, E, \rho_s, \xi_0) = 0$$

11 variables related to solid and fluid domains

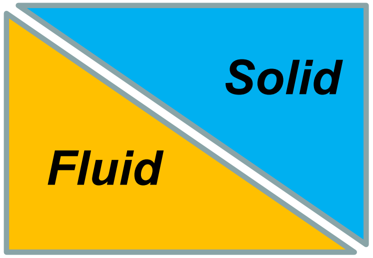
$$\begin{array}{cc} & \begin{array}{ccccccccccc} U & x & t & \mu & \rho & U_0 & L & g & E & \rho_s & \xi_0 \end{array} \\ Rank & \begin{bmatrix} 1 & 1 & 0 & -1 & -3 & 1 & 1 & 1 & -1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 & 0 & -2 & -2 & 0 & 0 \end{bmatrix} = 3 \end{array}$$

→ $\textit{Problem dimension} = 11 - 3 = 8$

Fluid-Solid Interaction: General Formulation

Coupled formulation of fluid-solid interaction:

- ***We consider a flow interacting with a moving/deforming solid:***



  - ***Dimensionless form:***

$$G\left(\frac{U}{U_0}, \frac{x}{L}, \frac{t}{L/U_0}, \frac{\rho U_0 L}{\mu}, \frac{U_0}{\sqrt{gL}}, \frac{\xi_0}{L}, \frac{\rho_s g L}{E}, A\right) = 0$$

  - ***Dimensionless parameter A ? (must combine fluid & solid)***

Fluid-Solid Interaction: General Formulation

Coupled formulation of fluid-solid interaction:

- ***Possible choices for the dimensionless number A:***

$$\boldsymbol{\frac{\rho}{\rho_s}}$$ *: Mass number (mass of the fluid vs mass of the solid)*

$$\boldsymbol{\frac{U_0}{\sqrt{E/\rho_s}}}$$ *: Reduced velocity (Ratio of fluid velocity & elastic wave celerity)*

$$\boldsymbol{\frac{\rho U_0^2}{E}}$$ *: Cauchy number (combines fluid forces and solid stiffness)*
High Cauchy number → The solid is more deformable under the action of the fluid forces

Fluid-Solid Interaction: General Formulation

Equations of motion of fluid and solid:

- ***On the Fluid side: Navier-Stokes equations***

Mass conservation or Continuity (incompressible): $\vec{\nabla}\vec{U} = 0$

Momentum conservation:

$$\rho \frac{d\vec{U}}{dt} = -\rho g \vec{e_z} - \vec{\nabla} p + \mu \Delta \vec{U}$$

Inertia

Gravity force

Viscous force

Pressure gradient

Equations of motion of fluid and solid:

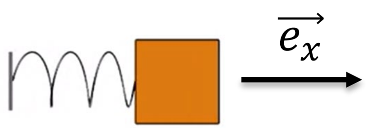
- ***On the Solid side:***

Single mode approximation: $\boldsymbol{\xi(x,t) = q(t)\phi(x)}$

Modal displacement

Modal shape
(Always known)

Example:



$\boldsymbol{\phi(x) = \overrightarrow{e_x}}$

Oscillator equation:

$$\boldsymbol{m\frac{d^2q}{dt^2} + kq = f}$$

Modal mass

Modal stiffness

Modal load

Fluid-Solid Interaction: General Formulation

Equations of motion of fluid and solid:

- ***At the interface between fluid and solid (Continuity equations):***
  - *Kinematic condition*
    (no mixing and no sliding between solid and fluid):

$$\boldsymbol{U} = \frac{d\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{t})}{d\boldsymbol{t}} = \frac{d\boldsymbol{q}(\boldsymbol{t})}{d\boldsymbol{t}}\boldsymbol{\phi}(\boldsymbol{x})$$

  - *Dynamic conditions: Continuity of forces at the interface*

$$\int \{[-\boldsymbol{p}\boldsymbol{I} + \boldsymbol{\mu}(\boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{\nabla}^{\boldsymbol{t}}\boldsymbol{U})].\boldsymbol{n}\}.\boldsymbol{\phi}\boldsymbol{d}\boldsymbol{S} = \boldsymbol{f}$$

Modal load

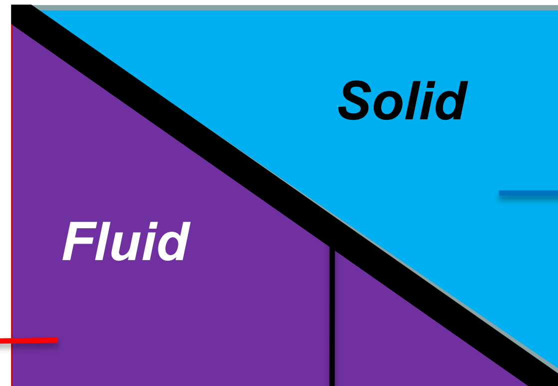
Fluid-Solid Interaction: General Formulation

Equations of motion of fluid and solid:

Navier-Stokes

$$\vec{\nabla}\vec{U} = 0$$

$$\rho\frac{d\vec{U}}{dt} = -\rho g\vec{e_z} - \vec{\nabla}p + \mu\Delta\vec{U}$$



$$m\frac{d^2q}{dt^2} + kq = f$$

Oscillator equation

At the interface:

$$U = \frac{dq(t)}{dt}\phi(x)$$

$$\int\{[-pI + \mu(\nabla U + \nabla^t U)].n\}.\phi dS = f$$

+ boundary conditions in the fluid and solid domains

Fluid-Solid Interaction: General Formulation

Dimensionless equations of fluid and solid motions

- ***Dimensionless variables in the fluid:***

$$x_f = \frac{x}{L} \qquad U_f = \frac{U}{U_0} \qquad p_f = \frac{p}{\rho U_0^2} \qquad t_f = \frac{t}{T_{fluid}}$$

- ***Dimensionless variables in the solid:***

$$x_s = \frac{x}{L} \qquad q_s = \frac{q}{\xi_0} \qquad f_s = \frac{f}{k\xi_0} \qquad t_s = \frac{t}{T_{solid}}$$

Fluid-Solid Interaction: General Formulation

Dimensionless equations of fluid and solid motions

- ***Dimensionless time ?***

  - ***In the solid:*** $T_{solid} = \sqrt{m/k} \;\rightarrow\; t_s \frac{t}{\sqrt{m/k}}$

    $$or \quad T_{solid} = \frac{L}{c} \;\rightarrow\; t_s = \frac{t}{L/c}$$

  - ***In the fluid:*** $T_{fluid} = \frac{L}{U_0} \;\rightarrow\; t_f = \frac{t}{L/U_0}$

  - ***The time references (clocks) in the solid and in the fluid are not necessarily the same. We may have:***

    $$T_{fluid} \gg T_{solid}\,, \qquad T_{fluid} \ll T_{solid} \quad or \quad T_{fluid} \sim T_{solid}$$

  - ***Arbitrary choice:*** $t_s = \frac{t}{T_{solid}}$

Fluid-Solid Interaction: General Formulation

Dimensionless equations of fluid and solid motions

- ***On the Fluid side: Navier-Stokes equations***
  - ***Continuity:*** $\vec{\nabla}\vec{U} = 0 \Rightarrow \vec{\nabla}\overrightarrow{U_f} = 0$
  - ***Momentum conservation:*** $\rho\dfrac{d\vec{U}}{dt} = -\rho g\overrightarrow{e_z} - \vec{\nabla}p + \mu\Delta\vec{U}$

$$\Rightarrow \quad \frac{c}{U_0}\frac{d\overrightarrow{U_f}}{dt_s} = -\frac{gL}{U_0^2}\overrightarrow{e_z} - \vec{\nabla}p_f + \frac{\mu}{\rho U_0 L}\Delta\overrightarrow{U_f}$$

$$\frac{1}{U_R}\frac{d\overrightarrow{U_f}}{dt_s} = -\frac{1}{Fr^2}\overrightarrow{e_z} - \vec{\nabla}p_f + \frac{1}{Re}\Delta\overrightarrow{U_f}$$

$U_R = \dfrac{T_{solid}}{T_{fluid}}$: *Reduced velocity. This term appears because the reference time is taken in the solid*

Dimensionless equations of fluid and solid motions

- ***On the solid side:***

$$m\frac{d^2q}{dt^2} + kq = f$$

$$q_s = \frac{q}{\xi_0} \qquad\qquad f_s = \frac{f}{k\xi_0}$$

Dimensionless time ? $$t_s = \frac{t}{\sqrt{\frac{m}{k}}}$$

Fluid-Solid Interaction: General Formulation

Dimensionless equations of fluid and solid motions

- ***On the solid side:***

$$m\frac{d^2q}{dt^2}+kq=f \quad \Rightarrow \quad m\left[\sqrt{\frac{k}{m}}\right]^2\xi_0\frac{d^2q_s}{dt_s^2}+k\xi_0q_s=k\xi_0f_s$$

$$\Rightarrow \quad \frac{d^2q_s}{dt_s^2}+q_s=f_s$$

$\Rightarrow$ *Dimensionless oscillator's frequency = 1*

Obvious, since the oscillation period is taken as the reference time

Fluid-Solid Interaction: General Formulation

Dimensionless equations of fluid and solid motions

- ***At the fluid-solid interface:***
  - ***Kinematic condition:***

$$\boldsymbol{U} = \frac{\boldsymbol{dq(t)}}{\boldsymbol{dt}}\boldsymbol{\phi(x)} \quad \Rightarrow \quad \frac{\boldsymbol{U_0 T_{solid}}}{\boldsymbol{L}}\boldsymbol{U_f} = \frac{\boldsymbol{\xi_0}}{\boldsymbol{L}}\frac{\boldsymbol{dq_s}}{\boldsymbol{dt_s}}\boldsymbol{\phi(x)}$$

$$\Rightarrow \quad \boldsymbol{U_R U_f} = \underbrace{\boldsymbol{D}}_{\text{Displacement Nb}}\frac{\boldsymbol{dq_s}}{\boldsymbol{dt_s}}\boldsymbol{\phi(x)}$$

Fluid-Solid Interaction: General Formulation

Dimensionless equations of fluid and solid motions

- ***On the fluid-solid interface***

  - ***Dynamic condition:***

$$\int_{Interface} \{[-\boldsymbol{p}\boldsymbol{I} + \boldsymbol{\mu}(\nabla \boldsymbol{U} + \nabla^t \boldsymbol{U})].\boldsymbol{n}\}.\boldsymbol{\phi} d\boldsymbol{S} = \boldsymbol{f}$$

$$\Downarrow$$

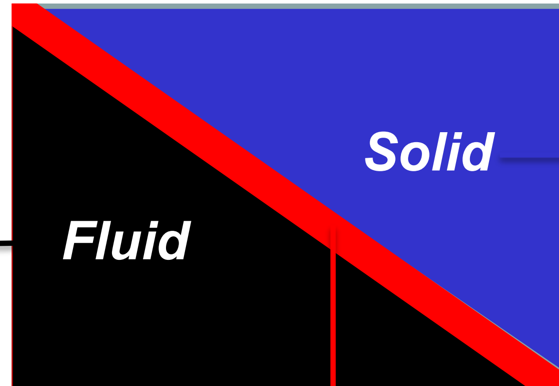
$$\int_{Interface} \left\{\frac{\rho U_0^2 L}{k}\left[-p_f I + \frac{\mu}{\rho U_0 L}\left(\nabla U_f + \nabla^t U_f\right)\right].n\right\}.\phi dS = \frac{\xi_0}{L} f_s$$

$$\Downarrow$$

$$\int_{Interface} \left\{Cy\left[-p_f I + \frac{1}{Re}\left(\nabla U_f + \nabla^t U_f\right)\right].n\right\}.\phi dS = Df_s$$

Fluid-Solid Interaction: General Formulation

Summary: Dimensionless equations



Navier-Stokes

$$\vec{\nabla}\overrightarrow{U_f} = 0$$

$$\frac{1}{U_R}\frac{d\overrightarrow{U_f}}{dt_s} = -\frac{1}{Fr^2}\overrightarrow{e_z} - \vec{\nabla}p_f + \frac{1}{Re}\Delta\overrightarrow{U_f}$$

+ fluid boundary conditions

Oscillator equation

$$\frac{d^2q_s}{dt_s^2} + q_s = f_s$$

+ solid boundary conditions

At the interface:

$$U_R U_f = D\frac{dq_s}{dt_s}\phi(x)$$

$$\int_{Interface}\left\{Cy\left[-p_f I + \frac{1}{Re}\left(\nabla U_f + \nabla^t U_f\right)\right].n\right\}.\phi dS = Df_s$$