
AEROELASTICITY AND FLUID-STRUCTURE INTERACTION

Chapter 5:

Quasi-Static Aeroelasticity: Dynamic Instability - Flutter

Dynamic Instability - Flutter

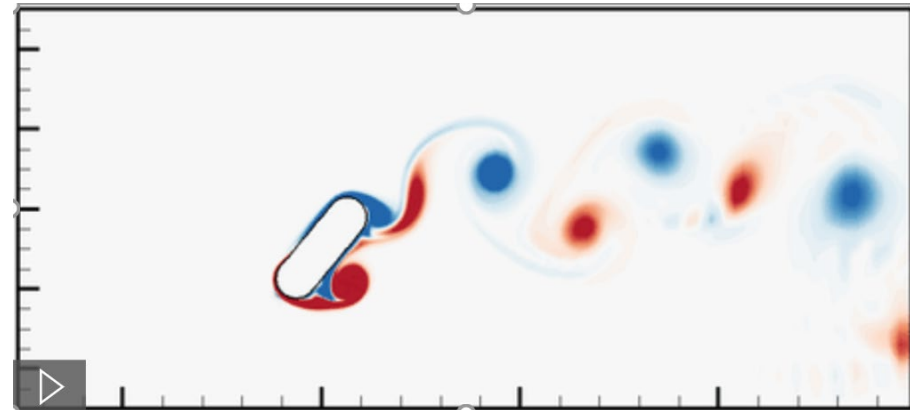
Definition (Cambridge Dictionary):

Flutter: *To make a series of quick delicate movements up and down or from side to side, or cause something to do this*

- **Examples:**
 - *Brightly colored flags were fluttering in the breeze*
 - *A white bird poised on a wire and fluttered its wings*
 - *Every time I think about my exams my stomach flutters!*
- **En Français: Flottement**

In engineering:

“Flutter is an unstable, self-excited structural oscillation at a definite frequency where energy is extracted from the flow by the motion of the structure”



Dynamic Instability - Flutter

Quasi-static aeroelasticity framework :

- **Large reduced velocity:**
$$U_R = \frac{T_{solid}}{T_{fluid}} \gg 1$$

→ *The fluid evolves at the contact of a solid almost at rest*

- **Static instability (divergence):**
 - *Always involves one vibration mode*
 - *Example: Torsional divergence of an airfoil*
- **Dynamic instability (Flutter):**
 - *Oscillations with increasing amplitude*
 - *Always involves two vibration modes (e.g. torsional & bending modes)*

Dynamic Instability - Flutter

Large reduced velocity: $U_R = \frac{T_{solid}}{T_{fluid}} \gg 1$

- **Dynamic instability : Oscillations with increasing amplitude**

Example: Instability induced by torsion and bending modes

airplane stabilizers



Glider wings



Dynamic Instability - Flutter

Large reduced velocity: $U_R = \frac{T_{solid}}{T_{fluid}} \gg 1$

- **Dynamic instability : Oscillations with increasing amplitude**

**Example: Flapping flag
(Flottement de drapeau)**

**Energy harvesting from air-induced
motion of a piezoelectric flag**



Reference: Simultaneous wind and solar energy harvesting with inverted flags

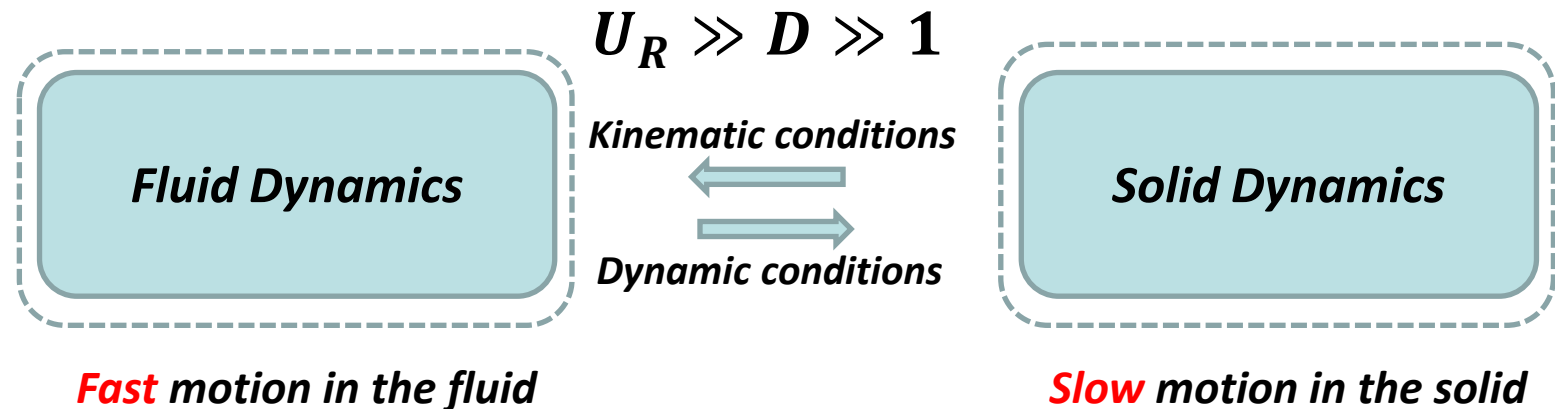
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Quasi-Static aeroelasticity – 2 modes approximation



Hypothesis: The solid motion is the combination of two modes:

$$\xi_s(x, t) = Dq_{1s}(t)\phi_1(x) + Dq_{2s}(t)\phi_2(x)$$

Two modal equations of motion:

$$\begin{cases} m_1 \ddot{q}_1 + k_1 q_1 = f\phi_1(x) \\ m_2 \ddot{q}_2 + k_2 q_2 = f\phi_2(x) \end{cases}$$

$f\phi_1(x), f\phi_2(x)$: the projections of the fluid loading (the same) on mode shapes ϕ_1 and ϕ_2

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Quasi-Static aeroelasticity – 2 modes approximation

Steady state fluid dynamics:

Depends on instantaneous position of the interface (q_1 and q_2)

$$p_f(Dq_{1s}, Dq_{2s}) \text{ and } U_f(Dq_{1s}, Dq_{2s})$$

Projection of the fluid loading on the modes $\phi_1(x)$ and $\phi_2(x)$:

$$f_s \phi_1(x) = C_y F_1(Re, Dq_{1s}, Dq_{2s})$$

$$f_s \phi_2(x) = C_y F_2(Re, Dq_{1s}, Dq_{2s})$$

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Quasi-Static aeroelasticity – 2 modes approximation

Remember (Chapter 4):

- **Single mode approximation (quasi-static aeroelasticity):**

$$\xi_s(x_s, t_s) = Dq_s(t_s)\phi(x_s) \quad p_f(Dq_s) \text{ and } U_f(Dq_s)$$
$$Df_s = C_y \int_I \left\{ \left[-p_f I + \frac{1}{Re} (\nabla U_f + \nabla^t U_f) \right] \cdot n \right\} \cdot \phi dS = C_y F(Re, Dq_s)$$

- **Expansion of fluid loading (small displacement Dq_s):**

$$C_y F(Re, Dq_s) = C_y F^0 + C_y \left(\frac{\partial F}{\partial Dq_s} \right) Dq_s + \dots$$

→ The solid behaves like if it was attached to a spring with a stiffness

$k_f = -C_y \left(\frac{\partial F}{\partial Dq_s} \right)$ with a risk of static instability (divergence), when $k_f < 0$

A similar approach may be applied to the case of 2 modes

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Quasi-Static aeroelasticity – 2 modes approximation

- **Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):**
 - **Flow induced force from the motion of the interface:**

$$f_s \phi_1(x) = C_y F_1^0 + C_y \left(\frac{\partial F_1}{\partial Dq_{1s}} \right) Dq_{1s} + C_y \left(\frac{\partial F_1}{\partial Dq_{2s}} \right) Dq_{2s} + \dots$$

$$f_s \phi_2(x) = C_y F_2^0 + C_y \left(\frac{\partial F_2}{\partial Dq_{1s}} \right) Dq_{1s} + C_y \left(\frac{\partial F_2}{\partial Dq_{2s}} \right) Dq_{2s} + \dots$$

- **The terms $C_y F_1^0$ and $C_y F_2^0$ represent the static part of the loading projections, which are not relevant for the vibration**
- **In the following, we will only keep the 1st order terms of the expansion**

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Quasi-Static aeroelasticity – 2 modes approximation

- **Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):**
 - **Flow induced force from the motion of the interface:**

$$\begin{cases} f_s \phi_1(x) = C_y \left(\frac{\partial F_1}{\partial Dq_{1s}} \right) Dq_{1s} + C_y \left(\frac{\partial F_1}{\partial Dq_{2s}} \right) Dq_{2s} + \dots \\ f_s \phi_2(x) = C_y \left(\frac{\partial F_2}{\partial Dq_{1s}} \right) Dq_{1s} + C_y \left(\frac{\partial F_2}{\partial Dq_{2s}} \right) Dq_{2s} + \dots \end{cases}$$



$$m_1 \ddot{q}_1 + k_1 q_1 = C_y \left(\frac{\partial F_1}{\partial q_1} \right) q_1 + C_y \left(\frac{\partial F_1}{\partial q_2} \right) q_2$$

$$m_2 \ddot{q}_2 + k_2 q_2 = C_y \left(\frac{\partial F_2}{\partial q_1} \right) q_1 + C_y \left(\frac{\partial F_2}{\partial q_2} \right) q_2$$

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Quasi-Static aeroelasticity – 2 modes approximation

- **Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):**
 - **Flow induced stiffness (proportional to C_y):**

$$\begin{cases} \ddot{q}_1 + \frac{k_1}{m_1} q_1 = \frac{C_y}{m_1} \left(\frac{\partial F_1}{\partial q_1} \right) q_1 + \frac{C_y}{m_1} \left(\frac{\partial F_1}{\partial q_2} \right) q_2 \\ \ddot{q}_2 + \frac{k_2}{m_2} q_2 = \frac{C_y}{m_2} \left(\frac{\partial F_2}{\partial q_1} \right) q_1 + \frac{C_y}{m_2} \left(\frac{\partial F_2}{\partial q_2} \right) q_2 \end{cases}$$

$$\left(\frac{\partial F_i}{\partial q_j} \right) = K_{ij} \quad i, j = 1, 2 \quad \omega_1 = \sqrt{\frac{k_1}{m_1}} \quad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

- **The 2 modes are coupled with flow-induced stiffness forces**

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Quasi-Static aeroelasticity – 2 modes approximation

- **Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):**
 - **Flow induced stiffness (proportional to C_y):**

$$\Rightarrow \begin{cases} \ddot{q}_1 + \omega_1^2 q_1 = C_y \frac{K_{11}}{m_1} q_1 + C_y \frac{K_{12}}{m_1} q_2 \\ \ddot{q}_2 + \omega_2^2 q_2 = C_y \frac{K_{21}}{m_2} q_1 + C_y \frac{K_{22}}{m_2} q_2 \end{cases}$$

K_{11}, K_{22} : *Similar to single mode approximation*

K_{12}, K_{21} : *Coupled stiffness resulting from the interaction between the 2 modes*

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Quasi-Static aeroelasticity – 2 modes approximation

- **Solid motion under coupled flow-induced stiffness forces:**

$$\begin{cases} \ddot{q}_1 + \left(\omega_1^2 - C_y \frac{K_{11}}{m_1} \right) q_1 = C_y \frac{K_{12}}{m_1} q_2 \\ \ddot{q}_2 + \left(\omega_2^2 - C_y \frac{K_{22}}{m_2} \right) q_2 = C_y \frac{K_{21}}{m_2} q_1 \end{cases}$$

- **As the flow velocity is increased, the frequencies of mode 1 and 2 are altered in different ways**
 - **These frequencies may come close to each other or move away from each other**

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Quasi-Static aeroelasticity – 2 modes approximation

- **Solid motion under coupled flow-induced stiffness forces:**
 - **We consider the case of “coincidence of frequencies”:**

$$\omega_1^2 - C_y \frac{K_{11}}{m_1} = \omega_2^2 - C_y \frac{K_{22}}{m_2}$$

- **In this case, a common reference time allows writing the equations of solid motion in non dimensional way.**
Coincidence \rightarrow Non dimensional frequency =1 for both modes

$$\begin{cases} \ddot{q}_{1s} + q_{1s} = C_y K'_{12} q_{2s} \\ \ddot{q}_{2s} + q_{2s} = C_y K'_{21} q_{1s} \end{cases}$$

K'_{12} and K'_{21} are non-dimensional forms of coupled stiffnesses K_{12} and K_{21}

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Quasi-Static aeroelasticity – 2 modes approximation

- **Solid motion in the case of “coincidence of frequencies”:**
 - **Assumptions (for simplicity):**
 - **Coupling stiffness very small with respect of modes stiffness:**
 - **Same magnitude of coupling stiffness forces:**

$$|C_y K'_{12}| = |C_y K'_{21}| = \varepsilon \ll 1$$

- **The equations for the solid motion read:**

$$\begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = \varepsilon q_{1s} \end{cases}$$

Symmetric stiffness coupling

or

$$\begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = -\varepsilon q_{1s} \end{cases}$$

Anti-symmetric stiffness coupling

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Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:**
$$\begin{cases} \ddot{\mathbf{q}}_{1s} + \mathbf{q}_{1s} = \varepsilon \mathbf{q}_{2s} \\ \ddot{\mathbf{q}}_{2s} + \mathbf{q}_{2s} = \varepsilon \mathbf{q}_{1s} \end{cases}$$
- We look for harmonic solutions \mathbf{q}_{1s} and \mathbf{q}_{2s} in the following form:

$$\begin{pmatrix} \mathbf{q}_{1s} \\ \mathbf{q}_{2s} \end{pmatrix} = \text{Re} \left[\begin{pmatrix} \mathbf{q}_{1s,0} \\ \mathbf{q}_{2s,0} \end{pmatrix} e^{i\omega t} \right]$$

$$\Rightarrow \begin{pmatrix} \ddot{\mathbf{q}}_{1s} \\ \ddot{\mathbf{q}}_{2s} \end{pmatrix} = -\omega^2 \begin{pmatrix} \mathbf{q}_{1s} \\ \mathbf{q}_{2s} \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 - \omega^2 & -\varepsilon \\ -\varepsilon & 1 - \omega^2 \end{pmatrix}}_D \begin{pmatrix} \mathbf{q}_{1s,0} \\ \mathbf{q}_{2s,0} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(Dynamic Matrix)

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Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:**

$$D \begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- ***D* stands for the dynamic matrix (not displacement number !). For non trivial solutions $(q_{1s}, q_{2s}) \neq (0, 0)$, the determinant of the dynamic matrix must vanish:**

$$\text{Det}(D) = 0 \Rightarrow (1 - \omega^2)^2 = \varepsilon^2 \Rightarrow \begin{cases} \omega_A = \sqrt{1 + \varepsilon} \approx 1 + \frac{\varepsilon}{2} \\ \omega_B = \sqrt{1 - \varepsilon} \approx 1 - \frac{\varepsilon}{2} \end{cases}$$

$\varepsilon \ll 1 \Rightarrow$ The new frequencies are real positive numbers

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Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:**

→ The eigenvectors may be obtained for ω_A and ω_B by solving:

$$\begin{pmatrix} 1 - \omega_A^2 & -\varepsilon \\ -\varepsilon & 1 - \omega_A^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 - \omega_B^2 & -\varepsilon \\ -\varepsilon & 1 - \omega_B^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By using : $1 - \omega_A^2 = -\varepsilon$; $1 - \omega_B^2 = \varepsilon$ and by setting $q_1 = 1$, we obtain 2 eigenvectors A and B:

$$Q_A = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad Q_B = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

→ In presence of symmetric coupled stiffness, the new modes A and B are combinations of original modes 1 and 2 with slightly altered frequencies (weak coupling)

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Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:**

- **Modal masses**

$$M_A = \overset{\text{Transpose}}{Q_A^t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_A = 2 \quad M_B = Q_B^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_B = 2$$

- **Modal stiffnesses**

$$K_A = Q_A^t \begin{pmatrix} 1 - \omega^2 & -\varepsilon \\ -\varepsilon & 1 - \omega^2 \end{pmatrix} Q_A = 2 + 2\varepsilon$$

$$K_B = Q_B^t \begin{pmatrix} 1 - \omega^2 & -\varepsilon \\ -\varepsilon & 1 - \omega^2 \end{pmatrix} Q_B = 2 - 2\varepsilon$$

We verify that :

$$\omega_A^2 = \frac{K_A}{M_A} = 1 + \varepsilon \quad \text{and} \quad \omega_B^2 = \frac{K_B}{M_B} = 1 - \varepsilon$$

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Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling:**
$$\begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = -\varepsilon q_{1s} \end{cases}$$
- We look for solutions q_{1s} and q_{2s} in the following forms:

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \text{Re} \left[\begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t} \right]$$

$$\Rightarrow \begin{pmatrix} 1 - \omega^2 & -\varepsilon \\ \varepsilon & 1 - \omega^2 \end{pmatrix} \begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of the dynamic matrix must vanish:

$$\Rightarrow (1 - \omega^2)^2 = -\varepsilon^2 \Rightarrow \begin{cases} \omega_A = \sqrt{1 + i\varepsilon} \approx 1 + i\frac{\varepsilon}{2} \\ \omega_B = \sqrt{1 - i\varepsilon} \approx 1 - i\frac{\varepsilon}{2} \end{cases}$$

The new frequencies are complexe numbers

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling: Solid motion**

→ We obtain 2 modes A and B with complex frequencies and complex eigenvectors (as we did for symmetric case):

$$\left\{ \begin{array}{l} \omega_A = 1 + i \frac{\varepsilon}{2} \\ Q_A = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \omega_B = 1 - i \frac{\varepsilon}{2} \\ Q_B = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B = \begin{pmatrix} 1 \\ i \end{pmatrix} \end{array} \right.$$

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Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling: Solid motion**
 - **Modal masses and stiffness**

$$M_A = \underset{\substack{\nearrow \\ \text{Conjugate}}}{Q_A}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_A = 2 \quad M_B = Q_B^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_B = 2$$

$$K_A = Q_A^* \begin{pmatrix} 1 - \omega^2 & -\varepsilon \\ \varepsilon & 1 - \omega^2 \end{pmatrix} Q_A = 2 + 2i\varepsilon$$

$$K_B = Q_B^* \begin{pmatrix} 1 - \omega^2 & -\varepsilon \\ \varepsilon & 1 - \omega^2 \end{pmatrix} Q_B = 2 - 2i\varepsilon$$

We verify that :

$$\omega_A^2 = \frac{K_A}{M_A} = 1 + i\varepsilon \quad \text{and} \quad \omega_B^2 = \frac{K_B}{M_B} = 1 - i\varepsilon$$

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling: Solid motion**

- **The first mode A:**
$$\begin{cases} \omega_A = 1 + i \frac{\epsilon}{2} \\ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{cases}$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \text{Re} \left[\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A e^{i\omega_A t} \right] = \text{Re} \left[\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i(1+i\frac{\epsilon}{2}) t} \right]$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{-\frac{\epsilon t}{2}}$$

→ **The mode A is a damped mode with the vibration amplitude vanishing in exponential way with time**

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling: Solid motion**

- The second mode B :
$$\begin{cases} \omega_B = 1 - i \frac{\epsilon}{2} \\ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B = \begin{pmatrix} 1 \\ i \end{pmatrix} \end{cases}$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \text{Re} \left[\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B e^{i\omega_B t} \right] = \text{Re} \left[\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i(1 - i\frac{\epsilon}{2}) t} \right]$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{\frac{\epsilon t}{2}}$$

→ The second mode B is an unstable mode with the vibration amplitude increasing exponentially without a limit

→ **Dynamic instability or Flutter**

Dynamic Instability - Flutter

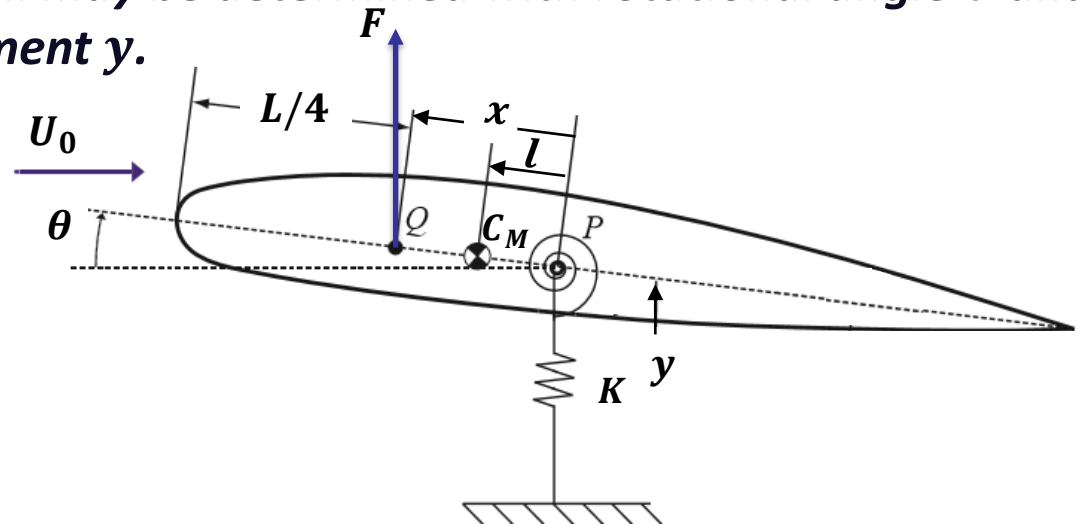
Quasi-Static aeroelasticity – 2 modes approximation

- **Dynamic Instability – Summary:**
 - ***In presence of coincidence of frequencies of 2 vibration modes :***
 - ***Symmetric stiffness coupling leads to new modes (combination of original modes) with a slightly altered frequencies***
This is a conservative coupling with no energy transfer between the modes within one cycle
 - ***Anti-symmetric stiffness coupling leads to new modes with complex frequencies and eigenvectors: a damped mode and an unstable mode***
This unstable mode is non conservative: Energy is exchanged between the two modes and accumulates in time.
This instability is called dynamic instability or flutter
 - ***Quasi-static elasticity***
→ Prediction of static & dynamic instabilities (Divergence & flutter)

Dynamic Instability - Flutter

Example: 2 modes approximation of an airfoil

- **We consider a 2D airfoil with torsion and plunge modes:**
 - **Attached at P to torsional and translational springs with stiffness C and K :**
 - L : Chord length
 - C_M : center of mass, located at a distance l from elastic center (l positive towards the leading edge)
 - Q : aerodynamic center, located at a distance x from elastic center (x positive towards the leading edge)
 - **The position of the foil may be determined with rotational angle θ and the vertical displacement y .**



Dynamic Instability - Flutter

2 modes approximation of an airfoil

- **In absence of flow ($U_0=0$)**

→ Classical problem of solid mechanics. May be solved with Lagrange equation

- **Kinetic energy:** $E_c = \frac{1}{2} M(\dot{y} + l\dot{\theta})^2 + \frac{1}{2} J\dot{\theta}^2$ M : Mass
 J : Moment of inertia
- **Potential energy (Elastic energy):** $E_p = \frac{1}{2} Ky^2 + \frac{1}{2} C\theta^2$
- **Lagrangian:** $\mathcal{L} = E_c - E_p$

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) - \frac{\partial \mathcal{L}}{\partial y} = 0 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \end{cases} \Rightarrow \begin{cases} M\ddot{y} + Ml\ddot{\theta} + Ky = 0 \\ J\ddot{\theta} + C\theta + (M\ddot{y} + Ml\ddot{\theta})l = 0 \end{cases}$$

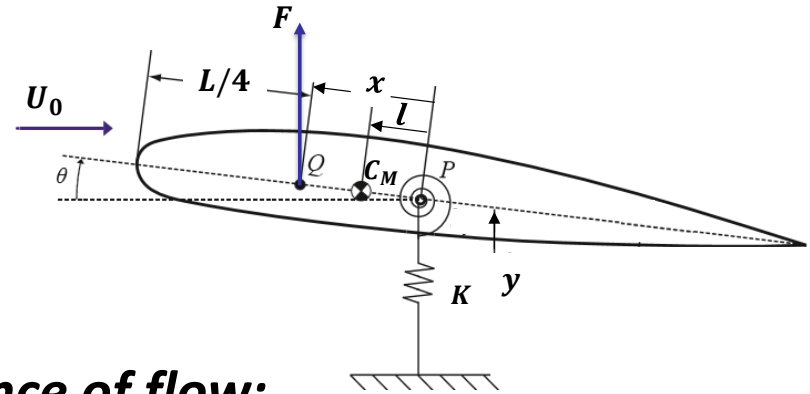
$$\Rightarrow \begin{cases} M\ddot{y} + Ml\ddot{\theta} + Ky = 0 \\ J\ddot{\theta} + C\theta - Kyl = 0 \end{cases}$$

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- In presence of a flow and with large reduced velocity ($UR \gg 1$)
 - The fluid force acting on the surface of the airfoil, at any frozen position defined by θ and y :

$$F = \frac{1}{2} \rho U_0^2 L C_L(\theta) \quad L: \text{Chord length}$$



- Equations of solid motion in presence of flow:

$$\begin{cases} M\ddot{y} + Ml\ddot{\theta} + Ky = \frac{1}{2} \rho U_0^2 L C_L(\theta) \\ J\ddot{\theta} + C\theta - Kyl = + \frac{1}{2} \rho U_0^2 L(x - l) C_L(\theta) \end{cases}$$

Dynamic Instability - Flutter

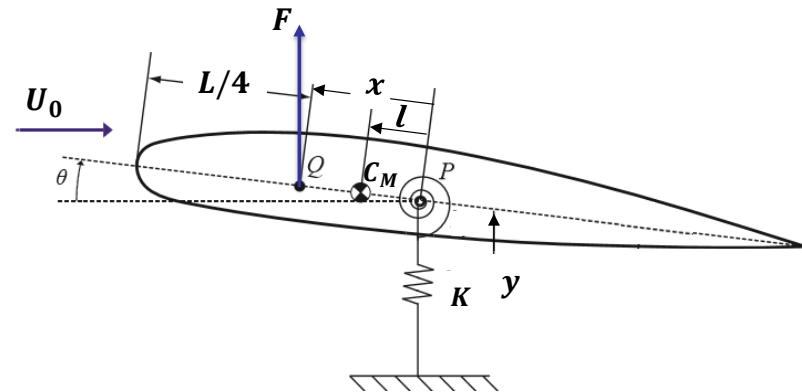
2 modes approximation of an airfoil

- **Thin airfoil hypothesis:**

$$C_L = 2\pi\theta \quad \Rightarrow \quad F = \frac{\rho U_0^2}{2} (2\pi\theta)L = \pi\rho U_0^2\theta L$$

- **Dimensionless variables:**

$$q_{1s} = \frac{y}{L}; \quad q_{2s} = \theta; \quad \epsilon = \frac{l}{L}; \quad x_s = \frac{x}{L}$$



$$\Omega = \sqrt{\frac{KJ}{CM}} \quad : \text{Ratio of translational and rotational frequencies}$$

$$\kappa = \frac{KL^2}{C} \quad : \text{Stiffness ratio}$$

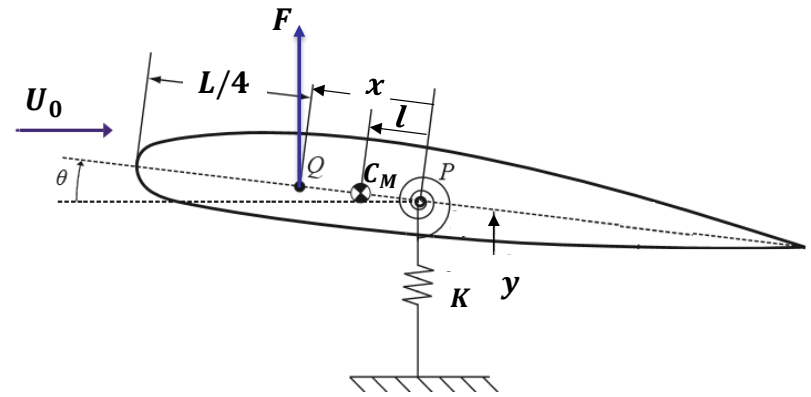
$$C_y = \frac{\rho U_0^2 L^2}{2C} \quad : \text{Cauchy Number} \quad t_s = \sqrt{\frac{C}{J}} t \quad : \text{dimensionless time, scaled by torsional period}$$

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- Dimensionless form of coupled equations of solid motion in presence of flow:

$$\begin{cases} \ddot{q}_{1s} = \frac{d^2 q_{1s}}{dt_s^2} = \frac{J\ddot{y}}{LC} \\ \ddot{q}_{2s} = \frac{d^2 q_{2s}}{dt_s^2} = \frac{J\ddot{\theta}}{C} \end{cases}$$



$$\Rightarrow \begin{cases} \ddot{q}_{1s} + \epsilon \ddot{q}_{2s} + \Omega^2 q_{1s} = 2\pi C_Y \frac{\Omega^2}{\kappa} q_{2s} \\ \ddot{q}_{2s} + (1 - 2\pi C_Y (x_s - \epsilon)) q_{2s} = +\kappa \epsilon q_{1s} \end{cases}$$

$x_s - \epsilon > 0 \Rightarrow$ The torsional frequency decreases with increasing C_Y

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- We look for harmonic solutions in the following form:

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \text{Re} \left[\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}_0 e^{i\omega t} \right]$$

\Rightarrow 2 modes: $\left\{ \begin{array}{l} \text{Frequencies: } \omega_A, \omega_B \\ \text{Modal vectors: } \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}_A, \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}_B \end{array} \right.$

$$\Rightarrow \begin{pmatrix} \Omega^2 & -2\pi C_Y \frac{\Omega^2}{\kappa} \\ -\kappa \epsilon & 1 - 2\pi C_Y (x_s - \epsilon) \end{pmatrix} \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \omega^2 \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}$$

- If all the parameters are fixed except the Cauchy number, we may solve numerically the linear system to derive the two roots :

$$\begin{aligned} \omega_A(C_Y) &= \omega_{A,real}(C_Y) + i\omega_{A,im}(C_Y) \\ \omega_B(C_Y) &= \omega_{B,real}(C_Y) + i\omega_{B,im}(C_Y) \end{aligned}$$

Dynamic Instability - Flutter

2 modes approximation of an airfoil

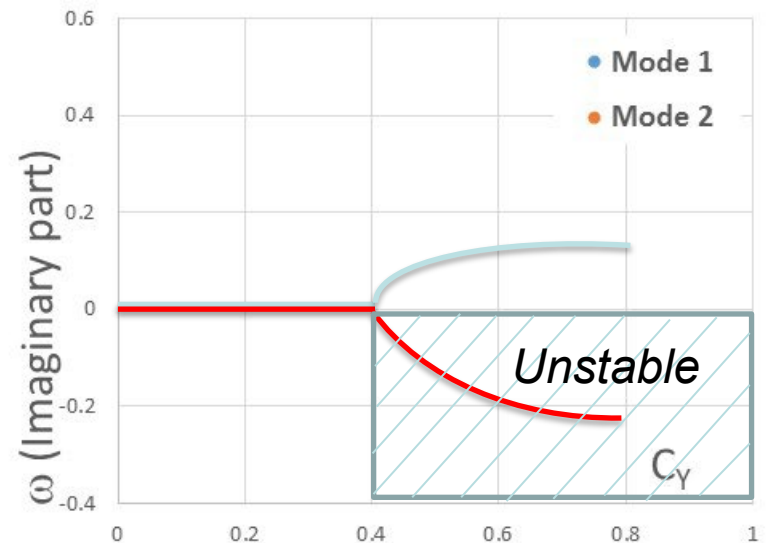
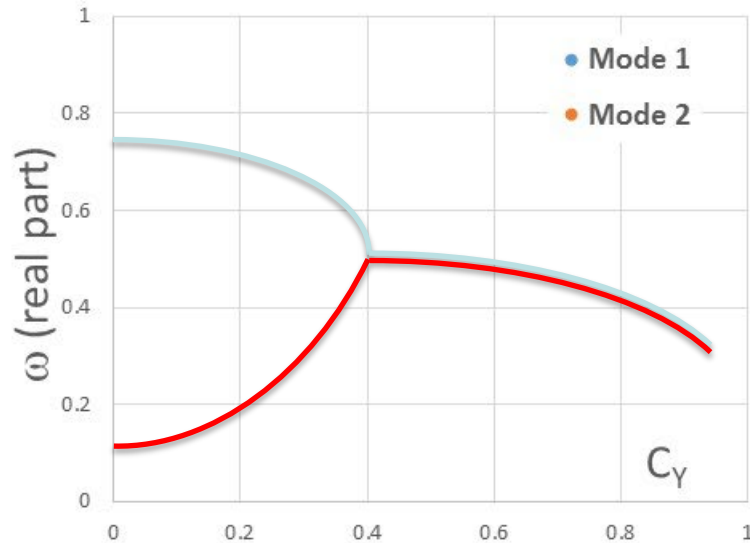
- **Coupled equations of solid motion in presence of flow:**
 - **By increasing the Cauchy number, the complex frequencies may be used to predict the dynamic instability of the system:**
 - **Dynamic instability occurs when:**

$$(1) \quad \begin{array}{l} \omega_{A,im}(C_Y) < 0 \\ \text{or} \\ \omega_{B,im}(C_Y) < 0 \end{array} \quad \text{Anti-symmetric stiffness coupling} \\ \text{(amplitude increases like } e^t \text{)}$$

Dynamic Instability - Flutter

2 modes approximation of an airfoil

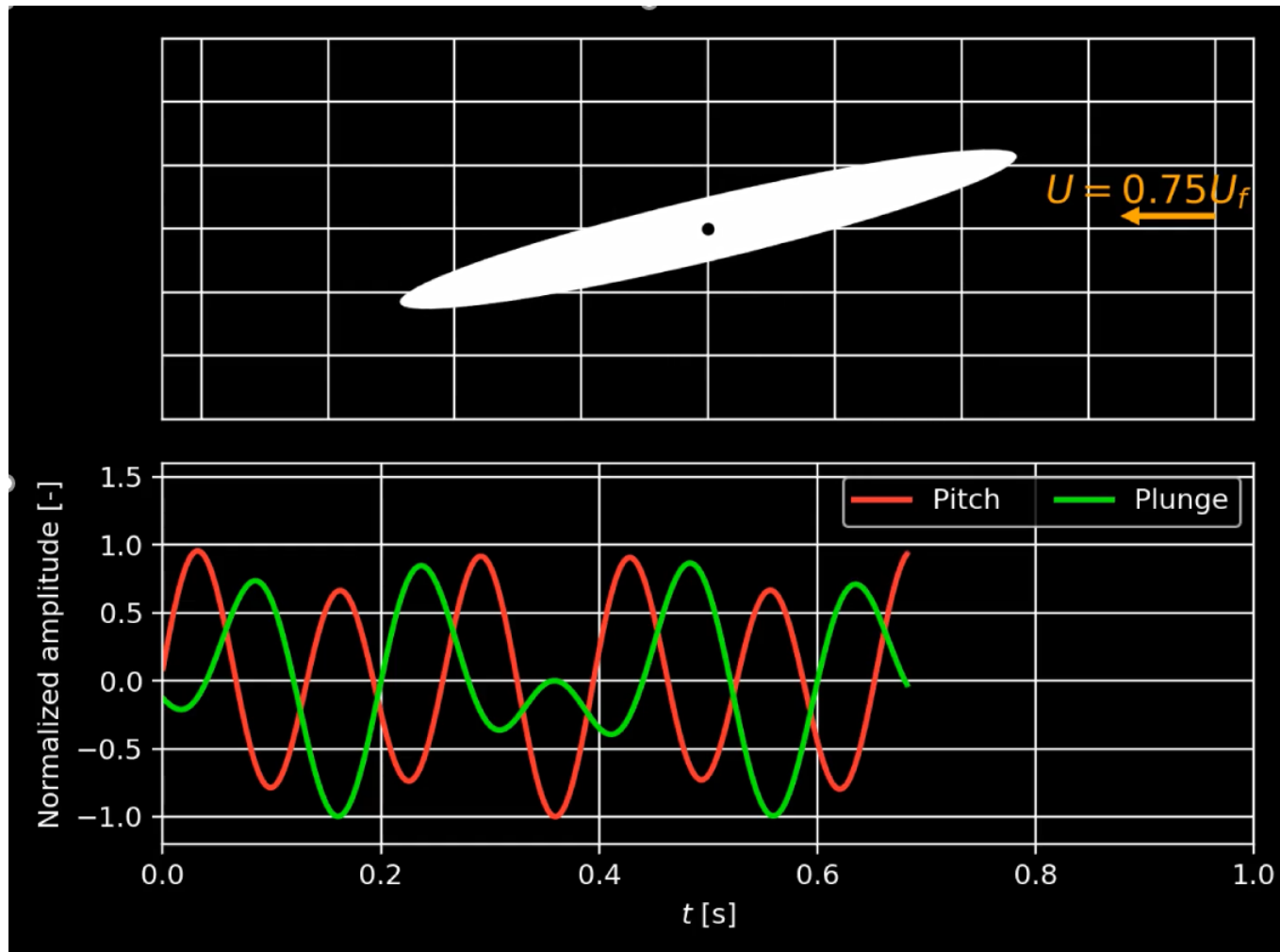
- **Procedure: Plot the real and imaginary parts of the frequencies as functions of Cauchy number and check the condition (1)**
- **Example of frequency evolution for a given set of parameters:**



- $C_Y < 0.4 \rightarrow 2$ modes with real values of frequencies (weak coupling)
- $C_Y > 0.4 \rightarrow$ Mode 2 has a frequency with negative imaginary part
 \rightarrow Dynamic instability

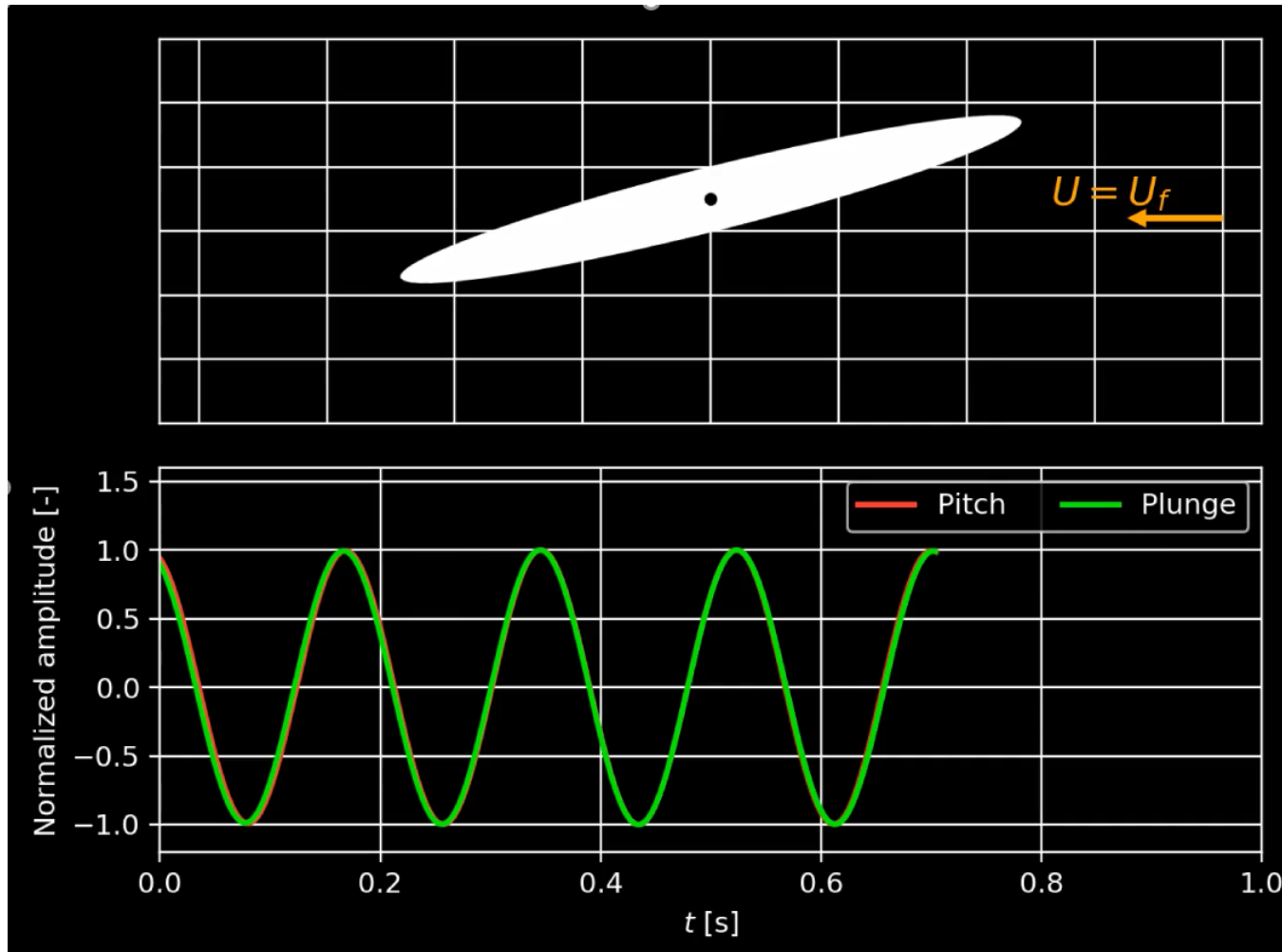
Dynamic Instability - Flutter

Numerical solution: Below flutter speed



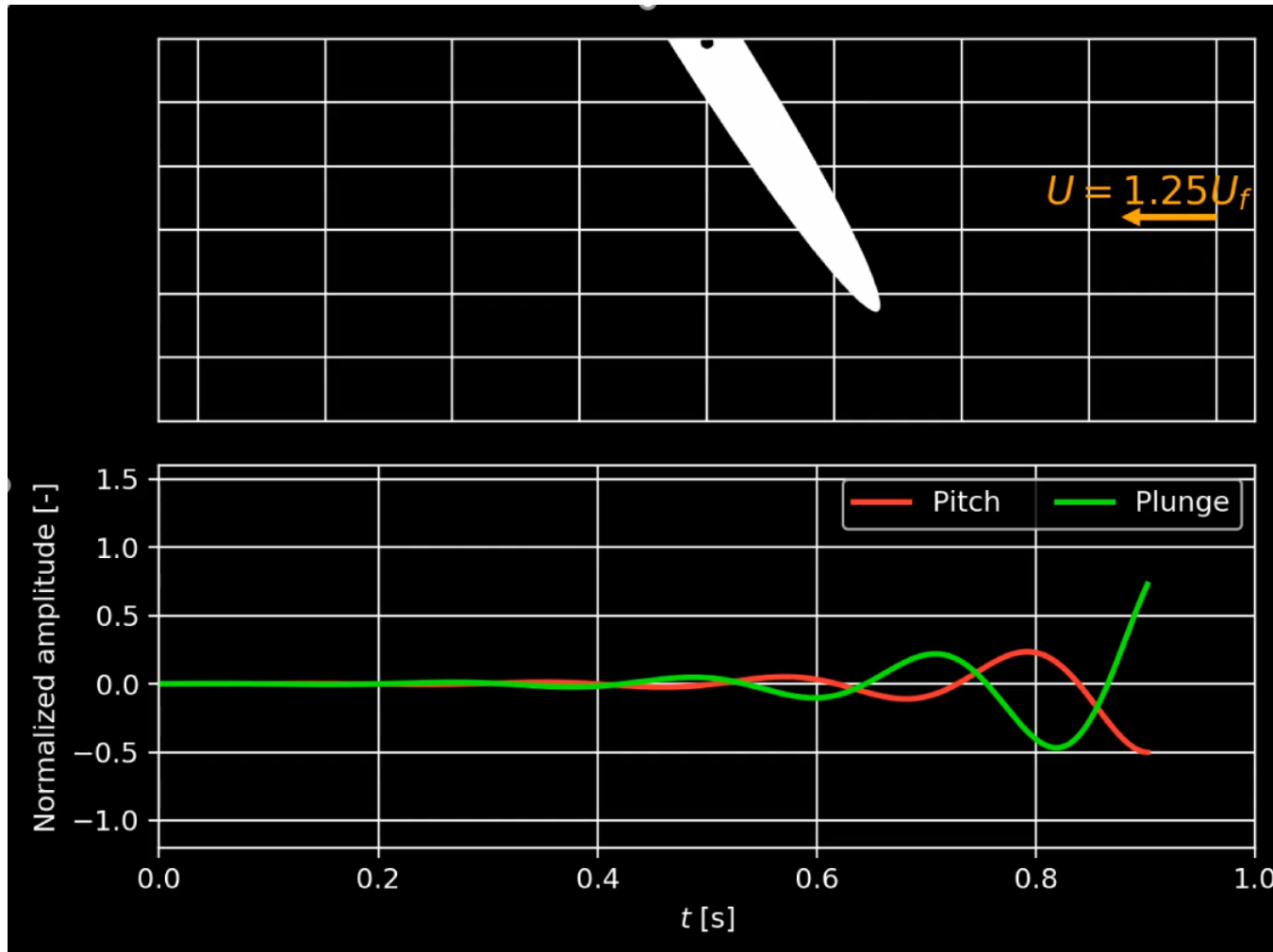
Dynamic Instability - Flutter

Numerical solution: At flutter speed



Dynamic Instability - Flutter

Numerical solution: Beyond flutter speed



Dynamic Instability - Flutter

Summary

- *We have seen how quasi-static elasticity framework allows to predict dynamic instability (flutter) involving 2 vibration modes of a foil in a flow*
- *Remarks:*
 - *The hypothesis of $U_R \gg 1$ is not always true. What happens at $U_R \approx 1$?*
 - *In the case of an airfoil, at $U_R \approx 1$, the motion of the solid may not be neglected and may have significant impacts:*
 - *Effective incidence angle depends on the foil speed*
→ damping effect
 - *The fast motion of the foil generates vortices at LE and TE*
 - *The flutter is not limited to 2 modes, we do see instabilities involving one mode only. This is not covered within the quasi-static aeroelasticity*