Lecture 7

Consensus: convergence rate and digraphs that are not strongly connected

Textbook §3.3.3, §5.1, §5.2

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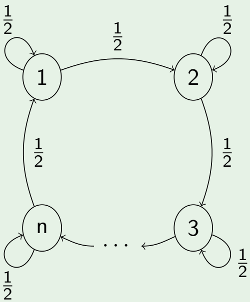
Analysis of $\rho_{\text{ess}}(A)$

- The smaller ρ_{ess}, the faster consensus
- $\rho_{\text{ess}}(A)$ depends upon the topology/weights of the digraph G associated to A

  $\hookrightarrow$ very difficult to compute, in general

Next: notable examples

Ring graph



to $\longrightarrow$

from $\downarrow$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \\ \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}$$

(A is a circulant matrix)

$$\text{Spec}(A) = \left\{ \frac{1}{2} + \frac{1}{2} e^{i\frac{2\pi h}{n}}, \quad h = 0, 1, \dots, n-1 \right\}$$

- G strongly connected, aperiodic $\rightarrow A$ primitive
- A doubly stochastic

$$\hookrightarrow \text{consensus theorem} \Rightarrow x(k) \rightarrow < x(0) > \mathbb{1}_n \text{ as } k \rightarrow +\infty$$

- $\lambda = 1$ is obtained for $h = 0$
- $\rho_{\text{ess}}(A)$ is obtained for $h = 1$

$$\rho_{\text{ess}}(A) = \left| \frac{1}{2} + \frac{1}{2} e^{i\frac{2\pi}{n}} \right| = \cdots = \sqrt{\frac{1}{2}\left(1 + \cos\frac{2\pi}{n}\right)} \simeq$$

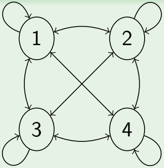
$\cos(\theta) = 1 - \frac{\theta^2}{2} + O(\theta^4)$

$$\underset{\text{Taylor approx. for } \cos\theta}{\simeq} \sqrt{1 - \frac{\pi^2}{n^2} + O\left(\frac{1}{n^2}\right)}$$

Remarks

- $\rho_{\text{ess}} \rightarrow 1$ as n increases. Large $n \Rightarrow$ slow consensus
- The ring graph is one of the worst cases for consensus

Example: complete graph



weights: $\frac{1}{n}$

$$A = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix}$$

- $\text{Spec}(A) = \{1, 0, \dots, 0\} \Rightarrow \rho_{\text{ess}}(A) = 0 \Rightarrow$ consensus in one step!
- Optimal for consensus but the worst for communication

Problem (weight assignment)

Given the digraph G, find weights ≥ 0 such that

- A is doubly stochastic and primitive
- A minimises $\rho_{\text{ess}}(A)$

$\hookrightarrow$ Very hard, in general. Requires numerical optimization

Main result from the previous lectures

Let $G = (V, E, w)$ be a digraph. Consider the consensus algorithm

$$x^+ = Ax \Rightarrow x(k) = A^k x(0) \quad A = \text{adjacency matrix of } G$$

Theorem (consensus with primitive, stochastic matrices)

If A is primitive and stochastic, the state trajectory $x(k)$ verifies

$$\lim_{k \to +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \tag{1}$$

where w is defined in the usual way.

If, in addition, A is double stochastic, then $w = \frac{1}{n} \mathbb{1}_n$ and hence

$$\lim_{k \to +\infty} x(k) = < x(0) > \mathbb{1}_n \tag{2}$$

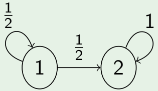
where $< x >= \frac{1}{n} \sum_{i=1}^{n} x_i$

- A primitive $\Rightarrow$ G strongly connected
- Problem: what happens if G is not strongly connected?

Consensus in digraphs that are not Strongly Connected (SC)

$$x^+ = Ax = \begin{bmatrix} \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ x_2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example: opinion dynamics with a single Globally Reachable (GR) node



$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \geq 0 \quad \text{stochastic, reducible}$$
$(n=2,\ r=1)$

Powers of A

$$A^2 = \begin{bmatrix} \frac{1}{4} & 1-\frac{1}{4} \\ 0 & 1 \end{bmatrix} \quad \dots \quad A^k = \begin{bmatrix} \frac{1}{2^k} & 1-\frac{1}{2^k} \\ 0 & 1 \end{bmatrix}$$

$$x(k) = A^k x(0) = A^k \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2^k} x_1(0) + \left(1 - \frac{1}{2^k}\right) x_2(0) \\ x_2(0) \end{bmatrix}$$

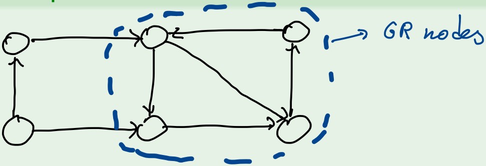
→ goes to $x_2(0)$

↳ stubborn agent not changing his opinion

Remarks

- There is a node not GR $\leftrightarrow$ A is reducible
- $A_{ij} > 0$ means agent j influences agent i (opposite direction of edges)
- 2 is GR. 2 acts as a leader. GR nodes = leaders? Yes, in general.

Example - multiple GR nodes



The "leader" is a subgraph. Consensus?

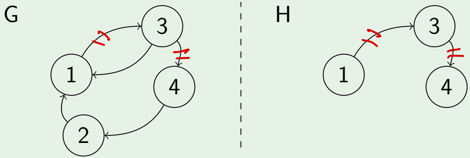
Subgraphs

Let $G = (V, E)$ be a digraph

Recall

$H = (V', E')$ is a subgraph of G if $V' \subseteq V$, $E' \subseteq E$ and E' connects only nodes in V'

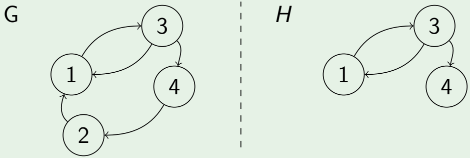
Example



Definition

H is induced by $V' \subseteq V$ if E' contains <u>all</u> edges in E between vertices in V'

Example (ctd.) H induced by $V' = \{1, 3, 4\}$

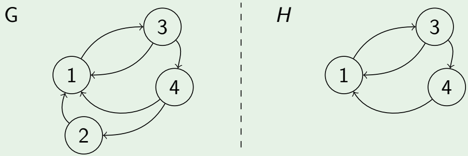


Definition

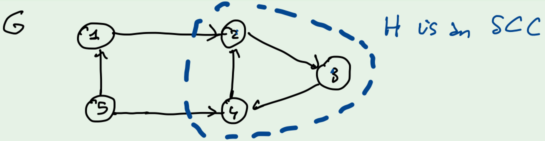
A subgraph H is a strongly connected component (SCC) of G if H is strongly connected (SC) and any other subgraph of G stricly containing H is not SC

with more nodes or edges

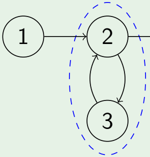
Example of a subgraph which is not an SCC



Example of an SCC



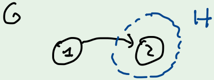
Example



H is an SCC

SCC with more than one node: once a path enters the SCC, it can stay in there forever

Example: a sink is an SCC



- $H = (\{2\}, \emptyset)$ is SC
- any other node cannot be reached from 2

  ↳ H cannot be included in a larger SC subgraph

Lemma

If a digraph G has a GR node, then the subgraph H induced by all GR nodes is an SCC of G.

Proof

- If H is made by a single node, it is SC and then an SCC
- If H is made by $k > 1$ nodes, let v and v', $v \neq v'$ be two nodes. There is a path $v \ldots v'$ and a path $v' \ldots v$. Assume by contradiction that a path $v \ldots v'$ crosses a node $\tilde{v} \notin H$. Then, $\tilde{v}$ is GR, which contradicts $\tilde{v} \notin H$. We have shown that each path connecting nodes in H is contained in H. Therefore there is no digraph H', SC and stricly including H

Should be not globally reachable (because not in H)

but, because of the red path it is globally reachable

The condensation graph

Goal: given a digraph $G = (V, E)$ obtain a simplified digraph where each SCC has been collapsed into a node.

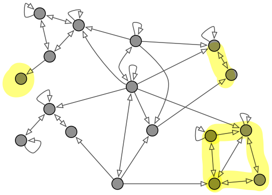
Definition

The condensation digraph $C(G)$ is defined as follows

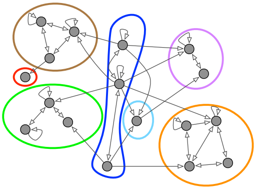
- Nodes $V(G)$ of $C(G)$ are the SCCs $H_1, \ldots, H_s$ of G
- There is an edge (H_i, H_j) in $C(G)$ if there is an edge in G from H_i to H_j.

Example of condensation digraph

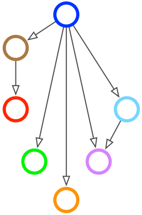
G



SCCs of G

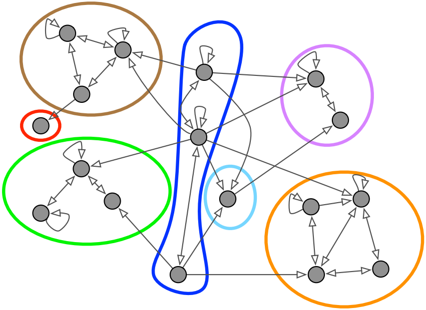


$C(G)$

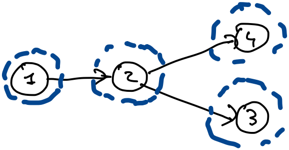


SCCs of G

A digraph can be always partitioned into SCCs, i.e. $\forall v \in V, \exists H \in V(G)$ s.t. $v \in H$



$\hookrightarrow$ In the "worst case" SCCs are composed by one node each



Lemma (properties of $C(G)$)

By construction,

1. (i) $C(G)$ is acyclic
2. (ii) $C(G)$ is weakly connected only if G is weakly connected
3. (iii) The following statements are equivalent
   - (a) G has a GR node
   - (b) $C(G)$ has a GR node
   - (c) $C(G)$ has a unique sink

Remarks

Under (a), let H be the SCC of G induced by all GR nodes.

- Then H is the unique sink of $C(G)$
- Once in H, there is no way out

Lemma (properties of $C(G)$)

By construction,

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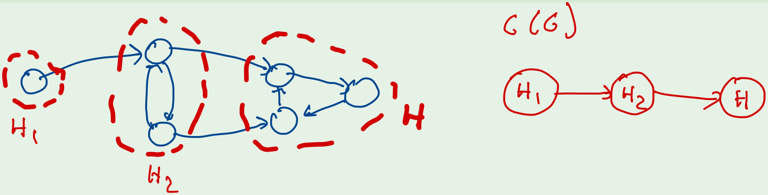
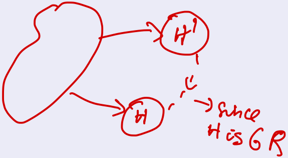
Example



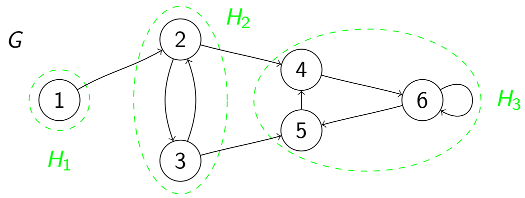
Proof $(b) \Rightarrow (c)$ (using the fact that $C(G)$ is acyclic)

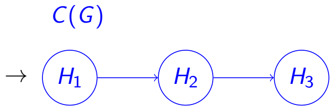
$\exists H \in C(G)$ which is GR. We show it is a sink. By contradiction, we assume H is not a sink. Then, there is an edge $(H, \tilde{H})$ and also a path from $\tilde{H}$ to H (as H is GR). But $C(G)$ has no cycles.

We now prove that the sink is unique. Assume, by contradiction, that there are 2 sinks H and H'. Since H is GR there is a path $H' \dots H$. But this cannot happen as H' has no outgoing edges.

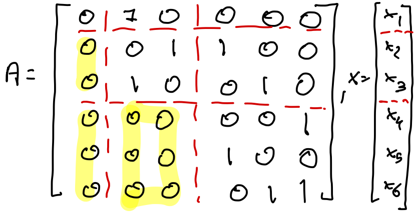


Condensation digraphs and consensus





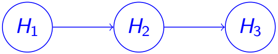
- Let A be the adjacency matrix of G
- Consensus algorithm $x^+ = Ax$
  $\hookrightarrow x_i^+$ depends on $x_j \Leftrightarrow A_{ij} \neq 0$



From $C(G)$ we deduce relations on blocks of states

$$\underbrace{\begin{bmatrix} x_4^+ \\ x_5^+ \\ x_6^+ \end{bmatrix}}_{H_3} \text{ does not depend on } \underbrace{\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}_{H_2} \text{ and } \underbrace{x_1}_{H_1}$$

$C(G)$



- $\begin{bmatrix} x_2^+ \\ x_3^+ \end{bmatrix}$ does not depend on x_1
- $\begin{bmatrix} x_2^+ \\ x_3^+ \end{bmatrix}$ depends on $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$
- ...

Conclusion: states in H_3 evolve autonomously and, over time, they will influence all other states.

- Let A_{H_3} be the adjacency matrix of H_3
  $\hookrightarrow$ If it is primitive and stochastic

$$\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \rightarrow \alpha \mathbb{1}_3 \text{ as } k \rightarrow +\infty$$

  for some α depending only on $x_4(0)$,$x_5(0)$, $x_6(0)$

- For $k \rightarrow +\infty$ nodes 2 and 3 receive α by the leaders $\{4, 5, 6\}$

  One expects $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \rightarrow \alpha \mathbb{1}_2$. True?

  Similarly, node 1 receives α, as $k \rightarrow +\infty$. Is $x_1 \rightarrow \alpha$ as well?

Main result

Theorem (consensus with GR nodes)

Let $G = (V, E, w)$ be a digraph with row-stochastic adjacency matrix A. Assume G has a GR node and that the SCC H induced by all GR nodes is aperiodic. Then

1. $\lambda = 1$ is a simple eigenvalue of A and it is strictly dominant, that is

$$\lambda = 1 > |\mu| \quad \forall \mu \in \text{Spec}(A), \mu \neq \lambda$$

2. $\lim_{k \to +\infty} A^k = \mathbb{1}_n w^T$, where w is the left eigenvector of $\lambda = 1$ verifying $w^T \mathbb{1}_n = 1$

3. the elements of w verify

$$\begin{cases} w_i > 0 \text{ if } i \in V \text{ is GR} \\ w_i = 0 \text{ otherwise} \end{cases} \qquad (*)$$

Moreover, one has

$$\lim_{k \to +\infty} x(k) = \left(w^T x(0)\right) \mathbb{1}_n \qquad (**)$$

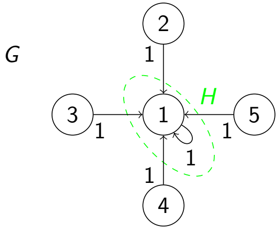
Remarks

- The theorem provides a generalization of the consensus theorem for primitive and stochastic matrices.

$$A \text{ primitive } \Rightarrow G \text{ strongly connected } \Rightarrow H = G$$

- From $(*)$ and $(**)$ one has $w^T x(0)$ is a weighted average of the initial states of the leaders (leaders=GRNs)
- Generalisations to multiple subgraphs of "leaders"
  $\hookrightarrow$ see the textbook

Example



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   - $w_i = 0$ otherwise $\quad (*)$

Moreover, one has
$$\lim_{k \to +\infty} x(k) = \left(w^T x(0)\right) \mathbb{1}_n \qquad (**)$$

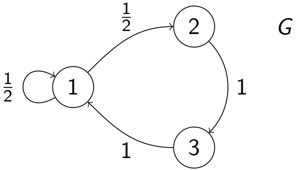
(1) is GR → $H = (\{1\}, (1,1))$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ → stochastic and aperiodic

Theorem ⇒ • $w^T = [*, 0, 0, 0, 0]$ $\xrightarrow{\text{normalization}}$ $w^T = [1 \; 0 \; 0 \; 0 \; 0]$

• $(**)$ $x(k) \to [x_1(0), x_1(0), x_1(0), x_1(0), x_1(0)]^T$

Example



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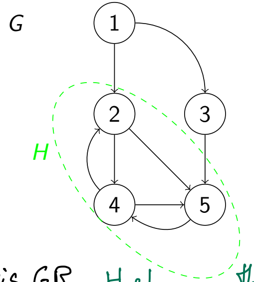
$$\lim_{k\to+\infty} x(k) = \left(w^T x(0)\right) \mathbb{1}_n \qquad (**)$$

- ① is GR, also ② and ③
  ↳ $H = G \rightarrow C(G) = [\{H\}, \emptyset]$
- $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ → stochastic. Moreover H is aperiodic

Theorem ⇒
- $w^T = [* \; * \; *]^T$, $* > 0$, $w^T \mathbb{1} = 1$
- $(**)$ $\quad x(k) \rightarrow \left([* \; * \; *]\, x(0)\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example



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- (5) is GR. H shown in the figure
- Assume A stochastic. Since H is aperiodic, then

Theorem ⇒ • $w^T = [0, *, 0, *, *]^T$ $\quad * > 0$, $\quad w^T \mathbb{1} = 1$

• $x(k) \to (w^T x(0)) \mathbb{1}_5$

Summary on consensus (so far)

Properties of $x(\infty)$	**Properties of A, stochastic**	**Properties of the digraph G**
Px1: Average consensus	PA1: Doubly stochastic and primitive	PG1: SC (strongly connected) and aperiodic
Px2: Consensus depending on all nodes ($w > 0$)	PA2: Primitive	
Px3: Consensus that might depend only on some nodes		PG2: One aperiodic GR SCC

We have shown the following implications

- PA1 $\Rightarrow$ Px1 and PA1 $\Rightarrow$ PG1
- PG1 $\Rightarrow$ PA2
- PA2 $\Rightarrow$ Px2
- PG2 $\Rightarrow$ Px3