# Lecture 3

## Stability of NCS: sampling and delay

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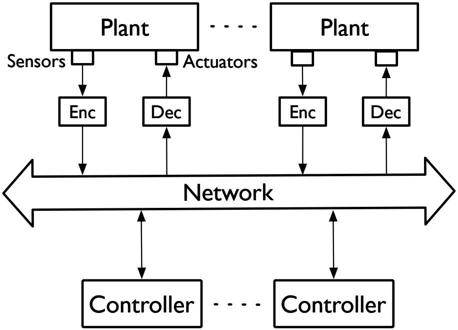
# Recap from the last lecture




**Control networks**

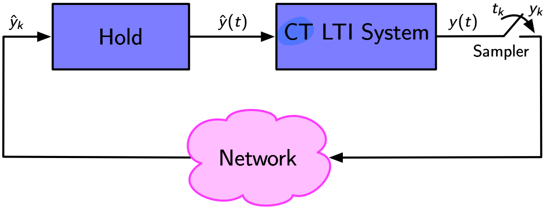
- Packet-networks designed for real-time operations
- Delays induced by: the physical layer, the transmission of complete packets, queuing at source nodes, decoding at the destination nodes, the MAC protocol, and the network load
  - Time-varying, often stochastic delays
- Packet dropout due to collisions + no retransmission of old packets

# Outline

How much sampling time and network delays can deteriorate stability and performance?

- Analysis
  - Discrete-time (DT) models of NCS with linear dynamics
  - Examples of the effect of sampling and delays on stability
  - The Maximum Allowable Transfer Interval (MATI): definition and estimation
- Control Design
  - Delay compensation for remote control

# NCS - collocated control

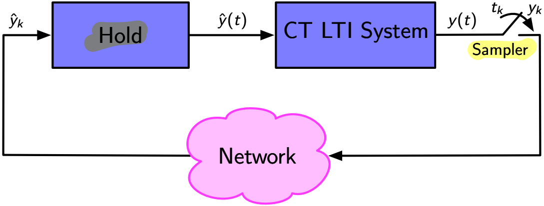



Previous lecture: "encoder"=sampler, "decoder"=hold

Assumption: One-plant-one-controller setting. The controller and system are

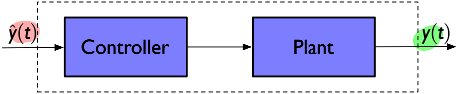
- collocated
- represented by a continuous-time (CT) LTI system

# NCS - collocated control



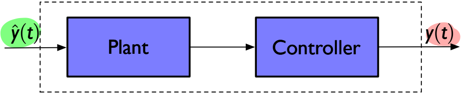

Previous lecture: "encoder"=sampler, "decoder"=hold

- Two possible arrangements:




- $y$: plant measurements
- $\hat{y}$: input to controller
- Controller collocated with actuators

- Controller gets "corrupted" measurements



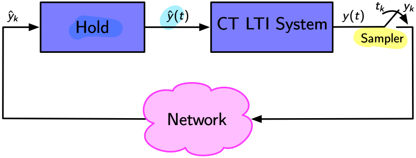

- $y$: control signals
- $\hat{y}$: input to the actuators
- controller collocated with sensors

- Plant gets "corrupted" control actions

# Model of sample and hold

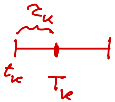
- System CT output $y(t)$ sampled at $\{t_k, k \in \mathbb{N}\}$
  - $y_k = y(t_k)$, $T_k \triangleq t_{k+1} - t_k$
- Hold + ideal network (no delay)

$$\hat{y}_k = y_k \qquad k \in \mathbb{N}$$
$$\hat{y}(t) = \hat{y}_k \quad t \in [t_k, t_{k+1})$$

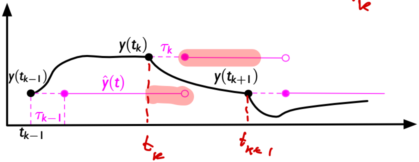



- Hold + network time varying delay: $y_k$ arrives at $t_k + \tau_k$

$\rightarrow$ Assumption (for simplicity) : $\tau_k < T_k$




$$\hat{y}_k = y_k, \quad k \in \mathbb{N}$$
$$\hat{y}(t) = \begin{cases} \hat{y}_{k-1} & t \in [t_k, t_k + \tau_k) \\ \hat{y}_k & t \in [t_k + \tau_k, t_{k+1}) \end{cases}$$

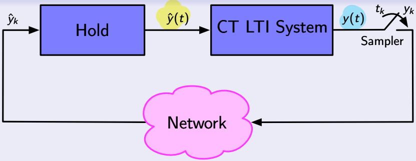


# Discrete-time model of the LTI system (1/2)

## Goal: compute the discrete-time dynamics for $x_k$

LTI system

$$\dot{x} = Ax + Bu, \quad u = \hat{y}$$
$$y = Cx$$




- Set $x_k = x(t_k)$, $y_k = y(t_k)$ etc.
- Recall the Lagrange formula for the above system with $x(t_0) = x_0$

$$x(t) = e^{A(t-t_0)}x_0 + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{(b)}$$

**State transition operator** $e^{As}$: pushes $x_0$ ahead by $s$ seconds

# Discrete-time model of the LTI system (1/2)

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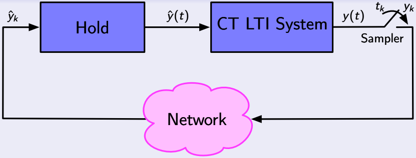


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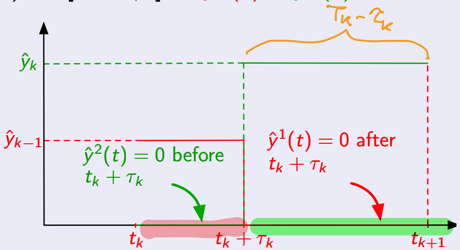
$$x(t) = e^{A(t-t_0)}x_0 + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{(b)}$$

**Constant-input transmission operator** $\Gamma(s) \triangleq \int_0^s e^{Az}dz$

- (b)$= \Gamma(t-t_0)B\bar{u}$ if $u(.) = \bar{u}$ on $[t_0, t]$.
  Proof: (b)$=\int_{t_0}^{t} e^{A(t-\tau)}d\tau B\bar{u} = -\int_{t-t_0}^{0} e^{Az}dzB\bar{u} = \int_0^{t-t_0} e^{Az}dzB\bar{u}$, where we have set $z = t - \tau$.
- $\Gamma(t-t_0)B$ "pushes" $\bar{u}$ ahead by $t - t_0$ seconds

# Discrete-time model of the LTI system (2/2)

Represent $u(t) = \hat{y}(t)$ on $[t_k, t_{k+1}]$ as $\hat{y}^1(t) + \hat{y}^2(t)$




For computing $x_{k+1}$, use the superposition principle with three causes: $x_k$, $\hat{y}^1(t)$, and $\hat{y}^2(t)$. Three experiments on $[t_k, t_{k+1}]$:

- $\phi(t_{k+1}, t_k, x_k, 0) = e^{AT_k} x_k$
- $\phi(t_{k+1}, t_k, 0, \hat{y}^2(t)) = e^{A(T_k - \tau_k)}\Gamma(\tau_k)B \cdot 0 + \Gamma(T_k - \tau_k)B\hat{y}_k = \Gamma(T_k - \tau_k)BCx_k$
- $\phi(t_{k+1}, t_k, 0, \hat{y}^1(t)) = e^{A(T_k - \tau_k)} x(t_k + \tau_k) = e^{A(T_k - \tau_k)}\Gamma(\tau_k)B\hat{y}_{k-1}$

$x_{k+1}$ is the linear combination (with unit coefficients) of the three experiments:

$$x_{k+1} = e^{AT_k} x_k + e^{A(T_k - \tau_k)}\Gamma(\tau_k)B\hat{y}_{k-1} + \Gamma(T_k - \tau_k)BCx_k$$

# Alternative derivation of the NCS (check @ home)

Use only the Lagrange formula on $[t_k, t_{k+1}]$

$$x_{k+1} = e^{A(t_{k+1}-t_k)}x_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-s)}B\hat{y}(s)ds$$

$$= e^{A(t_{k+1}-t_k)}x_k + \left(\int_{t_k}^{t_k+\tau_k} e^{A(t_{k+1}-s)}ds\right) B\hat{y}_{k-1}ds$$

$$+ \left(\int_{t_k+\tau_k}^{t_{k+1}} e^{A(t_{k+1}-s)}ds\right) B\hat{y}_k$$

$$= e^{A(t_{k+1}-t_k)}x_k + e^{A(t_{k+1})}e^{-A(t_k+\tau_k)} \int_{t_k}^{t_k+\tau_k} e^{A(t_k+\tau_k-s)}dsB\hat{y}_{k-1}$$

$$+ \left(\int_{t_k+\tau_k}^{t_{k+1}} e^{A(t_{k+1}-s)}ds\right) B\hat{y}_k$$

Changing variables in the integrals, so as to make $\Gamma(\cdot)$ appear:

$$x_{k+1} = e^{AT_k}x_k + e^{A(T_k-\tau_k)}\Gamma(\tau_k)B\hat{y}_{k-1} + \Gamma(T_k-\tau_k)BCx_k$$

# NCS global model

Need $x_k$ and $\hat{y}_{k-1}$ for computing $x_{k+1}$. Define the augmented state

$$z_k \triangleq \begin{bmatrix} x(t_k)^T & \hat{y}_{k-1}^T \end{bmatrix}^T.$$

## NCS dynamics

$$z_{k+1} = \Psi(T_k, \tau_k) z_k$$

$$\Psi(T_k, \tau_k) = \begin{bmatrix} e^{AT_k} + \Gamma(T_k - \tau_k)BC & e^{A(T_k - \tau_k)}\Gamma(\tau_k)B \\ C & 0 \end{bmatrix}$$

## Remarks

- The second line of $\Psi$ represents $\hat{y}_k = Cx(t_k)$
- $\Psi$ embodies the effect of sampling, network delay, and feedback interconnection
- Ideal network: $\tau_k = 0 \rightarrow \Gamma(0) = 0 \rightarrow$ The red part disappears
  - Simplified dynamics: $x_{k+1} = (e^{AT_k} + \Gamma(T_k)BC)x_k$

# NCS global model

Need $x_k$ and $\hat{y}_{k-1}$ for computing $x_{k+1}$. Define the augmented state

$$z_k \triangleq \left[ x(t_k)^T \quad \hat{y}_{k-1}^T \right]^T .$$

## NCS dynamics

$$z_{k+1} = \Psi(T_k, \tau_k) z_k$$

$$\Psi(T_k, \tau_k) = \begin{bmatrix} e^{AT_k} + \Gamma(T_k - \tau_k)BC & e^{A(T_k - \tau_k)}\Gamma(\tau_k)B \\ C & 0 \end{bmatrix}$$

## Problems

- How to calculate block matrices in $\Psi$?

# Computations of $e^{As}$

$$\Psi(T_k, \tau_k) = \begin{bmatrix} e^{AT_k} + \Gamma(T_k - \tau_k)BC & e^{A(T_k - \tau_k)}\Gamma(\tau_k)B \\ C & 0 \end{bmatrix}$$

- Closed-form of $e^{As}$: simple for only special A (e.g. diagonal)
- Symbolic computations. In MatLab (requires the symbolic toolbox)

```
syms s
A = [-1 0.9; 0 -0.2]
E = expm(A*s)
```

gives

$$E = \begin{bmatrix} e^{-s} & \frac{9}{8}(e^{\frac{s}{5}} - e^{-s}) \\ 0 & e^{\frac{s}{5}} \end{bmatrix}$$

- Numerical computation for given $A$ and $s$

```
A = [-1 0.9; 0 -0.2]
s=0.5
E = expm(A*s)
```

# Computation of $\Gamma(s) = \int_0^s e^{A\tau} d\tau$

- If $det(A) \neq 0$, then $\Gamma(s) = A^{-1}(e^{As} - I)$
  Proof:
  $$\int_0^s e^{A\tau} d\tau = \int_0^s (I + A\tau + \frac{(A\tau)^2}{2!} + ..)d\tau = (sI + \frac{As^2}{2!} + \frac{A^2 s^3}{3!} + ..)$$
  Then,
  $$A \int_0^s e^{A\tau} d\tau = (e^{As} - I)$$
- If $\det(A) = 0$, other methods exist
  In MatLab, compute $\Gamma(0.5)$ as

```
s=0.5
EXPO=(@(X)(expm(A*X))
Gamma=integral(EXPO,0,0,s,'ArrayValued',time)
```

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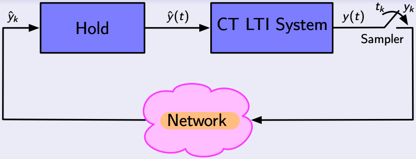
# Summary of the model (1/2)

## NCS with collocated control

LTI system

$$\dot{x} = Ax + B\hat{y}$$
$$y = Cx$$

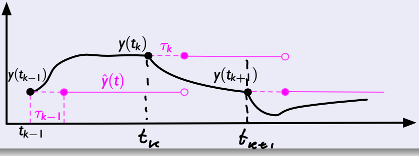



- $T_k = t_{k+1} - t_k$

Network model with delays: $y_k$ arrives at $t_k + \tau_k$ ($\tau_k < T_k$ by assumption)

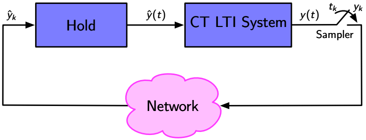
$$\hat{y}_k = y_k, \; k \in \mathbb{N}$$

$$\hat{y}(t) = \begin{cases} \hat{y}_{k-1} & t \in [t_k, t_k + \tau_k) \\ \hat{y}_k & t \in [t_k + \tau_k, t_{k+1}) \end{cases}$$




**Augmented state:** $z_k \triangleq \begin{bmatrix} x(t_k)^T & \hat{y}_{k-1}^T \end{bmatrix}^T$

# Summary of the model 2/2




## The NCS DT system

$$z_{k+1} = \Psi(T_k, \tau_k) z_k$$

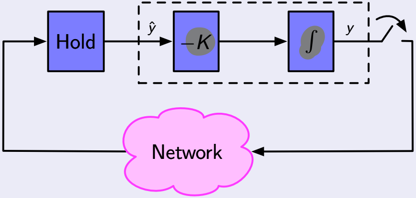
$$\Psi(T_k, \tau_k) = \begin{bmatrix} e^{AT_k} + \Gamma(T_k - \tau_k)BC & e^{A(T_k - \tau_k)}\Gamma(\tau_k)B \\ C & 0 \end{bmatrix}$$

- $\Gamma(s) = \int_0^s e^{A\tau} d\tau$
- The NCS is an LTV system
- Nonlinear ans nontrivial dependence of $\Psi$ on sampling intervals $T_k$ and delays $\tau_k$
- **Effect of $T_k$ and $\tau_k$ on stability?**

# Examples: effect of sampling and delays on NCS stability

# Effect of sampling and delay on stability (1/3)

## Example: controlled integrator




LTI system in the box

by assumption

$$\begin{cases} \dot{x} = 0 \cdot x - K\hat{y} \\ y = x \end{cases} \qquad A = 0,\ B = -K < 0 \qquad C = 1$$

$$z_{k+1} = \Psi(T_k, \tau_k) z_k$$

$$\Psi(T_k, \tau_k) = \begin{bmatrix} e^{AT_k} + \Gamma(T_k - \tau_k)BC & e^{A(T_k - \tau_k)}\Gamma(\tau_k)B \\ C & 0 \end{bmatrix}$$

Assumptions: $T_k = T$, $\tau_k = \tau$, $k = 0, 1, \cdots$

One has: $e^{AT} = 1$, $\Gamma(s) = \int_0^s \underbrace{e^{A\tau}}_{1} d\tau = s$

$$\Psi(T, \tau) = \begin{bmatrix} 1 - K(T - \tau) & -K\tau \\ 1 & 0 \end{bmatrix}$$

The NCS is a DT LTI system. Check stability using the eigenvalues of $\Psi$

$$\chi(\lambda) = \det(\lambda I - \Psi(T, \tau)) = \det\left( \begin{bmatrix} \lambda - 1 + K(T - \tau) & K\tau \\ -1 & \lambda \end{bmatrix} \right)$$

$$= \lambda^2 - \lambda(1 - K(T - \tau)) + K\tau$$

# Effect of sampling and delay on stability (2/3)

- Recall: Jury's criterion for $\chi(\lambda) = \lambda^2 + \alpha\lambda + \beta$

  All roots of $\chi(\lambda)$ have modulus $< 1 \Leftrightarrow \begin{cases} \beta > -\alpha - 1 \\ \beta > \alpha - 1 \\ \beta < 1 \end{cases}$

For $\chi(\lambda) = \lambda^2 - \lambda(1 - K(T - \tau)) + K\tau$ we have the conditions

$$+K\tau > (1 - K(T-\tau)) - 1 \quad \rightarrow \quad K\tau > -K(T-\tau) \tag{1}$$

$$+K\tau > -(1 - K(T-\tau)) - 1 \quad \rightarrow \quad K\tau > -2 + K(T-\tau) \tag{2}$$

$$K\tau < 1 \quad \xrightarrow{K>0} \quad \tau < \frac{1}{K} \tag{3}$$

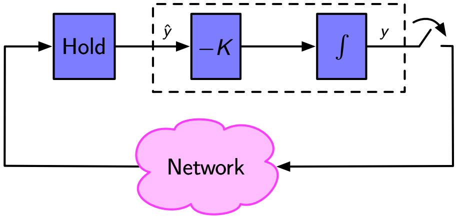
Since $K > 0$

(1) $\rightarrow \tau > -T + \tau \rightarrow 0 > -T$ always OK

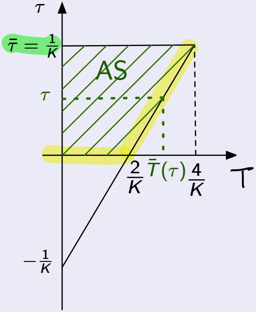
(2) $\rightarrow \tau > -\frac{2}{K} + T - \tau \rightarrow \tau > -\frac{1}{K} + \frac{T}{2}$

**Conclusion:**
From (2), (3), and $\tau > 0$, the NCS is AS iff $\max\left(0, -\frac{1}{K} + \frac{T}{2}\right) < \tau < \frac{1}{K}$

# Effect of sampling and delay on stability (3/3)




## Region of asymptotic stability in the $(T, \tau)$-plane

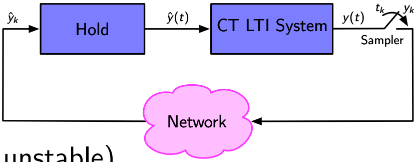



- Maximal tolerable delay: $\bar{\tau} = 1 \backslash K$
  - Aggressive controller ($K$ big) implies small $\bar{\tau}$
- For a given $\tau \in [0, \bar{\tau}]$, there is a Maximum Allowable Transfer Interval (MATI), i.e. a maximal $\bar{T}(\tau)$

# Other examples (constant $T_k$ and $\tau_k$): first-order system

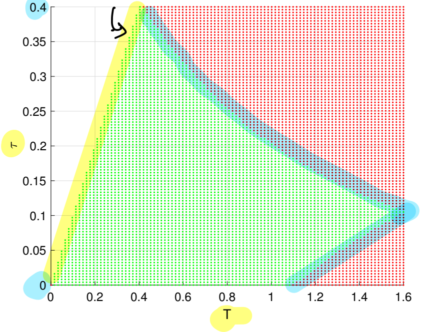
LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$




- $A = 1, B = -2, C = 1$ (open-loop unstable)

$T > \tau$




• = AS $\Psi(T,\tau)$

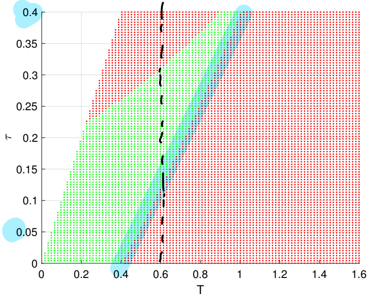
• = not AS $\Psi(T,\tau)$

- Green=$\Psi(T, \tau)$ is Schur. Non-obvious shape of the region ...
  - still for $\tau \in [0, 0.4]$, there is a MATI

# Other examples (constant $T_k$ and $\tau_k$): first-order system

- $A = -1, B = -5, C = 1$ (open-loop stable)

$\dot{x} = -x - 5\hat{y}$
$\hat{y} = x$




- There is a MATI but also a lower bound for $T$ for high enough $\tau$.

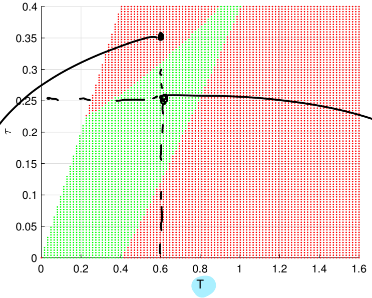
**Remark**

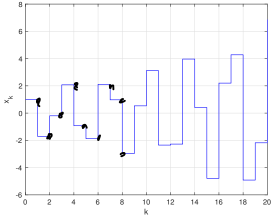
For $T = 0.6$, a delay large enough (but not too much) is stabilizing $\rightarrow$ not obvious


E. Fridman, "Introduction to time-delay and sampled-data systems," 2014 European Control Conference (ECC), Strasbourg, 2014, pp. 1428-1433.

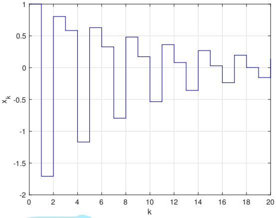
# Simulations

- $A = -1, B = -5, C = 1$ (open-loop stable)




$T = 0.6, \tau = 0.35$ : unstable

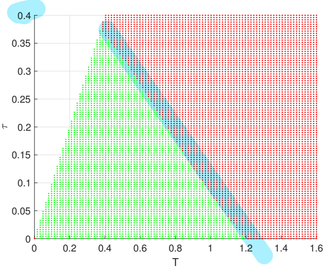



$T = 0.6, \tau = 0.25$ : stable

# Other examples : second-order system

$$A = \begin{bmatrix} -1 & 0.9 \\ 0.2 & -0.2 \end{bmatrix} \quad B = \begin{bmatrix} -5 \\ -0.5 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.1 & 1 \end{bmatrix} \quad \text{spec}(A) = \{-1.183, -0.017\}$$




## Conclusions

The analysis of pairs $(T, \tau)$ guaranteeing asymptotic stability is not trivial

- In all cases, there is a MATI
- formal methods needed: see next!

# Estimation of the MATI

# The MATI

## Definition

For a given $\tau_{min}, \tau_{max} \in \mathbb{R}$, the MATI is the largest $T \in \mathbb{R}$ such that

$$T > T_k,\ k = 0, 1, \ldots \Rightarrow \text{the NCS is AS for all } \tau_k \in [\tau_{min}, \tau_{max}]$$

## Remarks

Knowledge of MATI allows one to set the sampler for

- preserving stability
- avoiding small $T_k$, which increases the network load

# MATI estimation - constant $T, \tau$

**Assumption 1** : $\tau_k = \tau,\ T_k = T,\ \forall k \geq 0$ and $\tau < T$

- Realistic for:
  - Controlled Area Network (CAN) protocol (the maximal $\tau_k$ is constant for high-priority messages) and token-ring bus
  - protocols where $\tau_k$ are equalized using a buffer at the receiver $\rightarrow$ careful: all messages will appear to have the worst-case delay

Under Assumption 1, one can compute a stability region in the plane $(T, \tau)$, as done in previous examples.

# MATI estimation - variable $T_k, \tau_k$

Sufficient stability condition using the candidate Lyapunov function $V(z) = z^T P z$ for the NCS dynamics

$$z_{k+1} = \Psi(T_k, \tau_k) z_k$$

**Theorem:** Assume that $\forall k \in \mathbb{N}$

$$T_k \in [T_{min}, T_{max}] \text{ and } \tau_k \in [\tau_{min}, \tau_{max}] \tag{4}$$

where $T_{min} > \tau_{max}$. The NCS is exponentially stable if $\exists P = P^T > 0$ and $\nu > 0$ such that

$$\Psi(T,\tau)^T P \Psi(T,\tau) - P \leq -\nu I \quad \forall T \in [T_{min}, T_{max}] \; \forall \tau \in [\tau_{min}, \tau_{max}] \tag{5}$$

- Checking if there is $P$ such that (5) holds amounts to solving a set of LMIs, after gridding the box $[T_{min}, T_{max}] \times [\tau_{min}, \tau_{max}]$
- $V(z)$ is a *common* Lyapunov function for all $(T, \tau)$ in the box $\rightarrow$ **sufficient** condition only.

# MATI estimation - variable $T_k, \tau_k$

Sufficient stability condition using the candidate Lyapunov function $V(z) = z^T Pz$ for the NCS dynamics

$$z_{k+1} = \Psi(T_k, \tau_k) z_k$$

**Theorem:** Assume that $\forall k \in \mathbb{N}$

$$T_k \in [T_{min}, T_{max}] \text{ and } \tau_k \in [\tau_{min}, \tau_{max}] \tag{4}$$

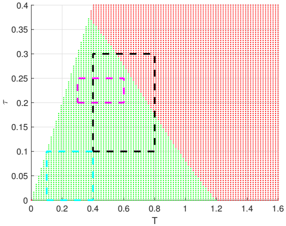
where $T_{min} > \tau_{max}$.The NCS is exponentially stable if $\exists P = P^T > 0$ and $\nu > 0$ such that

$$\Psi(T, \tau)^T P \Psi(T, \tau) - P \leq -\nu I \quad \forall T \in [T_{min}, T_{max}] \ \forall \tau \in [\tau_{min}, \tau_{max}] \tag{5}$$

- For estimating the MATI
  - $\tau_{min}, \tau_{max}, T_{min}$ are given by the network technology
  - Reduce $T_{max}$ until (5) is feasible $\rightarrow$ (conservative) estimate of MATI

# Example: second-order system (ctd.)

Green/red regions=stable/unstable NCS for constant $T$ and $\tau$



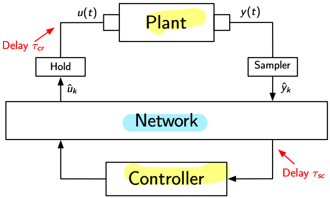

## Conservativity of the theorem

Through numerical computations one finds that the LMIs are:

- unfeasible for $(T, \tau)$ in the black region (expected as $T_k = T, \tau_k = \tau$, and $(T, \tau) \in$(red region) causes instability)
- unfeasible for $(T, \tau)$ in the magenta region
  - LMIs are just sufficient for stability
  - stability condition for variable $T_k, \tau_k$ are expected to be more restrictive than those for constant $T_k$ and $\tau_k$
- feasible in the cyan region

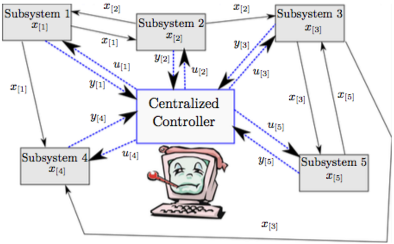
## Compensation of network-induced delay (remote control setting)

# NCS - remote control setting




- Plant and controller on different sides of the network

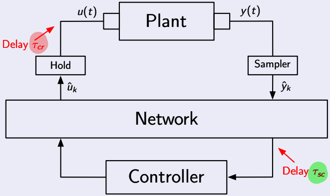
## Centralized control




## Remote control by human

# Compensation for network-induced delay

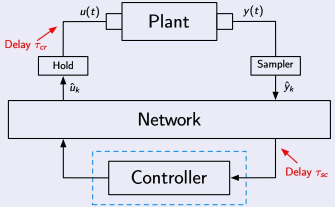
## NCS - remote control setting




$$\text{Plant}: \begin{cases} \dot{x} = Ax + Bu \\ y = x \end{cases}$$

- The controller gets delayed (full-state) measurements and the plant gets delayed control actions.
- $\tau_{sc}$ : sensor-to-controller delay. Using time-stamped measurements the controller CAN KNOW $\tau_{sc}$ at the time of computation of $\hat{u}_k$. Idea: compensate for it!
- $\tau_{cr}$ : controller-to-actuator delay. Unknown to the controller at the time of computation of $\hat{u}_k$

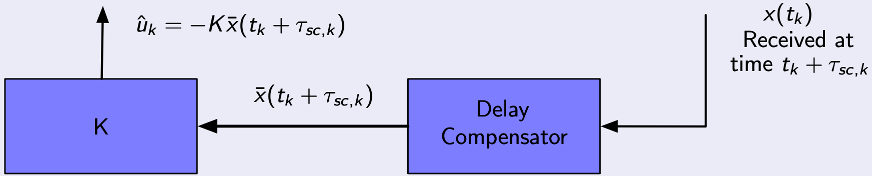
# Timing diagram and structure of the controller



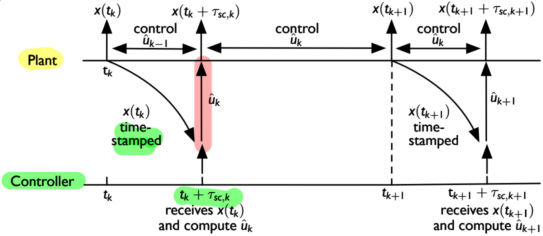

$$\text{Plant}: \begin{cases} \dot{x} = Ax + Bu \\ y = x \end{cases}$$

↳ the whole state is measured

## Zoom of the controller



# Timing diagram and structure of the controller



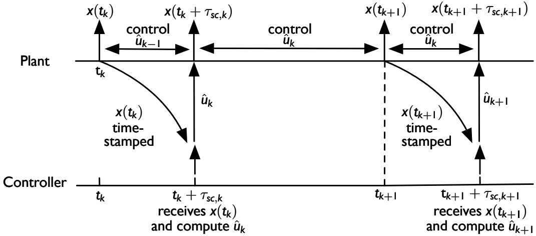

## Assumptions (for simplicity)

- No controller-to-actuator delay: $\tau_{cr,k}=0$
- Constant sampling period $T$ and $\tau_{sc,k}<T$, $k=0,1,\dots$

## How the controller computes $\hat{u}_k$?

- Option 1 : $\hat{u}_k=-Kx(t_k)$, but the state is already $x(t_k+\tau_{sc,k})\rightarrow$ delay
- Option 2 : $\hat{u}_k=-K\bar{x}(t_k+\tau_{sc,k})$, where $\bar{x}(t_k+\tau_{sc,k})$ is an estimate of $x(t_k+\tau_{sc,k})\rightarrow$ much better

# Compensation of $\tau_{sc}$ and computation of $\hat{u}_k$

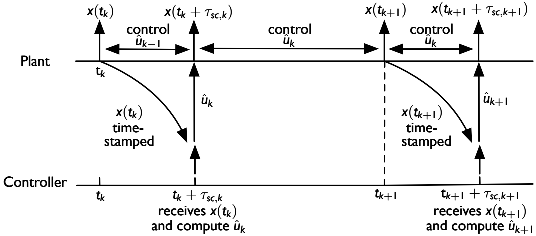



- Compute the prediction $\bar{x}(t_k+\tau_{sc,k})$ of $x(t_k+\tau_{sc,k})$
  - Setting $\Gamma(\bar{t}) = \int_0^{\bar{t}} e^{As} ds$, we define

$$\bar{x}(t_k+\tau_{sc,k}) = e^{A\tau_{sc,k}} x(t_k) + \Gamma(\tau_{sc,k}) B \hat{u}_{k-1}$$

  - $x_{t_k}$ is received at $t_k+\tau_{sc,k}$
  - $\hat{u}_{k-1}$ is the known constant input over $t \in [t_k, t_k+\tau_{sc,k}]$
  - $\bar{x}(t_k+\tau_{sc,k})$ coincides with the true state $x(t_k+\tau_{sc,k})$

# Compensation of $\tau_{sc}$ and computation of $\hat{u}_k$



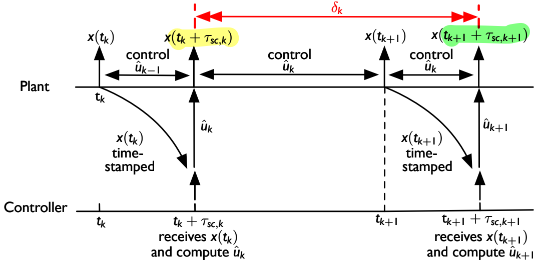

- Control law computed at $t_k + \tau_{sc,k}$

$$\hat{u}_k = -K\bar{x}(t_k + \tau_{sc,k})$$

Remark: We use the notation $\hat{u}_k$ even if the input is not computed at $t_k$ (but it is the $k^{th}$ computed value of $\hat{u}$ )

Hold $u(t) = \hat{u}_k$ for $t \in [t_k + \tau_{sc,k}, t_{k+1} + \tau_{sc,k+1}]$

# Closed-loop system from $t_k + \tau_{sc,k}$ to $t_{k+1} + \tau_{sc,k+1}$

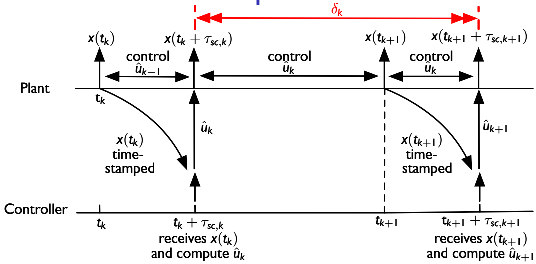



For $\delta_k = T + \tau_{sc,k+1} - \tau_{sc,k}$, one obtains

$$x(t_{k+1} + \tau_{sc,k+1}) = \underbrace{\left(e^{A\delta_k} - \Gamma(\delta_k)BK\right)}_{\tilde{A}_k} x(t_k + \tau_{sc,k})$$

- Defining the state $\tilde{x}_k = x(t_k + \tau_{sc,k})$, one has the LTV system $\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k$
- If $\tau_{sc,k}$ is constant, $\delta_k = T$. Hence, $\tilde{A}_k = \tilde{A} = (e^{AT} - \Gamma(T)BK)$ and the NCS dynamics are LTI

# Nominal controller with compensation




## Nominal design of K

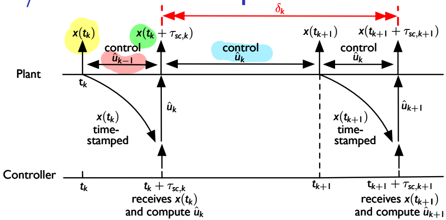
Assume $\tau_{sc,k} = \tau_{sc}$ (constant) and compute $K$ such that $e^{AT} - \Gamma(T)BK$ is Schur

- use, e.g., eigenvalue assignment
- the exact value of $\tau_{sc}$ is irrelevant. Closed-loop stability is guaranteed if delays are constant in the real network

## Summary: nominal controller with compensation

$$\bar{x}(t_k + \tau_{sc,k}) = e^{A\tau_{sc,k}} x(t_k) + \Gamma(\tau_{sc,k}) B \hat{u}_{k-1}$$
$$\hat{u}_k = -K \bar{x}(t_k + \tau_{sc,k})$$

# Comparison with/without compensation



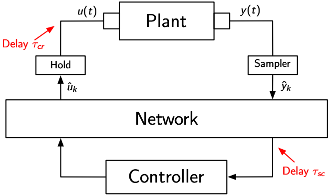

## Comparison with the uncompensated controller: $\hat{u}_k = -Kx(t_k)$

- Compensated controller:
  - "wrong" (=old) control action on $[t_k, t_k + \tau_{sc,k}]$
  - best possible control action on $[t_k + \tau_{sc,k}, t_{k+1}]$
- Uncompensated controller:
  - "wrong" (=old) control action on $[t_k, t_k + \tau_{sc,k}]$
  - "wrong" (=based on past state) control action on $[t_k + \tau_{sc,k}, t_{k+1}]$

## Performance and stability

- The compensated controller always outperforms the uncompensated one
- If, in the real network, $\tau_{sc,k}$ is not constant, closed-loop stability is not guaranteed using either controller

# Example: delay compensation vs no compensation (1/2)




Plant dynamics

$$\dot{x} = -2x + u, \quad T = 0.2$$

## Controller with compensation

τ constant

- Nominal design of K

  $$e^{-2T} - \Gamma(T) \cdot 1 \cdot K = \boxed{-0.9} \rightarrow \text{desired closed-loop eigenvalue}$$

  $$0.6703 - 0.1648 \cdot K = -0.9 \rightarrow K = 9.5263$$

- Predictor

  $$\bar{x}(t_k + \tau_{sc,k}) = e^{-2T}x(t_k) + \Gamma(\tau_{sc,k})B\hat{u}_{k-1}$$

- Control action

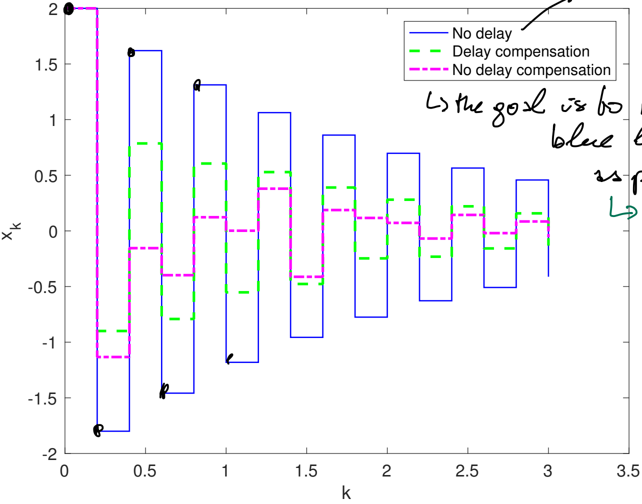
  $$\hat{u}_k = -K\bar{x}(t_k + \tau_{sc,k})$$

- Controller without compensation $\hat{u}_k = -Kx(t_k)$

# Example: delay compensation vs no compensation (2/2)

Simulation with delays chosen randomly in $[0, 0.07]$

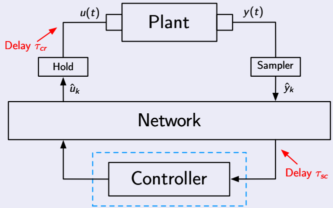
ideal behavior




↳ the goal is to mimic the blue line as close as possible

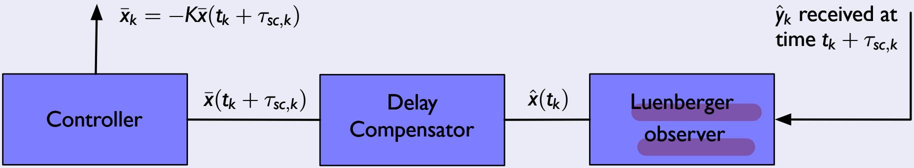
↳ green wins

# Generalization: NCS with output feedback




$$\text{Plant}: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

## Zoom of the controller




- $\bar{x}(t_k + \tau_{sc,k})$ is an estimate of the state $x(t_k + \tau_{sc,k})$
- No further details on this scheme

# Take-home messages

- The effect of time-varying sampling intervals and delays can be VERY difficult to analyze
  - Estimate MATI using simulations or Lyapunov theory
- For remote control and time-stamped sensor measurements, it is always beneficial to compensate for known delays

2h Lh