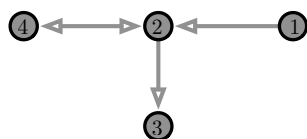


# Networked Control Systems (ME-427)- Exercise session 14

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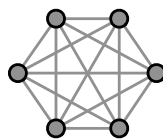
1. **Laplacian average consensus in directed networks.** [Textbook E7.4] Consider the directed network in figure below with arbitrary positive weights and its associated Laplacian flow  $\dot{x}(t) = -Lx(t)$ .



- (a) Can the network reach consensus, that is, as  $t \rightarrow \infty$  does  $x(t)$  converge to a limiting point in  $\text{span}\{\mathbb{1}_n\}$ ?
- (b) Does  $x(t)$  achieve average consensus, that is,  $\lim_{t \rightarrow \infty} x(t) = \text{average}(x(0))\mathbb{1}_n$ ?
- (c) Will your answer to point (b) change if you smartly add one directed edge and adapt the weights?

**Solution:** The following answer is based on the consensus theorems for digraphs with globally reachable node seen in the lectures.

- (a) Node 3 is globally reachable, hence the network will reach consensus.
  - (b) However, node 1 is not globally reachable. Hence, if we let  $w$  denote the left dominant eigenvector of  $L$ , then we know  $w_1 = 0$  and, therefore, it is not possible to reach average consensus.
  - (c) By adding a link from node 3 to node 1 the network becomes strongly connected and by choosing the weights appropriately (so that the associated adjacency matrix is doubly-stochastic) it is possible to reach average consensus.
2. **The adjacency and Laplacian matrices for the complete graph.** [Textbook E6.2] For any number  $n \in \mathbb{N}$ , the *complete graph* with  $n$  nodes, denoted by  $K(n)$ , is the undirected and unweighted graph in which any two distinct nodes are connected. For example, see  $K(6)$  in figure.



Compute, for arbitrary  $n$ ,

- (a) the adjacency matrix of  $K(n)$  and its eigenvalues; and
- (b) the Laplacian matrix of  $K(n)$  and its eigenvalues.

**Hint.** Note that if  $\lambda \in \text{Spec}(A)$  then  $\lambda + \alpha \in \text{Spec}(A + \alpha I)$ . Indeed, if  $v \neq 0$  verifies  $Av = \lambda v$ , then ...

**Solution:** The adjacency, degree and Laplacian matrices are, respectively,

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \ddots & \ddots & 1 \\ 1 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & 1 & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix} = \mathbb{1}_n \mathbb{1}_n^T - I_n, \quad D = (n-1)I_n,$$

$$L = (n-1)I_n - (\mathbb{1}_n \mathbb{1}_n^T - I_n) = nI_n - \mathbb{1}_n \mathbb{1}_n^T$$

To compute the spectra of  $Q$  and  $L$ , note the following two facts:

- the spectrum of  $\mathbb{1}_n \mathbb{1}_n^T$  is  $\{n, 0, \dots, 0\}$ ,
- if  $\lambda$  is an eigenvalue of  $\mathbb{1}_n \mathbb{1}_n^T$ , then  $\lambda - 1$  is an eigenvalue of  $A = \mathbb{1}_n \mathbb{1}_n^T - I_n$  and  $n - \lambda$  is an eigenvalue of  $L = nI_n - \mathbb{1}_n \mathbb{1}_n^T$ .

In summary, the spectrum of  $A$  is  $\{n-1, -1, \dots, -1\}$  and the spectrum of  $L$  is  $\{0, n, \dots, n\}$ .

3. **Linear spring networks with loads.** [Textbook E6.15] Consider the two (connected) spring networks with  $n$  moving masses in figure. For the right network, assume one of the masses is connected with a single stationary object with a spring. Refer to the left spring network as *free* and to the right network as *grounded*. Let  $F_{\text{load}}$  be a load force applied to the  $n$  moving masses.



For the left network, let  $L_{\text{free},n}$  be the  $n \times n$  Laplacian matrix describing the free spring network among the  $n$  moving masses, as defined the lectures. Let  $L_{\text{grounded}}$  be the  $n \times n$  *grounded Laplacian* of the  $n$  masses taking into account the elastic force of the stationary object.

For the free spring network subject to  $F_{\text{load}} = [F_{\text{load},1}, \dots, F_{\text{load},n}]$ ,

- Show that, for any  $x \in \mathbb{R}^n$ , the vector  $v = L_{\text{free},n}x$  is balanced, that is  $\mathbb{1}_n^T v = 0$ .
- Do equilibrium displacements exist for arbitrary loads? If the load force  $F_{\text{load}}$  is balanced, that is  $\mathbb{1}_n^T F_{\text{load}} = 0$ , is the resulting equilibrium displacement unique?  
**Hint:** Start by writing the system dynamics as  $M\ddot{x} = \dots$  where  $M = \text{diag}(M_1, \dots, M_n)$  collects the value of the masses and  $x = [x_1, \dots, x_n]^T$  the position of the masses.

For the grounded spring network,

- derive an expression relating  $L_{\text{grounded}}$  to  $L_{\text{free},n}$ ,
- one can show that  $L_{\text{grounded}}$  is invertible. Using this fact, compute the equilibrium displacement for the grounded spring network for arbitrary and constant load forces.

**Solution:** The displacement dynamics for the spring-mass system are given by:

$$M\ddot{x} + Lx = F_{\text{load}}$$

- For the free spring network  $L = L_{\text{free},n}$ . The vector  $L_{\text{free},n}x$  is a linear combination of the columns of  $L_{\text{free},n}$  (through the scalars  $x_1, \dots, x_n$ ). Since each column of  $L_{\text{free},n}$  is balanced the result easily follows.
- For equilibrium, we need  $\ddot{x}_i = 0$ . Hence, no equilibrium exists unless  $F_{\text{load}}$  is balanced (see the previous point). If  $F_{\text{load}}$  is balanced, there exists an equilibrium but it is not unique. The reason is that if  $\bar{x}$  is an equilibrium, then  $\bar{x} + \alpha \mathbb{1}_n$  is an equilibrium as well, for any  $\alpha \in \mathbb{R}$ . Indeed  $L_{\text{free},n}(\bar{x} + \alpha \mathbb{1}_n) = L_{\text{free},n}\bar{x} + \alpha L_{\text{free},n}\mathbb{1}_n = 0$ , where the last equation follows from  $L_{\text{free},n}\mathbb{1}_n = 0$ , which is a general property of Laplacian matrices.

(c) With the stationary object, mass  $M_1$  obeys to

$$M_1 \ddot{x}_i + k_{12}(x_1 - x_2) + k_{10}(x_1 - 0) = F_1$$

where  $k_{10} > 0$  is the stiffness of the spring connecting mass  $M_1$  to the stationary object. The first line of  $L_{\text{free},n}$  is then replaced by

$$[+k_{10} + k_{12}, -k_{12}, 0, \dots, 0].$$

Then  $L_{\text{grounded}} = L_{\text{free},n} + \text{diag}(k_{10}, 0, \dots, 0)$ .

(d) The equilibrium displacement for the grounded spring network is given by:  $x = L_{\text{grounded}}^{-1} F_{\text{load}}$ . (Using invertibility of grounded  $L_{\text{grounded}}$ ).

4. **Euler discretization of the Laplacian.** [Textbook E7.6] Given a weighted digraph  $G$  with Laplacian matrix  $L$  and maximum out-degree  $d_{\text{max}} = \max\{d_{\text{out}}(1), \dots, d_{\text{out}}(n)\}$ . Show that:

- (a) if  $\epsilon < 1/d_{\text{max}}$ , then the matrix  $I_n - \epsilon L$  is row-stochastic,
- (b) if  $\epsilon < 1/d_{\text{max}}$  and  $G$  is weight-balanced, then the matrix  $I_n - \epsilon L$  is doubly-stochastic, and
- (c) if  $\epsilon < 1/d_{\text{max}}$  and  $G$  is strongly connected, then  $I_n - \epsilon L$  is primitive (**Hint:**  $I_n - \epsilon L$  can be interpreted as the adjacency matrix of a new weighted digraph ...)

Given these results, we highlight that  $I_n - \epsilon L$  is the one-step Euler discretization (with sampling time  $\epsilon > 0$ ) of the continuous-time Laplacian flow and is a discrete-time consensus algorithm.

**Solution:** Regarding (a), note that  $L\mathbb{1}_n = \mathbb{0}_n$  implies  $(I_n - \epsilon L)\mathbb{1}_n = \mathbb{1}_n$ . Therefore, it suffices to show that  $(I_n - \epsilon L)$  is non-negative. By the definition of Laplacian matrix, all off-diagonal elements are non-negative. Moreover, because  $\epsilon < 1/d_{\text{max}}$ , the  $i$ th diagonal element satisfies  $1 - \epsilon L_{ii} \geq 1 - \epsilon d_{\text{out}}(i) > 1 - d_{\text{out}}(i)/d_{\text{max}} \geq 0$ , for each  $i \in \{1, \dots, n\}$ . Therefore, the diagonal elements are positive and the matrix  $I_n - \epsilon L$  is row stochastic.

Regarding (b), we have seen in the lectures that  $G$  is weight balanced if and only if  $\mathbb{1}_n^T L = \mathbb{0}_n^T$ . It is easy to derive the equality:  $(I_n - \epsilon L)^T \mathbb{1}_n = \mathbb{1}_n - \epsilon L^T \mathbb{1}_n = \mathbb{1}_n$ . This equality and the statement (a) together imply that  $I_n - \epsilon L$  is doubly-stochastic.

Regarding (c), let  $\tilde{A} = (I_n - \epsilon L)$ , for  $\epsilon < 1/d_{\text{max}}$ . Since  $\tilde{A}$  is non-negative and row stochastic, it can be interpreted as an adjacency matrix of a graph with associated weighted out-degree matrix equal to  $I_n$ . In particular, the Laplacian matrix associated with  $\tilde{A}$  is given by  $\tilde{L} = D_{\text{out}}(\tilde{A}) - \tilde{A} = \epsilon L$ . Since, for  $i \neq j$ ,  $(\tilde{L})_{i,j} = 0 \Leftrightarrow (L)_{i,j} = 0$ , the digraphs associated to  $L$  and  $\tilde{L}$  have the same edges (up to self-loops). Then, strong connectivity of  $G$  implies strong connectivity of  $\tilde{G}$  (which is also the digraph induced by  $\tilde{A}$ , up to self loops). Moreover, since each diagonal element of  $\tilde{A}$  is strictly positive (this has been shown in the proof of point (a)), the associated digraph has a self loop at each node and is therefore aperiodic. We have seen in the lectures that these conditions imply that  $\tilde{A}$  is primitive.