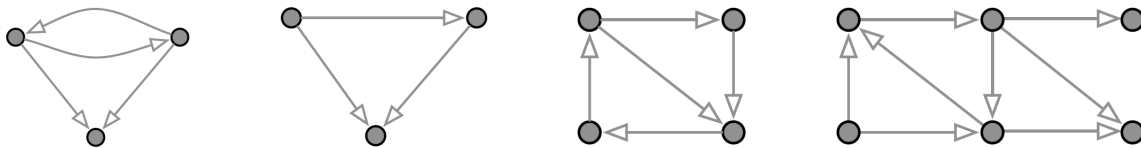


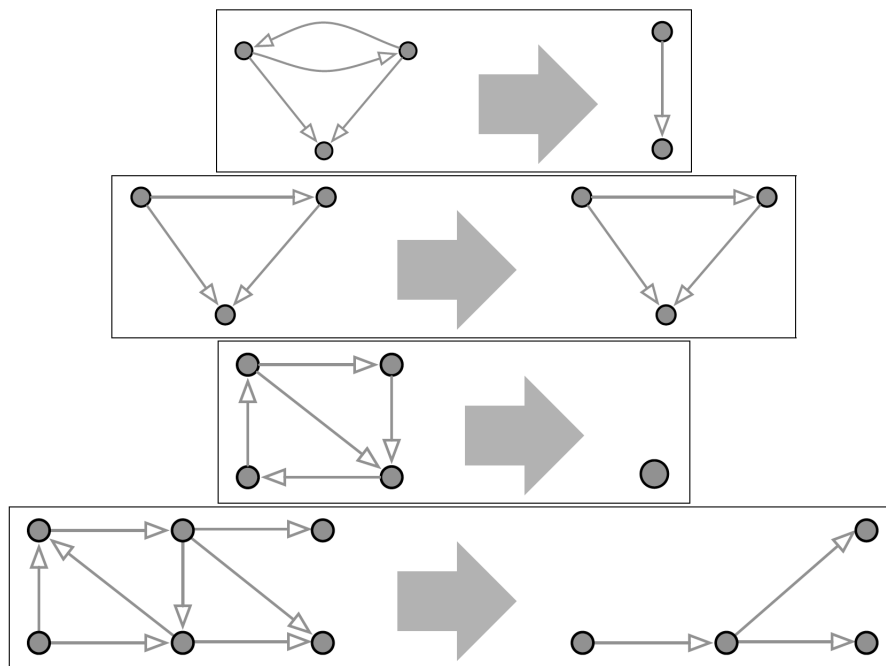
Networked Control Systems (ME-427)- Exercise session 11

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1. **Condensation digraphs.** [Textbook E3.2] Draw the condensation for each of the following digraphs.



Solution:



2. **Directed spanning trees in the condensation digraph.** [Textbook E3.3] For a digraph G and its condensation digraph $C(G)$, show that the following statements are equivalent:

- (a) G contains a directed spanning tree, and
- (b) $C(G)$ contains a directed spanning tree.

Hint: Reason on G_{rev} .

Solution: A digraph G contains a directed spanning tree if and only if the reverse digraph G_{rev} contains a globally reachable node. Thus, the equivalence between statement (a) and (b) follows from the results about condensation digraphs seen in the lectures and applied to G_{rev} and $C(G_{rev})$.

More specifically, G_{rev} contains a globally reachable node if and only if $C(G_{rev})$ contains a globally reachable node.

3. **Agents with self-confidence levels.** [Textbook E5.6] Consider 2 agents, labeled $+1$ and -1 , described by the self-confidence levels s_{+1} and s_{-1} . Assume $s_{+1} > 0$, $s_{-1} > 0$, and $s_{+1} + s_{-1} = 1$. For $i \in \{+1, -1\}$, define

$$x_i^+ := s_i x_i + (1 - s_i) x_{-i}.$$

Perform the following tasks:

- compute the matrix A representing this algorithm and verify it is row-stochastic,
- compute A^2 ,
- compute the eigenvalues, the right eigenvectors, and the left eigenvectors of A ,
- compute the final value of this algorithm as a function of the initial values and of the self-confidence levels. Is it true that an agent with higher self-confidence makes a larger contribution to the final value?

Solution: Regarding question (a), we compute

$$A = \begin{bmatrix} s_{+1} & (1 - s_{+1}) \\ (1 - s_{-1}) & s_{-1} \end{bmatrix} = \begin{bmatrix} s_{+1} & s_{-1} \\ s_{+1} & s_{-1} \end{bmatrix},$$

and verify that the row-sums are equal to 1.

Regarding question (ii), we compute

$$A^2 = \begin{bmatrix} s_{+1} & s_{-1} \\ s_{+1} & s_{-1} \end{bmatrix} \begin{bmatrix} s_{+1} & s_{-1} \\ s_{+1} & s_{-1} \end{bmatrix} = \begin{bmatrix} s_{+1}(s_{+1} + s_{-1}) & s_{-1}(s_{+1} + s_{-1}) \\ s_{+1}(s_{+1} + s_{-1}) & s_{-1}(s_{+1} + s_{-1}) \end{bmatrix} = A.$$

Regarding question (c), the eigenvalues are $\{+1, 0\}$ because A is row-stochastic and has rank 1. The corresponding right eigenvectors are $[1, 1]^T$ and $[s_{-1}, -s_{+1}]^T$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad A \begin{bmatrix} s_{-1} \\ -s_{+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The corresponding left eigenvectors are $[s_{+1}, s_{-1}]^T$ and $[1, -1]^T$, because

$$\begin{bmatrix} s_{+1} & s_{-1} \end{bmatrix} A = \begin{bmatrix} s_{+1} & s_{-1} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & -1 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Regarding question (d), if the initial values are $x_{+1}(0)$ and $x_{-1}(0)$, then it is easy to see

$$x_{\text{final}} = s_{+1} x_{+1}(0) + s_{-1} x_{-1}(0).$$

Clearly, if $s_{+1} > s_{-1}$, then agent $+1$ will have a greater impact on the final value than agent -1 .

4. **Two social influence networks.** [Textbook E5.2] Similarly to the DeGroot model introduced in the lectures, we consider n individuals with an initial opinion $x_i(0) \in \mathbb{R}$, $i = 1, \dots, n$. $x_i(k)$ is the updated opinion of individual i after k communications with its neighbors. Then, the vector of opinions evolves over time according to $x(k+1) = Ax(k)$ where the coefficient $a_{ij} \in [0, 1]$ is the influence of the opinion of individual j on the update of the opinion of agent i , subject to the constraint $\sum_j a_{ij} = 1$. Consider the following two scenarios:

- Everybody gives the same weight to the opinion of everybody else.
- There is a distinct agent (suppose the agent with index $i = 1$) that weights equally the opinion of all the others, and the remaining agents compute the mean between their opinion and the one of first agent.

In each case, derive the averaging matrix A , show that the opinions converge asymptotically to a final opinion vector, and characterize this final opinion vector.

Solution:

(a) The matrix A in the first case is given by

$$A = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

Clearly A is primitive since $A > 0$ is positive. Moreover A is doubly stochastic. By the results seen in the lectures, $\rho(A) = 1$ is a simple eigenvalue. By direct computation, the normalized right and left eigenvectors are $v = \mathbf{1}_n$ and $w = \frac{1}{n}\mathbf{1}_n$. Therefore,

$$x(\infty) = (w^T x(0))v = \text{average}(x(0))\mathbf{1}_n.$$

This means that all the agents asymptotically reach the same opinion which is the average of their initial opinions. In this particular case, it can be verified that $x(1) = x(2) = \dots = x(\infty)$, that is, the limiting behavior is actually reached after a single step.

(b) The matrix A in the second case is given by

$$A = \begin{bmatrix} \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & & \ddots & \\ \frac{1}{2} & 0 & \cdots & \frac{1}{2} \end{bmatrix}.$$

Observe that the associated digraph is strongly connected and aperiodic. According to the results seen in the lectures, A is primitive. Clearly, A is row-stochastic and has a self-loop attached to every node. By the consensus theorem seen in the lectures, $\rho(A) = 1$. Moreover, by direct calculation, the right dominant eigenvector is $v = \mathbf{1}_n$. The left eigenvector satisfies

$$[w_1 \quad w_2 \quad \dots \quad w_n] \begin{bmatrix} \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & & \ddots & \\ \frac{1}{2} & 0 & \cdots & \frac{1}{2} \end{bmatrix} = [w_1 \quad w_2 \quad \dots \quad w_n].$$

We can immediately derive $w_i = \frac{2}{n}w_1$ for all $i \neq 1$. By imposing $\sum w_i = 1$ we get $w_1 = \frac{n}{3n-2}$ and $w_i = \frac{2}{3n-2}$ for all $i \neq 1$. As in the previous case, from the consensus theorem we get

$$x(\infty) = (w^T x(0))v = c\mathbf{1}_n,$$

where $c \sum_i w_i x(0)_i = \frac{n}{3n-2}x_1(0) + \frac{2}{3n-2} \sum_{i=2}^n x_i(0)$. This means that the agents reach agreement (they all have the same opinion at the end), however the opinion to which they converge is not the average of the initial opinions anymore. In this case, the opinion of the first agent is $n/2$ times more influential than the one of the others.