

# Lecture 6

State observers

Output-feedback controllers

Giancarlo Ferrari Trecate<sup>1</sup>

<sup>1</sup>Dependable Control and Decision Group  
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland  
giancarlo.ferraritrecate@epfl.ch

# State observers

Motivations: in several applications

- not all scalar states are accessible
- sensors are costly  $\rightarrow$  not convenient to measure all scalar states

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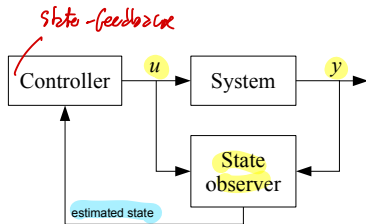
## Problems

Knowing just the output, but not the state, prevents from using state feedback controllers

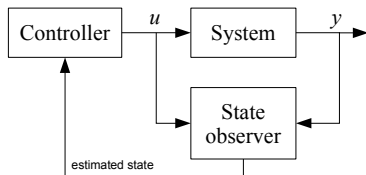
## Solutions

- Build an observer, that is a dynamical system with
  - ▶ **inputs:** the inputs and outputs of the system  $\Sigma$  under observation
  - ▶ **outputs:** the estimated state of  $\Sigma$
- Use the estimated state in the controller

# State observers



# State observers

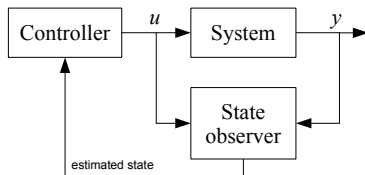


Terminology: state **observers** or state **estimators**

Challenges: stability of the closed-loop system? Performance?

Other uses: estimates of the internal state are also very useful for detecting malfunctioning and faults on components

# State observers



**Terminology:** state observers or state estimators

**Challenges:** stability of the closed-loop system? Performance?

**Other uses:** estimates of the internal state are also very useful for detecting malfunctioning and faults on components

## Outline of the lecture

- Full-order observers and filters
  - ▶ Output-feedback controllers: the separation principle
- Reduced-order observers
- Choice of the closed-loop eigenvalues

**Later:** Kalman filtering (stochastic framework)

# Full-order observer

$$\Sigma : \begin{cases} x^+ = Ax + Bu \\ y = Cx \quad \text{+ Du} \\ x(0) = x_0 \end{cases}$$

- **Full-order:** reconstruct the whole state  $x(k)$
- Define  $\hat{x}(k|k-1)$  the estimate available at time  $k$  of  $x(k)$  using data (inputs and outputs) known up to  $k-1$   
 $\hookrightarrow$  Notation for these sequences :  $u^{k-1}, y^{k-1}$


$$\begin{bmatrix} u^0 \\ u^1 \\ \vdots \\ u^{k-1} \end{bmatrix}$$

# Full-order observer

## Luenberger observer

$$\hat{\Sigma} = \begin{cases} \hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) - L[y(k) - C\hat{x}(k|k-1)] \\ \hat{y}(k) = C\hat{x}(k|k-1) \\ \hat{x}(0) = \hat{x}_0 \end{cases}$$

*measured*

# Full-order observer

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## Remarks

- $L \in \mathbb{R}^{n \times p}$ : **observer gain**. Multiplies the output estimation error
- Recursive update at  $\hat{x}$ : the observer is an **LTI** system
- The dynamic of  $\Sigma$  and  $\hat{\Sigma}$  are identical, up to the output-error term

# Full-order observer

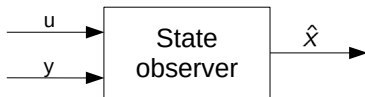
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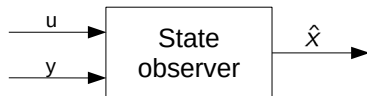
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- $L \in \mathbb{R}^{n \times p}$ : observer gain. Multiplies the output estimation error
- Recursive update at  $\hat{x}$ : the observer is an LTI system
- The dynamic of  $\Sigma$  and  $\hat{\Sigma}$  are identical, up to the output-error term
- If, at  $\bar{k}$ ,  $\hat{x}(\bar{k}|\bar{k}-1) = x(\bar{k})$ , then,  $\hat{x} = x, \forall k \geq \bar{k}$  (perfect reconstruction)
- $x_0$  is not known,  $\hat{x}_0$  is chosen using "common sense" and, unavoidably,  $\hat{x}_0 \neq x_0$

# Observer stability



# Observer stability



## Goal

Guarantee that the estimation error  $e(k|k-1) = x(k) - \hat{x}(k|k-1)$  goes to zero as  $k \rightarrow \infty$ .

↪ recovers from the mismatch  $x_0 \neq \hat{x}_0$ .

Key point for observers: analyse the error dynamics (not  $\hat{x}$ )!

# Observer stability

## System

$$x^+ = Ax + Bu$$

$$y = Cx$$

## Observer ( $\hat{x} = \hat{x}(k|k-1)$ )

$$\hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x})$$

$$\hat{y} = C\hat{x}$$

# Observer stability

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Error dynamics ( $e = e(k|k-1) \triangleq x(k) - \hat{x}(k|k-1)$ )

$$e^+ = Ae + \cancel{Bu} - \cancel{Bu} + L(Cx - C\hat{x}) = (A + LC)e$$

# Observer stability

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Error dynamics ( $e = e(k|k-1) = x(k) - \hat{x}(k|k-1)$ )

$$e^+ = Ae + \cancel{Bu} - \cancel{Bu} + L(Cx - C\hat{x}) = (A + LC)e$$

## Definition

The **observer** is asymptotically stable (AS) if the **error dynamics** has this property

- The **error dynamics is an autonomous system**
  - ▶ AS  $\Rightarrow \hat{x} \rightarrow x$  irrespectively of  $x_0$  and  $\hat{x}_0$

## Definition

The eigenvalues of  $A + LC$  are called the **eigenvalues of the observer**

2h 45  
↓ ✓

# Observer design

## Problem

Find  $L$  such that  $A + LC$  is Schur

- $\text{Spec}(A + LC) = \text{Spec}(A^T + C^T L^T)$

The problem is identical to the design of  $K$  such that  $(A + BK)$  has prescribed eigenvalues, up to the following replacements

$$A \rightarrow A^T$$

$$B \rightarrow C^T$$

$$K \rightarrow L^T$$

# Observer design

- For single-output systems,  $y \in \mathbb{R}$ , use Ackermann's formula: if

$$M_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T \quad \text{is full rank, set}$$

$$L^T = - [ 0 \ 0 \ \dots \ 0 \ 1 ] (M_0)^{-1} P^D (A^T)$$

where  $P^D$  is the desired characteristic polynomial of  $A + LC$

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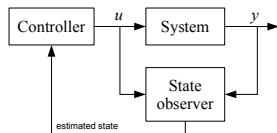
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- For multi-output systems all methods seen for MIMO control design can be applied to observer design

# Output-feedback controllers: the separation principle

# Observer & state feedback design: the separation principle



## System

$$x^+ = Ax + Bu$$

$$y = Cx$$

## Observer

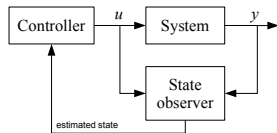
$$\hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x})$$

$$\text{Recall: } \hat{x} = \hat{x}(k|k-1)$$

## Controller

$$u = K\hat{x}$$

# Observer & state feedback design: the separation principle



## System

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## Controller

$$u = K\hat{x}$$

## Closed-loop system

$$\begin{bmatrix} x^+ \\ \hat{x}^+ \end{bmatrix} = \underbrace{\begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix}}_F \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

## Separation principle

$$\text{Spec}(F) = \text{Spec}(A + BK) \cup \text{Spec}(A + LC)$$

- If  $L$  stabilizes  $A + LC$  and  $K$  stabilizes  $A + BK$ , the closed-loop system is AS
- Each gain is designed independently of the other one

## Proof of the separation principle

Make the error  $e = x - \hat{x}$  appear through the change of variables

$$\begin{bmatrix} x \\ e \end{bmatrix} = T \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \quad T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \Rightarrow T = T^{-1}$$

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- Dynamics of  $\begin{bmatrix} x \\ e \end{bmatrix}$ :  $\begin{bmatrix} x^+ \\ e^+ \end{bmatrix} = \hat{F} \begin{bmatrix} x \\ e \end{bmatrix}$  where

$$\hat{F} = T \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} T^{-1} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix}$$

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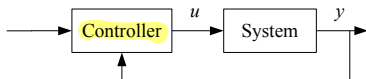
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- Block-diagonal structure  $\Rightarrow \text{Spec}(\hat{F}) = \text{Spec}(A + BK) \cup \text{Spec}(A + LC)$

# Output feedback controllers



State observer + state feedback provides a method for designing output feedback controllers

$$\begin{aligned}\hat{x}^+ &= A\hat{x} + Bu + L(y - C\hat{x}) \\ u &= K\hat{x}\end{aligned}$$

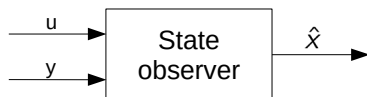
- The controller is a *dynamical system* of order  $n$

## Problem

Is it possible to *reduce the order of the controller*? Yes, by using reduced-order observers (see later).

# Full-order observers with no delay (filters)

# Observers with no delay



## Previously

Compute  $\hat{x}(k|k-1)$  using  $u^{k-1}$  (inputs up to  $k-1$ ) and  $y^{k-1}$  (outputs up to  $k-1$ )

## Next

Compute  $\hat{x}(k|k)$  using  $u^{k-1}$  and  $y^k$

- Controller  $u(k) = K\hat{x}(k|k)$

# Observers with no delay

## Pros

- $\hat{x}(k|k)$  is "better" than  $\hat{x}(k|k-1)$  as it uses more information.
  - ▶ Example: a disturbance at time  $k$  can be captured by  $y^k$  but not by  $y^{k-1}$

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## Cons

Timing of the computations (uniform sampling period  $T$ )



- Measure  $y(k) = y(kT)$  at time  $kT$
- $\hat{x}(k|k)$  is available at time  $kT + \epsilon$
- $u(k)$  is available at time  $kT + \epsilon$ , at earliest

Applicable if  $\epsilon$  is "small" compared to  $T$

# Observers with no delay

## Terminology

- $\hat{x}(k|k)$ : **filtered** estimate
- $\hat{x}(k+1|k)$ : **predicted** estimate
- $\hat{x}(k-1|k)$ : **smoothed** estimate



# Luenberger filter

Filter dynamics ( $\hat{x}(k) = \hat{x}(k|k)$ )

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) - L(y(k+1) - \underbrace{C(A\hat{x}(k) + Bu(k))}_{\text{estimate of } y(k+1)})$$

# Luenberger filter

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Error dynamics ( $e(k) = x(k) - \hat{x}(k)$ )

$$\begin{aligned}e(k+1) &= Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) + LCx(k+1) - LCA\hat{x}(k) \\ &\quad - LCBu(k) \\ &= Ae(k) + LC(Ax(k) + Bu(k)) - LCA\hat{x}(k) - LCBu(k) \\ &= (A + LCA)e(k)\end{aligned}$$

Filter design : find  $L$  such that  $A + LCA$  is Schur

- Eigenvalue assignment for the pair  $(A, CA)$ , if it is observable

## Definition

The eigenvalues of  $A + LCA$  are termed the filter eigenvalues

# Luenberger filter

## Remarks

- $(A, C)$  observable  $\not\Rightarrow (A, CA)$  observable
- The observability matrix for  $(A, CA)$  is

$$\tilde{M}_o = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}^T = A^T M_o$$

*obs matrix of CA C*

$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T$$

If  $\det(A) = 0$ ,  $\tilde{M}_o$  is not full rank, even if  $M_o$  is

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If  $\det(A) = 0$ ,  $\tilde{M}_o$  is not full rank, even if  $M_o$  is

- Possible to prove that if  $\lambda = 0$  is an eigenvalue of  $A$ , then it is an unobservable eigenvalue of  $(A, CA)$ 
  - ▶ Not critical, however, because the mode associated to  $\lambda = 0$  is AS (vanishes in  $n$  steps)

## Example

$$\begin{aligned}x^+ &= Ax + Bu \\ y &= Cx\end{aligned} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0]$$

Observability analysis:  $\det(M_o) \neq 0$  but  $\det(\tilde{M}_o) = 0$   
  $(CA, CA)$

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Assignment of the observer eigenvalues

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \quad A + LCA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 0 & l_1 + 1 \\ 0 & l_2 + 1 \end{bmatrix}$$

- $\text{Spec}(A + LCA) = \{0, l_2 + 1\}$ . The eigenvalue  $\lambda = 0$  cannot be modified. However, this is not a problem for the observer stability

## Separation principle

Proceeding as in the case of the observer with state  $\hat{x}(k|k-1)$ , one can prove that

$$\{\text{closed-loop eigenvalues}\} = \text{Spec}(A + BK) \cup \text{Spec}(A + LCA)$$

- If  $(A, B)$  is controllable and  $(A, CA)$  is observable, all eigenvalues of the closed-loop system can be assigned by choosing  $K$  and  $L$  independently

1h16 ↓

# Reduced-order observers

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**Goal:** Design observers with order strictly less than  $n$

## Idea

$y = Cx \in \mathbb{R}^p$  carries (partial) information on  $x$ .

Find a state transformation  $\bar{x} = T x$  such that  $p$  states coincide with  $y$  and reconstruct only the remaining  $n - p$  states through an observer of order  $n - p$ .

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## System dynamics

$$x^+ = Ax + Bu$$

$$y = Cx$$

Set

$$T = \begin{bmatrix} C \\ T_1 \end{bmatrix} \left. \begin{array}{l} \} p \text{ rows} \\ \} n - p \text{ rows} \end{array} \right\}$$

where  $T_1$  is such that  $\det(T) \neq 0$  ( $T_1$  is not unique)

## Change of coordinates

System with state  $\bar{x} = Tx$

$$\begin{aligned}\bar{x}^+ &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x}\end{aligned}\quad (*)$$

where  $\bar{A} = TAT^{-1}$ ,  $\bar{B} = TB$ ,  $\bar{C} = CT^{-1}$  and, by construction,

$$\begin{aligned}\bar{x} &= \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ w \end{bmatrix} \\ \bar{C} &= [ I \quad 0 ] \end{aligned}$$

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$$\bar{x} = \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ w \end{bmatrix} \begin{array}{l} \rightarrow \in \mathbb{R}^p \\ \rightarrow \in \mathbb{R}^{n-p} \end{array}$$
$$\bar{C} = \begin{bmatrix} I & 0 \end{bmatrix}$$

- Partition  $\bar{A}$ ,  $\bar{B}$  as

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{A}_{11} \in \mathbb{R}^{p \times p}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$$

- The system (\*) becomes

$$y(k+1) = \bar{A}_{11}y(k) + \bar{A}_{12}w(k) + \bar{B}_1u(k) \quad (1)$$

$$w(k+1) = \bar{A}_{21}y(k) + \bar{A}_{22}w(k) + \bar{B}_2u(k) \quad (2)$$

# Observer design

## Dynamics of $w(k)$

$$\Sigma_w = \begin{cases} \text{from (2)} \rightarrow w(k+1) = \bar{A}_{22}w(k) + \underbrace{[\bar{A}_{21}y(k) + \bar{B}_2u(k)]}_{\text{Known input } \bar{u}(k)} \\ \text{from (1)} \rightarrow \underbrace{y(k+1) - \bar{A}_{11}y(k) - \bar{B}_1u(k)}_{\text{Measured output } \bar{y}(k+1)} = \bar{A}_{12}w(k) \end{cases}$$

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## Design of the estimator $\bar{y}(k) \rightarrow \hat{w}(k|k)$

Full order observer (**without delay**) for  $\Sigma_w$ .

Set  $\hat{w}(k) = \hat{w}(k|k) \in \mathbb{R}^{n-p}$ .

$$\hat{w}(k+1) = \bar{A}_{22}\hat{w}(k) + \bar{u}(k) - L(\bar{y}(k+1) - \bar{A}_{12}\hat{w}(k))$$

# Observer design

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## Error dynamics (check at home)

$$\hat{e}(k+1) = w(k+1) - \hat{w}(k+1) = (\bar{A}_{22} + L\bar{A}_{12})\hat{e}(k)$$

- $(\bar{A}_{22}, \bar{A}_{12})$  observable  $\Rightarrow$  design  $L \in \mathbb{R}^{(n-p) \times p}$  for assigning the eigenvalues of  $\bar{A}_{22} + L\bar{A}_{12}$  with the usual procedures

# Observer design

## Lemma

If  $(A, C)$  is observable, then  $(\bar{A}_{22}, \bar{A}_{12})$  is observable.

The (not obvious) proof is omitted. . .

# Observer design

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If  $(A, C)$  is observable, then  $(\bar{A}_{22}, \bar{A}_{12})$  is observable.  
The (not obvious) proof is omitted...

## Reconstruction of the full state

$$\hat{x}(k|k) = T^{-1} \begin{bmatrix} y(k) \\ \hat{w}(k|k) \end{bmatrix}$$

## Remarks

Using the control law

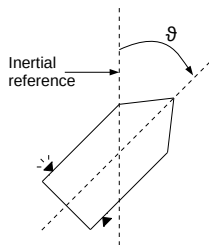
$$u(k) = K\hat{x}(k|k)$$

one can prove a separation principle: under the reachability and observability of suitable pairs,  $2n - p$  eigenvalues of the closed-loop system can be assigned through  $K$  and  $L$ .

*n eigenvalues assigned by the controller and  $n-p$  eig. assigned by the reduced-order observer*

## Example: satellite attitude control - output feedback

**Attitude control:** proper orientation of the satellite antenna with respect to earth



$$\ddot{\theta} = u + w, \quad u = \frac{M_C}{I}, \quad w = \frac{M_D}{I}$$

- $I$ : moment of inertia of the satellite (about the mass center)
- $M_C$ : control torque applied by thrusters
- $M_D$  disturbance torque
- $\theta$ =angle of satellite

DT model ( $x_1 = \theta$ ,  $x_2 = \dot{\theta}$ ,  $y = \theta$ , exact discretisation)

As seen in lecture 5,

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}}_B (u + w)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \quad \rightarrow \text{the angular velocity } x_2 \text{ is not measured}$$

In the sequel :

$$w = 0, \quad T = 0.1$$

## Problem

Design a state observer

# First design: standard Luenberger observer

## Goal

Synthesize a full-order observer with eigenvalues  $z_{1,2} = 0.4 \pm j0.4$

$$\hat{x} = \hat{x}(k|k-1)$$

$$\begin{cases} \hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$

- Design of  $L$  in Matlab for assigning the eigenvalues of  $A + LC$

$$T = 0.1$$

$$A = [1 \ T ; 0 \ 1]$$

$$C = [1 \ 0]$$

$$p = [0.4 + i*0.4 ; 0.4 - i*0.4]$$

$$L = \text{-acker}(A', C', p)'$$

$$L = \begin{bmatrix} 1.2 \\ 5.2 \end{bmatrix}$$

$\rightarrow \text{acker}(A, B, p) = K$  s.t.  
 $(A+BK)$  has the eigenvalues  
in  $p$

Now we want to assign the eig.  
of  $(A+LC)^T = A^T + C^T L^T$

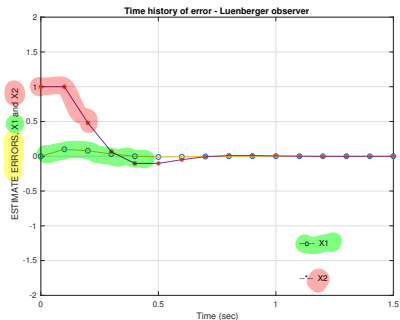
# First design: Luenberger observer

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$$\begin{cases} \hat{x}^+ = A\hat{x} + Bu - L(y - C\hat{x}) \\ \hat{y} = C\hat{x} \end{cases}$$



## Second design: observer with no delays (filter)

$$\hat{x} = \hat{x}(k|k)$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu - \hat{L}(y(k+1) - C(A\hat{x}(k) + Bu(k)))$$

- Design of  $\hat{L}$  in Matlab for assigning the eigenvalues of  $A + \hat{L}CA$

$$\hat{L} = -\text{acker}(A^T, A^T * C^T, p)^T$$

$$\hat{L} = \begin{bmatrix} 0.68 \\ 5.2 \end{bmatrix}$$

## Second design: observer with no delays (filter)

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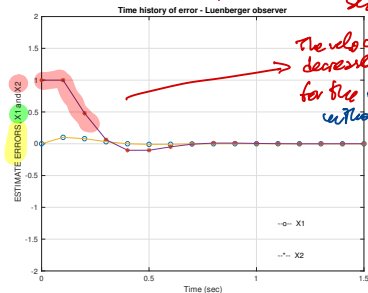
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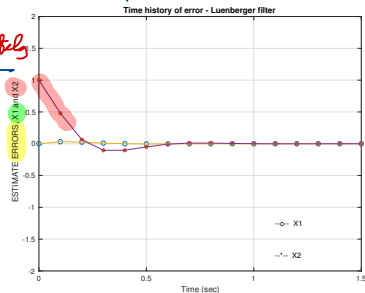
$$\hat{L} = \begin{bmatrix} 0.68 \\ 5.2 \end{bmatrix}$$

*observer with delay - see previous slide*

*observer without delay*



*The velocity error decreases immediately for this observer without delay*



## Third design: reduced-order observer

### Goal

Synthesize a reduced-order observer with eigenvalue  $z = 0.5$

First step: put the system dynamics in the reference form

$$\bar{x}^+ = \bar{A}\bar{x} + \bar{B}u$$

$$y = \bar{C}\bar{x}$$

$$\text{with } \bar{x} = \begin{bmatrix} y \\ w \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \mathbf{1} & 0 \end{bmatrix}$$

## Third design: reduced-order observer

### Goal

Synthesize a reduced-order observer with eigenvalue  $z = 0.5$

First step: put the system dynamics in the reference form

$$\begin{aligned}\bar{x}^+ &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x}\end{aligned}$$

with  $\bar{x} = \begin{bmatrix} y \\ w \end{bmatrix}$ ,  $\bar{C} = [1 \ 0]$

- The DT model is already in the reference form

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{T^2}{2} \\ -\frac{T}{1} \end{bmatrix}, \quad \bar{C} = [1 \ 0]$$

## Third design: reduced-order observer

### Goal

Synthesize a reduced-order observer with eigenvalue  $z = 0.5$

First step: put the system dynamics in the reference form

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- The DT model is already in the reference form

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}, \quad \bar{C} = [1 \ 0]$$

- $(\bar{A}_{22}, \bar{A}_{12})$  is observable  $\Rightarrow$  design  $\bar{L}$  such that

$$\begin{aligned}\bar{A}_{22} + \bar{L}\bar{A}_{12} &= 0.5 \quad \rightarrow \quad 1 + \bar{L}T = 0.5 \\ T &= 0.1 \quad \rightarrow \quad \bar{L} = -5\end{aligned}$$

# Estimator dynamics

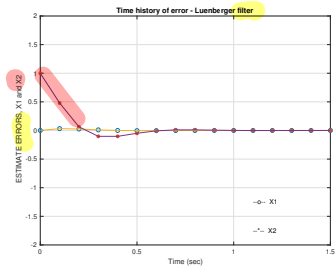
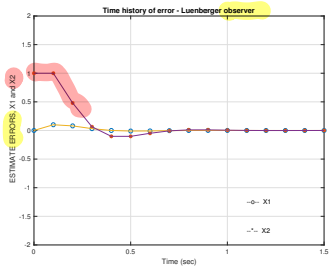
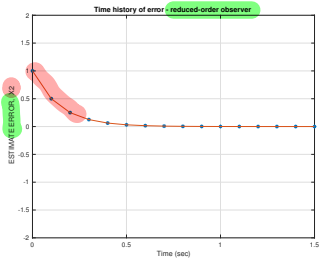
*↪ because  $\hat{w} = \hat{x}_2$*

Since  $w = x_2$ , by using the notation  $\hat{x}_2(k+1) = \hat{x}_2(k+1|k+1)$ , we have

$$\hat{x}_2(k+1) = \bar{A}_{22}\hat{x}_2(k) + \bar{u}(k) - \bar{L}(\bar{y}(k+1) - \bar{A}_{12}\hat{x}_2(k))$$

$$\hat{x}(k) = \begin{bmatrix} y(k) \\ \hat{x}_2(k) \end{bmatrix}$$

# Estimator dynamics



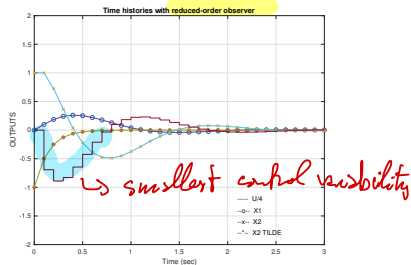
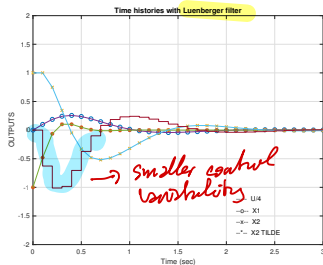
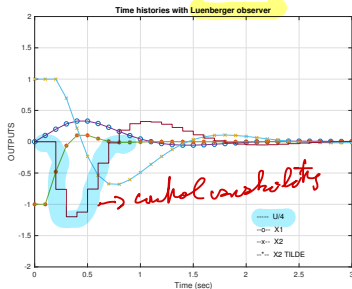
# Output-feedback controller

## Goal

Design  $u = K\hat{x}$  such that the remaining eigenvalues of the closed-loop system are  $z = 0.8 \pm j0.25$

- Already done (see Lecture 5)  $\rightarrow K = \begin{bmatrix} -10 & -3.5 \end{bmatrix}$

# Output-feedback controller



# Output-feedback controller

- Luenberger filter: faster response than Luenberger observer (as expected)
- Reduced-order observer: first-order response of the estimator → slightly reduced control effort compared to Luenberger observer

# Choice of the closed-loop eigenvalues

# How to choose the closed-loop eigenvalues ?

- State feedback + full-order observer : assign  $\nu = 2n$  eigenvalues
- State feedback + reduced-order observer : assign  $\nu = n + (n - p)$  eigenvalues

## First heuristic approach

- Assign  $\nu - 2$  eigenvalues to the origin
  - ▶ Deadbeat behaviour
- Set 2 "dominant" eigenvalues as desired

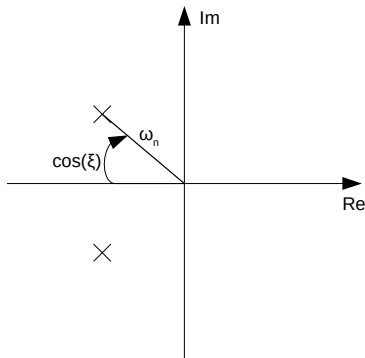
# Assignment of dominant eigenvalues: first heuristic method

Desired closed-loop continuous-time transfer function

*→ 2<sup>nd</sup> order*

$$G_c(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$\xi$ : damping  $\hookrightarrow [0, 1]$        $\omega_n$ : natural frequency  $\hookrightarrow > 0$



# Assignment of dominant eigenvalues: first heuristic method

Exact discretisation of  $G_c(s)$  with sampling time  $T_c$  results in a DT system with eigenvalues

$$p_1 = -2e^{-\xi\omega_n T_c} \cos(\omega_n T_c \sqrt{1 - \xi^2})$$
$$p_2 = e^{-2\xi\omega_n T_c}$$

## Algorithm

Choose desired  $\xi$  and  $\omega_n \rightarrow$  compute  $p_1$  and  $p_2$

## Recall the golden rules from basic control theory

- If the CT LTI system  $(A, B, C, D)$  is a low pass filter with pass-band  $[0, \bar{\omega}]$ , do not set  $\omega_n \gg \bar{\omega}$ . Otherwise
  - ▶ the magnitude of control variables might be large and actuator limits might be reached
  - ▶ high-frequency disturbances might start playing a significant role

## Second set of heuristic criteria

- Choose control eigenvalues no more than  $2 \div 6$  times faster than open-loop eigenvalues
  - ▶ Good for limiting the actuator effort
- Choose observer eigenvalues faster than control eigenvalues
  - ▶ They do not impact on actuators
    - ★ However, if estimation errors due to sensor noise are significant, one has to slow down the observer eigenvalues
  - ▶ Closed-loop performance will be dominated by control eigenvalues

# Take home messages

- Observers are essential for systems where not all states can be measured
- Duality + separation principle: eigenvalue assignment is the key tool for designing output-feedback controllers

# Take home messages

- Observers are essential for systems where not all states can be measured
- Duality + separation principle: eigenvalue assignment is the key tool for designing output-feedback controllers

## Problem

How stochastic disturbances affect state estimation ?

See later (Kalman filtering)

2h16 ↓