

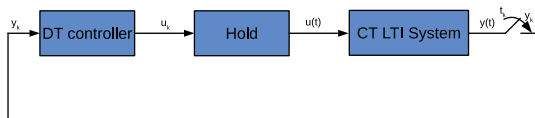
# Lecture 4

## Discretisation of Continuous Time (CT) systems

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# Motivation: digital control of CT systems



- $y(t)$ : plant measurements
- $y_k$ : input to controller
- Plant dynamics

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(0) = x_0$$

$$x(t) \in \mathbb{R}^n$$

$$u(t) \in \mathbb{R}^p$$

$$y(t) \in \mathbb{R}^m$$

# Model of sample and hold

- System CT output  $y(t)$  sampled at  $\{t_k, k \in \mathbb{N}\}$ 
  - ▶  $y_k = y(t_k)$ ,  $T_k \triangleq t_{k+1} - t_k$

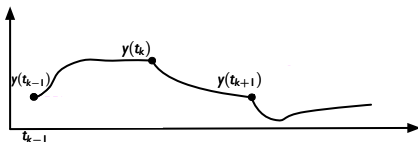
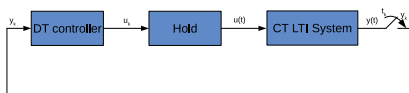
- Hold

$$u(t) = u_k \quad t \in [t_k, t_{k+1})$$

- **Problem:** obtain DT models representing the cascade Hold + (CT system) + Sampler

- **Next:** popular discretisation methods

- ▶ exact
- ▶ approximate



# Exact discretisation

Goal: compute the discrete-time dynamics for  $x_k$

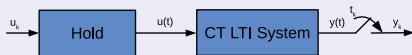
LTI system

$$\dot{x} = Ax + Bu$$

- Set  $x_k \stackrel{y}{=} \bar{x}(t_k)$ ,  $y_k \stackrel{y}{=} y(t_k)$  etc.

- Recall the Lagrange formula for the above system with  $x(t_k) = x_k$

$$x(t) = e^{A(t-t_k)}x_k + \underbrace{\int_{t_k}^t e^{A(t-\tau)}Bu(\tau)d\tau}_{(b)}$$



**State transition operator**  $e^{As}$ : pushes  $x_k$  ahead by  $s$  seconds

# Exact discretisation

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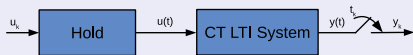
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**Constant-input transmission operator**  $\Gamma(s) \triangleq \int_0^s e^{Az} dz$

- ▶  $(b) = \Gamma(t - t_k)Bu_k$  if  $u(\cdot) = u_k$  on  $[t_k, t]$ .

**Proof:**  $(b) = \int_{t_k}^t e^{A(t-\tau)}d\tau Bu_k = -\int_{t-t_k}^0 e^{Az} dz Bu_k = \int_0^{t-t_k} e^{Az} dz Bu_k$ , where we have set  $z = t - \tau$ .

- ▶  $\Gamma(t - t_k)B$  pushes  $u_k$  ahead by  $t - t_k$  seconds

## Exact discretisation

- Sample at times  $t_0 = 0, t_1, t_2, \dots$  and set  $T_k = t_{k+1} - t_k$ .

- We have
$$x_{k+1} = x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-s)}Bu(s) ds$$
$$= \underbrace{e^{AT_k}}_{\hat{A}_k} x_k + \underbrace{\Gamma(T_k)B}_{\hat{B}_k} u_k \quad (1)$$

$$y_k = \hat{C}x_k + \hat{D}u_k \quad \text{for} \quad \hat{C} = C, \hat{D} = D \quad (2)$$

- (1) - (2) : linear time-varying system

Under uniform sampling (i.e,  $T_k = T, k = 0, 1, \dots$ )

$$\hat{A} = e^{AT}, \quad \hat{B} = \Gamma(T)B \rightarrow \text{invariant system}$$

$$x^+ = \hat{A}x + \hat{B}u$$

$$y = \hat{C}x + \hat{D}u$$

## Eigenvalues under exact sampling (1/2)

$$\hat{A} = e^{AT}$$

### Properties

One can show that

- a)  $\lambda \in \text{Spec}(A) \Rightarrow z = e^{\lambda T} \in \text{Spec}(\hat{A})$
- b)  $\det(\hat{A}) \neq 0$  always (even if  $\det(A) = 0$ )  
Moreover  $(e^{AT})^{-1} = e^{-AT} = e^{A(-T)}$  (inverse = "backward in time")

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- Implications of (b) :  
reachability and controllability coincide for DT systems obtained through exact sampling

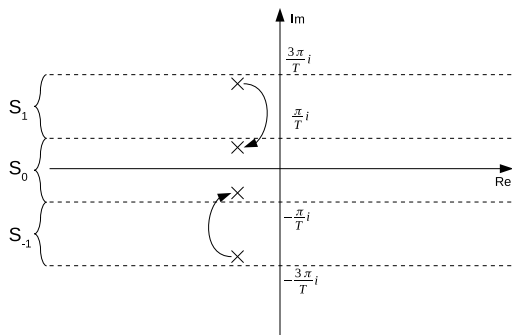
- Implications of (a) :

$\lambda_1 \neq \lambda_2$  does not imply that  $z_1 \neq z_2$

Check : take  $\lambda_2 = \lambda_1 + j\frac{2\pi}{T}N$ ,  $N = \pm 1, \pm 2, \pm 3, \dots$

$$\text{We have } z_2 = \underbrace{e^{\lambda_1 T}}_{z_1} \underbrace{e^{j\frac{2\pi}{T}NT}}_1$$

# Interpretation in the complex plane

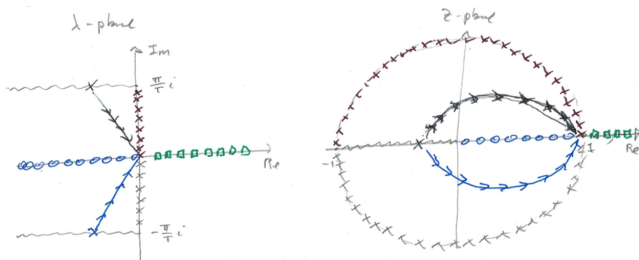


- $S_i =$  Strips of width  $2\pi/T$
- Due to sampling, all strips "collapse" into  $S_0$   
Sampling  $\Rightarrow$  loss of information: can not reconstruct eigenvalues of the CT system from those of the DT one

## Eigenvalues under exact sampling (2/2)

Map of regions in the  $\lambda$ -plane (continuous time) into regions of the  $z$ -plane (discrete-time).

$\lambda$ -plane	$z$ -plane
0	1
real negative	$[0, 1)$
real positive	$(1, +\infty)$
imaginary part $\pm \frac{\pi}{T}$ and real part $< 0$	$(-1, 0]$
constant damping	log spirals



- Remark: asymptotically stable eigenvalues  $\lambda$  (i.e.  $\text{Re}(\lambda) < 0$ ) are mapped into asymptotically stable eigenvalue  $z$  (i.e.  $|z| < 1$ ).

## Exact sampling: conclusions

- Pros
  - ▶  $x_k$  and  $y_k$  represent  $x(t_k)$  and  $y(t_k)$  with no approximations
  - ▶ Stability, AS and instability are preserved
- Cons
  - ▶  $\hat{A} = e^{AT} = I + AT + \frac{(AT)^2}{2} + \dots$
  - ▶ can be difficult to compute for large systems !  
In Matlab, `hA=expm(A*T)`
  - ▶ does not preserve the pattern of zeros in  $A$

### Example

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad T = 0.1$$

$$\rightarrow \hat{A} = e^{AT} = \begin{bmatrix} 0.905 & 0.0905 & 0.0045 \\ 0.0045 & 0.905 & 0.0905 \\ 0.0905 & 0.0045 & 0.905 \end{bmatrix}$$

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## Generalisations

What happens if  $u(t)$  or  $x(t)$  are affected by delays ? → see the class "Networked Control Systems"

# Approximate discretisation methods

- Goal: avoid the computation of  $e^{AT}$  and preserve the structure of  $A$

## CT LTI System

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx + Du \quad (2)$$

## Assumption

Uniform sampling period  $T$

- Integrating (1) on  $[kT, (k+1)T]$  and setting  $x_k = x(kT)$  gives

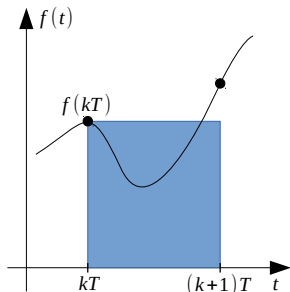
$$x_{k+1} - x_k = A \int_{kT}^{(k+1)T} x(t) dt + B \int_{kT}^{(k+1)T} u(t) dt$$

# Approximate discretisation methods

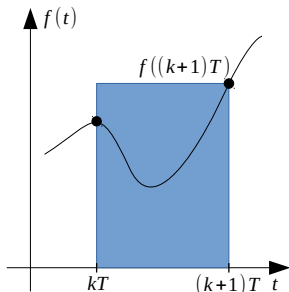
- To make  $u(kT)$  and  $x(kT)$  appear in the right hand side, we use the following numerical integration scheme, written for a generic function

$$f(t) : \mathbb{R} \rightarrow \mathbb{R}^n$$

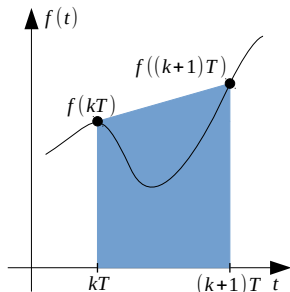
$$\int_{kT}^{(k+1)T} f(t) dt \simeq [(1 - \alpha)f(kT) + \alpha f((k+1)T)] T, \quad 0 \leq \alpha \leq 1$$



Forward Euler ( $\alpha=0$ )



Backward Euler ( $\alpha=1$ )



Tustin ( $\alpha=0.5$ )

# DT Models

- $\alpha = 0$  : Forward Euler (FE)
- $\alpha = 1$  : Backward Euler (BE)
- $\alpha = 0.5$  : Trapezoidal method or Tustin method

In all cases, loss of information. Is stability preserved ?

## Resulting DT system

$$x_{k+1} - x_k = A[(1 - \alpha)x_k + \alpha x_{k+1}]T + BT[(1 - \alpha)u_k + \alpha u_{k+1}]$$

- $\alpha = 0$  (FE)

$$x_{k+1} = (AT + I)x_k + BTu_k$$

- $\alpha = 1$  (BE)

$$x_{k+1} = ATx_{k+1} + x_k + BTu_{k+1}$$

- $\alpha = 0.5$  (Tustin)

$$x_{k+1} = \left(\frac{1}{2}AT + I\right)x_k + \frac{1}{2}ATx_{k+1} + BT\frac{1}{2}[u_k + u_{k+1}]$$

## Analysis of FE ( $\alpha = 0$ )

### Remark

FE provides a causal DT system with  $\hat{A} = AT + I$ ,  $\hat{B} = BT$

### Lemma (properties of FE)

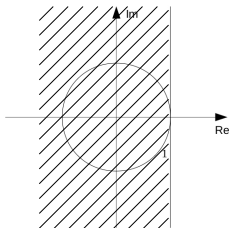
- Ⓐ The off-diagonal zero entries of  $A$  and  $AT + I$  are in the same positions
- Ⓑ if  $A$  is Hurwitz stable, then  $AT + I$  is Schur stable if and only if

$$T < T^* = \min_{i=1, \dots, n} \frac{-2 \operatorname{Re}(\lambda_i)}{|\lambda_i|^2} \quad (1)$$

where  $\lambda_i$  are the eigenvalues of  $A$ .

## Remark

If  $\lambda$  is an eigenvalue of  $A$ , then  $\lambda T + 1$  is an eigenvalue of  $AT + I$ . The function  $\lambda \mapsto \lambda T + 1$  maps the set  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\}$  in the shaded region below



## Remark

$$e^{AT} = I + AT + \frac{(AT)^2}{2} + \dots$$

Therefore FE can be seen as a first order Taylor approximation of  $e^{AT}$

## Effect of discretization on reachability and observability

- Discretisation  $\Rightarrow$  loss of information
- We expect that sampling may impair reachability/observability...

### CT LTI System

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\quad (*)$$

### Theorem

Assume that  $(*)$  is reachable and observable. The DT system obtained through exact discretisation is reachable and observable if and only if, for any pair of eigenvalues  $\lambda_i, \lambda_j$  of  $A$

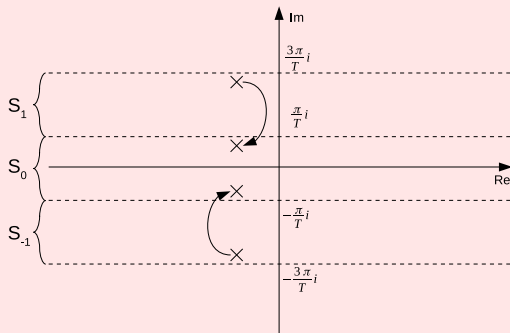
$$\operatorname{Re}(\lambda_i) = \operatorname{Re}(\lambda_j) \Rightarrow \operatorname{Im}(\lambda_i - \lambda_j) \neq \frac{2n\pi}{T} \quad n = \pm 1, \pm 2, \dots$$

where  $T$  is the sampling period.

# Effect of discretization on reachability and observability

## Remarks

- Recall the relations between eigenvalues under exact sampling



- The CT system has real eigenvalues only  $\Rightarrow$  no loss of reachability/observability
- Always possible to avoid losses of reachability/observability by choosing  $T > 0$  small enough

## Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (**)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Check at home that  $(**)$  is reachable and observable (the tests are the same for CT and DT systems)
- System eigenvalues  $\lambda_1 = j$ ,  $\lambda_2 = -j$

DT system (sampling time  $T > 0$ )

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}}_{\hat{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 - \cos(T) \\ \sin(T) \end{bmatrix}}_{\hat{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\hat{C}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

## Reachability of the DT system

$$M_r = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} \end{bmatrix} = \begin{bmatrix} 1 - \cos(T) & \cos(T) + 1 - 2\cos^2(T) \\ \sin(T) & -\sin(T) + 2\cos(T)\sin(T) \end{bmatrix}$$

$$\text{Rank}(M_r) = 2 \quad \Leftrightarrow \quad T \neq n\pi \quad n = 1, 2, 3, \dots$$

- By applying the Theorem

$\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$ . Then, the system is reachable if and only if

$$\underbrace{\text{Im}(\lambda_1 - \lambda_2)}_{\text{Im}(2j) = 2} \neq \frac{2n\pi}{T} \quad \Leftrightarrow \quad T \neq n\pi$$

# Conclusions

- Time-discretization  $\Rightarrow$  loss of information
- Exact sampling : best possible choice but it does not
  - ▶ preserve structure
  - ▶ apply to nonlinear systems

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$\rightarrow$  What about the simultaneous presence of sampling and time delays?  
Master course "Networked Control Systems"