# Lecture 2

## Stability, reachability and controllability

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Section 1

# Modes and stability of LTI systems

# Modes and stability of LTI systems

$$\begin{cases} x^+ = Ax + Bu \\ x(0) = x_0 \end{cases} \tag{1}$$

$$(\bar{x}, \bar{u}) \text{ equilibrium}$$

- Recall: stability of the equilibrium state $\bar{x}$.

**Definition (Lyapunov stability)**

The equilibrium state $\bar{x}$ is

- stable if $\forall \epsilon > 0 \ \exists \delta > 0 : \|\tilde{x}_0 - \bar{x}\| \leq \delta \Rightarrow \|\tilde{x}(k) - \bar{x}\| < \epsilon, \forall k \geq 0$
- (globally) asymptotically stable (AS) if it is stable and attractive, i.e.,

$$\lim_{k \to \infty} \|\tilde{x}(k) - \bar{x}\| = 0, \ \forall \tilde{x}_0 \in \mathbb{R}^n$$

- unstable if not stable

# Modes and stability of LTI Systems

- Key quantity to analyse: the error

$$e(k) = \tilde{x}(k) - \bar{x}$$

**Proposition**

Set $e_0 = e(0) = \tilde{x}(0) - \bar{x}$. The error verifies

$$\begin{aligned} e^+ &= Ae \\ e(0) &= e_0 \end{aligned} \tag{2}$$

# Modes and stability of LTI systems

$\bar{x}(k) = \phi(k, 0, \bar{x}, \bar{u})$

## Proof

$\tilde{x}(k) = \phi(k, 0, \tilde{x}_0, \bar{u})$

Note that $e(k) = \alpha\tilde{x}(k) + \beta\bar{x}(k)$ with $\alpha = 1$ and $\beta = -1$. From the superposition principle,

$$e(k) = \tilde{x}(k) - \bar{x}(k) = \phi(k, 0, \tilde{x}_0 - \bar{x}, \underbrace{\bar{u} - \bar{u}}_{0})$$

Since $\phi$ is the transition map of (1), the error satisfies (1) for zero input. This is (2).

## Proof (alternative)

Since $\bar{x}$ verifies $\bar{x}^+ = \bar{x} = A\bar{x} + B\bar{u}$, compute $e^+ = x^+ - \bar{x}^+$ explicitly and obtain (2).

# Remarks

- Stability/AS of $\bar{x}$ is the same as stability/AS of $\bar{e} = 0$ for (2).
  Check: stability of $\bar{e} = 0$ means

$$\forall \varepsilon > 0, \ \exists \delta : \ \|e_0 - 0\| < \delta \implies \|e(k) - 0\| < \varepsilon, \ \forall k \geq 0$$

  which, by using $e(k) = x(k) - \bar{x}$ coincides with the definition of stability for $\bar{x}$.
- (2) is independent of $\bar{u}$ and $\bar{x}$. This proves the following theorem.

**Theorem**

*An equilibrium state of an LTI system is stable/AS/unstable* ***if and only if*** *all other equilibria have the same properties.*

This is why we can say, for an LTI system, that « the system is stable ».

# Stability and free states

For stability analysis, setting $u(k) = 0$ in $x^+ = Ax + Bu$ is not conservative. $\implies$ stability depends only on free states.

## Theorem

*An LTI system*

1. *is stable* $\iff$ *all free states are* ***bounded***
2. *is AS* $\iff$ *all free states are* ***bounded and go to zero as*** $k \to +\infty$
3. *is unstable* $\iff$ *there is a free state which is* ***unbounded***

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# Stability and eigenvalues of $A$

Each free state $x(k) = A^k x_0$ is a linear combination of the modes of $A$.

| eigenvalues | modes |
|---|---|
| $\lambda_i \in \mathbb{R}$ | $\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \ldots, \eta_i - 1$ |
| $\lambda_i = \rho_i e^{j\theta_i}$ AND $\lambda_h = \lambda_i^*$ | $\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\rho_i^{k-p_i}\sin(\theta_i(k-p_i)+\varphi_i) & \text{for } k \geq p_i \end{cases}$ $p_i = 0, 1, \ldots, \eta_i - 1$ |

# Stability and eigenvalues of $A$

Recall the macroscopic behaviour of modes

## Lemma

- If $|\lambda_i| < 1$, all modes associated to $\lambda_i$ are **bounded** and **go to zero** as $k \to +\infty$.
- If $|\lambda_i| > 1$, all modes associated to $\lambda_i$ are **unbounded**.
- If $|\lambda_i| = 1$ and $\nu_i = n_i$, all modes associated to $\lambda_i$ are **bounded**. $p_i = 0$
- If $|\lambda_i| = 1$ and $\nu_i < n_i$, there's an **unbounded** mode associated to $\lambda_i$

$\to p_i = 0, 1, \ldots$

- **Terminology**: « eigenvalues of $A$ » = « eigenvalues of the system »
- Combining the previous Lemma and Theorem we have the three Theorems given next

# Stability and eigenvalues of $A$

**Theorem (test of AS)**

*An LTI system is AS* ***if and only if*** *all its eigenvalues have modulus $< 1$.*

**Theorem (test of instability)**

*An LTI system is unstable* ***if and only if*** *one of the following conditions occurs.*

1. *A system eigenvalue has modulus $> 1$.*
2. *All system eigenvalues have modulus $\leq 1$ and there is an eigenvalue $\lambda_i$ with modulus $= 1$, algebraic multiplicity $n_i \geq 2$ and $\dim(V_{\lambda_i}) < n_i$.*

# Stability and eigenvalues of $A$

**Theorem (test of simple stability)**

*An LTI system is simply stable **if and only if** all its eigenvalues have modulus $\leq 1$ and, for each eigenvalue $\lambda_i$ with modulus 1 and algebraic multiplicity $n_i \geq 2$, one has*

$$dim(V_{\lambda_i}) = n_i$$

- **Recall:** $\dim(V_{\lambda_i})$ is the geometric multiplicity of $\lambda_i$.
- **Remark**: AS is the most important property in engineering applications
- **Terminology**: we say that « $A$ is Schur » if all its eigenvalues have modulus $< 1$.

# Exponential Stability

Recall that the equilibrium $\bar{x}$ of $x^+ = Ax$ is **exponentially stable** if there are $\alpha > 0, \rho \in [0,1)$ such that

$$\|\tilde{x}(k) - \bar{x}\| \leq \alpha\rho^k \|\tilde{x}_0 - \bar{x}\|, \quad \forall \tilde{x}_0 \in \mathbb{R}^n$$

### Lemma

An LTI systems is AS **if and only if** it is ES.

### Sketch of the Proof

AS $\iff$ all modes of the system go to zero as $k \to \infty$. But if a mode goes to zero, it does so exponentially fast. This implies that $\exists\ \alpha, \rho$ verifying the definition of ES

# Examples

Analyse the stability of $x^+ = Ax + Bu$.

## For $x \in \mathbb{R}^2$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \implies \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 0 \end{cases} \implies \text{unstable}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases} \implies \text{stable but not AS}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \implies \lambda_1 = \lambda_2 = 1 \implies \text{alg. multiplictity } n_1 = 2$$

$(A - \lambda I)v = 0$

$$V_1 = \left\{ v : \; A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} = \left\{ v : \; \begin{matrix} v_1 + v_2 = v_1 \\ v_2 = v_2 \end{matrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$

$\implies \dim(V_1) = 1 < n_1$, therefore the system is unstable

# Section 2

## Reachability and controllability

# Key properties of dynamical systems

## Reachability

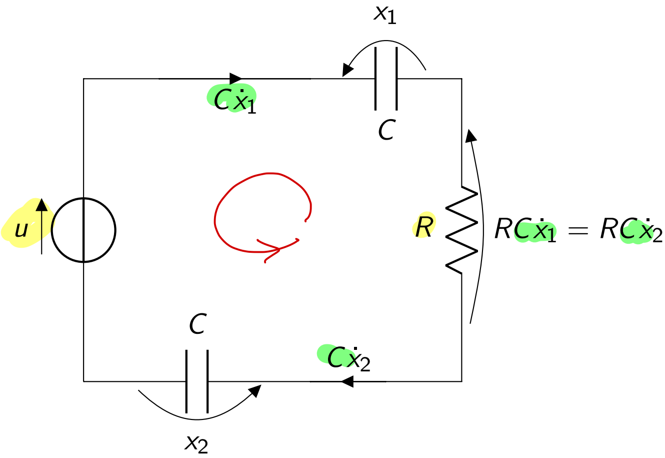
Is it possible to steer the state $x_0 = 0$ **to a desired value** by acting on the inputs?

## Controllability

Is it possible to steer the state $x_0 \in \mathbb{R}^n$ **to the origin** by acting on the inputs?

# Reachability

## Example




$C = 1$
$R = 1$

# Reachability

## Example

### Model

$$\begin{cases} u = x_1 + x_2 + \dot{x}_1 \\ u = x_1 + x_2 + \dot{x}_2 \\ y = x_1 \end{cases}$$

### Discretization

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x(k+1) - x(k)}{T}, \quad T = 0.1$$

$$\Longrightarrow \begin{cases} x_1^+ = -9x_1 - 10x_2 + 10u \\ x_2^+ = -10x_1 - 9x_2 + 10u \\ y = x_1 \end{cases}$$

Can we modify $x_1$ independently of $x_2$? It seems not: in the CT model the currents in the upper and lower branches are identical.

# Change of coordinates to highlight this phenomenon

$$\hat{x} = Tx, \quad T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \implies \begin{cases} \hat{x}_1 = x_1 - x_2 \\ \hat{x}_2 = x_1 + x_2 \end{cases}$$

$$\hat{x}^+ = TAT^{-1}\hat{x} + TBu, \quad T^{-1} = \begin{bmatrix} +0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

By direct calculation:

$$\hat{x}_1^+ = \hat{x}_1$$

$$\hat{x}_2^+ = -19\hat{x}_2 + 20u$$

→ for $\hat{x}_{1,0}$ given, one has $\hat{x}_1(k) = \hat{x}_{1,0}$ $\forall k \geq 0$

The difference of the voltages is constant and it cannot be affected by the input. $\implies$ states $\tilde{x} \in \mathbb{R}^2$ with $\tilde{x}_1 - \tilde{x}_2 \neq x_1(0) - x_2(0)$ cannot be reached.

# Reachability: definitions

$$x^+ = Ax + Bu \tag{3}$$
$$y = Cx + Du \tag{4}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$

## Definition

A state $\tilde{x}$ is **reachable** if $\exists\, \tilde{k} > 0$ and $\tilde{u}(k),\ k = 0, 1, \ldots, \tilde{k}$ such that

initial condition

$$x(\tilde{k}) = \phi(\tilde{k}, 0, 0, \tilde{u}) = \tilde{x} \tag{5}$$

If all states are reachable, then the system is termed « reachable ».

# Reachability: definitions

## Remarks

- Reachability = reachability from the origin as $x(0) = 0$ in (5)
- Reachability = property of the pair $(A, B)$ only
- Problem: Difficult to check if a system is reachable using the definition (infinitely many $\tilde{x}$ should be tested)

# Reachability test

## Definition

The reachability matrix is defined as

$A \in \mathbb{R}^{n \times n}$

$$M_r = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times mn} \tag{6}$$

**Remark**: powers of $A$ from 0 to $n-1$ only.

## Theorem

1. *$\tilde{x} \in \mathbb{R}^n$ is reachable only if it belongs to*

   the reachable subspace

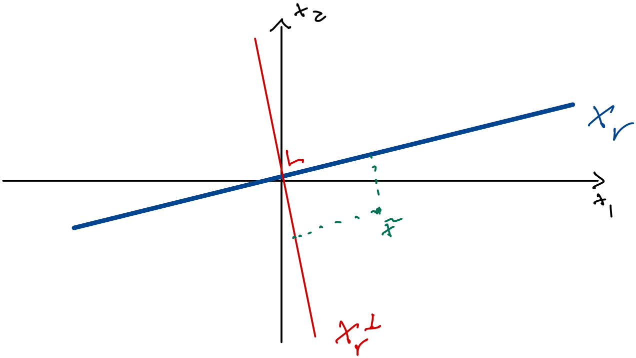
   $$X_r = span(M_r) \tag{7}$$

   *where $span(M_r)$ is the subspace spanned by the columns of $M_r$.*
2. *If $\tilde{x}$ is reachable, it can be reached in $\tilde{k} \le n$ steps.*
3. *The system is reachable if and only if $rank(M_r) = n$*

# Reachability test

## Remarks

- For LTI systems, reachability is a finitely determined property.
- The orthogonal subspace $X_r^\perp$ is termed the « unreachable subspace ».
- The point 3 in the above theorem is a maximal rank condition



# Reachability test

$[b_1 \ldots b_m]$

## Proof

Setting $x(0) = 0$ one has

$$B \begin{bmatrix} u_1(0) \\ u_m(0) \end{bmatrix} = b_1 u_1(0) + \ldots + b_m u_m(0)$$

$$x(1) = Bu(0) \implies \text{lin. comb. of columns of } B$$

$x(2) = Ax(1) + Bu(1)$

$$x(2) = Bu(1) + ABu(0) = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\implies \text{lin. comb. of columns of } \begin{bmatrix} B & AB \end{bmatrix}$$

$$\vdots$$

$$x(k) = \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

# Reachability test

## Proof cont.

From the theorem of Cayley-Hamilton, if $\psi(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n$ is the characteristic polynomial of $A$, then $\psi(A) = 0$, *i.e.* $A^n + \alpha_1 A^{n-1} + \ldots + \alpha_n I = 0$

$$A^n = -(\alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_{n-1} A + \alpha_n I)$$

Therefore, the columns of $A^n B$ are a linear combination of the columns of matrices $A^i B$, $i = 0, 1, \ldots, n-1$.
This shows that a state is reachable only if it can be reached in at most $n$ steps and that the set of reachable states is given by (7).

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# Unreachable systems

If the system is not reachable, the unreachable part can be isolated.

## Theorem

*Let $n_r = rank(M_r) \geq 1$. There is a suitable and non-unique change of state coordinates*

$$\hat{x}(k) = T_r x(k), \quad \det(T_r) \neq 0$$

*such that $x^+ = Ax + Bu$ is equivalent to*

$$\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u$$

*where*

$$\hat{A} = \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} \hat{x}_a \\ \hat{x}_b \end{bmatrix} \qquad \hat{A}_a \in \mathbb{R}^{n_r \times n_r}$$

$$\hat{B} = \begin{bmatrix} \hat{B}_a \\ 0 \end{bmatrix} \qquad \hat{B}_a \in \mathbb{R}^{n_r \times m}$$

→ the pair $(\hat{A}_a, \hat{B}_a)$ is reachable

$$rank([\hat{B}_a \quad \hat{A}_a\hat{B}_a \quad \cdots \quad \hat{A}_a^{n_r-1}\hat{B}_a]) = n_r \tag{8}$$

**Terminology:** $(\hat{A}, \hat{B})$ is called the « reachability form » of $(A, B)$

# Unreachable systems

The zero blocks in $(\hat{A}, \hat{B})$ reveal the **unreachable** part. Setting $\hat{x} = \begin{bmatrix} \hat{x}_a^T & \hat{x}_b^T \end{bmatrix}^T$, $\hat{x}_a \in \mathbb{R}^{n_r}$, we have

$u \rightarrow \boxed{\hat{x}_a^+ = \ldots}$

$\boxed{\hat{x}_b^+ = \ldots}$

$$\hat{x}_a^+ = \hat{A}_a \hat{x}_a + \hat{A}_{ab} \hat{x}_b + \hat{B}_a u \tag{9}$$

$$\hat{x}_b^+ = \hat{A}_b \hat{x}_b \tag{10}$$

- (9) is the reachable part. Since $(\hat{A}_a, \hat{B}_a)$ is reachable, one can steer $\hat{x}_a$ to an arbitrary position; (*c.f.* (8)).
- (10) is the unreachable part: $\hat{x}_b$ is not affected by $u$, neither directly, nor through $\hat{x}_a$.
- Terminology: the eigenvalues of $\hat{A}_a$ are termed « reachable ». Those of $\hat{A}_b$, « unreachable ». The same for the corresponding modes.

# How to build $T_r$?

- Build $M_r = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, $\mathrm{rank}[M_r] = n_r$. Let $v_1, v_2, \ldots, v_{n_r}$ be linearly independent columns of $M_r$.
- Build $T_r^{-1} = \left[ v_1 \; \cdots \; v_{n_r} \,\middle|\, z_1 \; \cdots \; z_{n-n_r} \right]$, where $z_i$ are arbitrary vectors guaranteeing that $\det\left(T_r^{-1}\right) \neq 0$.

## Example (ctd)

$$x^+ = \begin{bmatrix} -9 & -10 \\ -10 & -9 \end{bmatrix} x + \begin{bmatrix} 10 \\ 10 \end{bmatrix} u$$

$$M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 10 & -190 \\ 10 & -190 \end{bmatrix}, \quad n_r = 1$$

$$T_r^{-1} = \left[\begin{array}{c|c} 10 & -10 \\ 10 & 10 \end{array}\right] \quad \Longrightarrow \quad T_r = \frac{1}{20} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = T_r A T_r^{-1} = \begin{bmatrix} -19 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{B} = T_r B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Section 3

# Controllability

# Controllability

$$x^+ = Ax + Bu \tag{11}$$
$$y = Cx + Du \tag{12}$$

## Definition

A state $\hat{x}$ is **controllable** if $\exists\, \hat{k} > 0$ and $\hat{u}(k)$, $k = 0, 1, \dots, \hat{k}$ such that

initial state

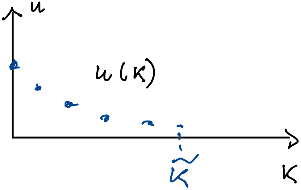
$$\color{red}{0} = \phi(\hat{k}, 0, \hat{x}, \hat{u})$$

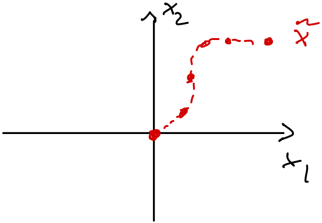
If all states are controllable, then the system is termed «controllable».

## Remarks

- Controllability = controllability to the origin
- Property of $(A, B)$ only

# Controllability



# Controllability and reachability: do they coincide?

## Example

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$x^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad \begin{array}{l} x(k) \in \mathbb{R}^2 \\ u(k) \in \mathbb{R} \end{array}$$

- Every state $\hat{x}$ is controllable using $u(\cdot) = \hat{u}(\cdot) = 0$

$$x(0) = \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \implies x(1) = \begin{bmatrix} \hat{x}_2 \\ 0 \end{bmatrix} \implies x(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The state $\tilde{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ cannot be reached by $x(0) = 0$.

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x(1) = \begin{bmatrix} u(0) \\ 0 \end{bmatrix} \implies x(2) = \begin{bmatrix} u(1) \\ 0 \end{bmatrix} \implies \dots$$

- Intuition: all eigenvalues of $A$ are zero $\implies$ free states go naturally to zero.

# Controllability and reachability: do they coincide?

**Lemma**

One has:

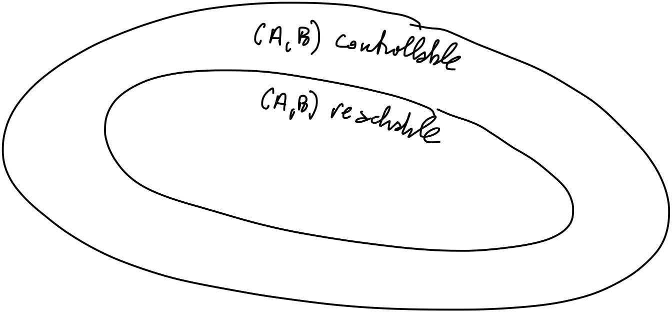
1. $(A, B)$ reachable $\implies$ $(A, B)$ controllable
2. if $\det(A) \neq 0$:

$$(A, B) \text{ controllable } \implies (A, B) \text{ reachable}$$

**Remark**

LTI systems with $\det(A) \neq 0$ are termed **reversible**.

# Controllability and reachability: do they coincide?




If A is invertible, the two sets coincide

# Controllability and reachability: do they coincide?

## Proof of (i) "(A,B) reachable ⇒ (A,B) controllable"

For $x(0) = \hat{x}$, setting $\boldsymbol{u}^k = \begin{bmatrix} u^T(k-1) & \cdots & u^T(0) \end{bmatrix}^T$, one has

$$x(k) = A^k \hat{x} + \underbrace{\begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix}}_{M_r^k} \boldsymbol{u}^k$$

By setting $x(k) = 0$ one has

$$\hat{x} \text{ is controllable} \Leftrightarrow \exists \boldsymbol{u}^k \text{ such that } \overbrace{-A^k \hat{x} = \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \boldsymbol{u}^k}^{(*)} \quad (13)$$

Formula (13) is also the same as requiring that $-A^k \hat{x}$ is reachable from the origin in $k$ steps. Equivalently,

$$-A^k \hat{x} \in \underbrace{\text{span}(M_r^k)}_{X_r^k} \quad (14)$$

If $(A, B)$ is reachable, then $X_r^n = \mathbb{R}^n$ and (14) is verified for all $\hat{x} \in \mathbb{R}^n$ if $k = n$. This proves point (i).

# Controllability and reachability: do they coincide?

## Proof of (ii)

As for point (ii), by assumption, for any $\hat{x} \in \mathbb{R}^n$, $\exists\, k, \boldsymbol{u}^k$ s.t.

$$-A^k \hat{x} = M_r^{k} \boldsymbol{u}^k$$ ← This is (*)

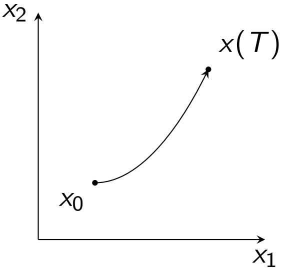
Therefore, any $\bar{x}$ that can be written as

$$\bar{x} = -A^k \hat{x} \quad \text{for some } \hat{x}$$

is reachable. But since $\det(A) \neq 0$, the previous equation always has a solution $\hat{x}$ for any given $\bar{x} \in \mathbb{R}^n$.

# Final remarks

A continuous-time LTI system is always reversible, meaning that $\det\left(e^{At}\right) \neq 0$ for any $A \in \mathbb{R}^{n\times n}$.




If $\exists\, u(t) \in [0, T]$ transferring $x_0$ into $x(T)$, then there is $u'(t) \in [0, T]$ transferring $x(T)$ into $x_0$.

- Controllability and reachability coincide for continuous-time LTI systems
- Discrete-time LTI systems are substantially different