Lecture 2

Stability, reachability and controllability

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Section 1

Modes and stability of LTI systems

Modes and stability of LTI systems

$$\begin{cases} x^+ = Ax + Bu \\ x(0) = x_0 \end{cases} \tag{1}$$

$$(\bar{x}, \bar{u}) \text{ equilibrium}$$

- Recall: stability of the equilibrium state $\bar{x}$.

Definition (Lyapunov stability)

The equilibrium state $\bar{x}$ is

- stable if $\forall \epsilon > 0 \ \exists \delta > 0 : \|\tilde{x}_0 - \bar{x}\| \leq \delta \Rightarrow \|\tilde{x}(k) - \bar{x}\| < \epsilon, \forall k \geq 0$
- (globally) asymptotically stable (AS) if it is stable and attractive, i.e.,

$$\lim_{k \to \infty} \|\tilde{x}(k) - \bar{x}\| = 0, \ \forall \tilde{x}_0 \in \mathbb{R}^n$$

- unstable if not stable

Modes and stability of LTI Systems

- Key quantity to analyse: the error

$$e(k) = \tilde{x}(k) - \bar{x}$$

Proposition

Set $e_0 = e(0) = \tilde{x}(0) - \bar{x}$. The error verifies

$$\begin{aligned} e^+ &= Ae \\ e(0) &= e_0 \end{aligned} \tag{2}$$

Modes and stability of LTI systems

Proof

Note that $e(k) = \alpha\tilde{x}(k) + \beta\bar{x}(k)$ with $\alpha = 1$ and $\beta = -1$. From the superposition principle,

$$e(k) = \tilde{x}(k) - \bar{x}(k) = \phi(k, 0, \tilde{x}_0 - \bar{x}, \underbrace{\bar{u} - \bar{u}}_{0})$$

Since ϕ is the transition map of (1), the error satisfies (1) for zero input. This is (2).

Proof (alternative)

Since $\bar{x}$ verifies $\bar{x}^+ = \bar{x} = A\bar{x} + B\bar{u}$, compute $e^+ = x^+ - \bar{x}^+$ explicitly and obtain (2).

Remarks

- Stability/AS of $\bar{x}$ is the same as stability/AS of $\bar{e} = 0$ for (2).
  Check: stability of $\bar{e} = 0$ means

  $$\forall \varepsilon > 0,\ \exists \delta :\ \|e_0 - 0\| < \delta \implies \|e(k) - 0\| < \varepsilon,\ \forall k \geq 0$$

  which, by using $e(k) = x(k) - \bar{x}$ coincides with the definition of stability for $\bar{x}$.
- (2) is independent of $\bar{u}$ and $\bar{x}$. This proves the following theorem.

Theorem

An equilibrium state of an LTI system is stable/AS/unstable ***if and only if*** *all other equilibria have the same properties.*

This is why we can say, for an LTI system, that « the system is stable ».

Stability and free states

For stability analysis, setting $u(k) = 0$ in $x^+ = Ax + Bu$ is not conservative. $\implies$ stability depends only on free states.

Theorem

An LTI system

1. *is stable* $\iff$ *all free states are* ***bounded***
2. *is AS* $\iff$ *all free states are* ***bounded and go to zero as*** $k \to +\infty$
3. *is unstable* $\iff$ *there is a free state which is* ***unbounded***

Stability and eigenvalues of A

Each free state $x(k) = A^k x_0$ is a linear combination of the modes of A.

eigenvalues	modes
$\lambda_i \in \mathbb{R}$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \ldots, \eta_i - 1$
$\lambda_i = \rho_i e^{j\theta_i}$ AND $\lambda_h = \lambda_i^*$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i}\rho_i^{k-p_i}\sin(\theta_i(k-p_i)+\varphi_i) & \text{for } k \geq p_i \end{cases}$ $p_i = 0, 1, \ldots, \eta_i - 1$

Stability and eigenvalues of A

Recall the macroscopic behaviour of modes

Lemma

- If $|\lambda_i| < 1$, all modes associated to λ_i are **bounded** and **go to zero** as $k \to +\infty$.
- If $|\lambda_i| > 1$, all modes associated to λ_i are **unbounded**.
- If $|\lambda_i| = 1$ and $\nu_i = n_i$, all modes associated to λ_i are **bounded**.
- If $|\lambda_i| = 1$ and $\nu_i < n_i$, there's an **unbounded** mode associated to λ_i

- **Terminology**: « eigenvalues of A » = « eigenvalues of the system »
- Combining the previous Lemma and Theorem we have the three Theorems given next

Stability and eigenvalues of A

Theorem (test of AS)

An LTI system is AS ***if and only if*** *all its eigenvalues have modulus* < 1.

Theorem (test of instability)

An LTI system is unstable ***if and only if*** *one of the following conditions occurs.*

1. *A system eigenvalue has modulus* > 1.
2. *All system eigenvalues have modulus* ≤ 1 *and there is an eigenvalue* λ_i *with modulus* $= 1$, *algebraic multiplicity* $n_i \geq 2$ *and* $\dim(V_{\lambda_i}) < n_i$.

Stability and eigenvalues of A

Theorem (test of simple stability)

An LTI system is simply stable ***if and only if*** *all its eigenvalues have modulus* ≤ 1 *and, for each eigenvalue* λ_i *with modulus 1 and algebraic multiplicity* $n_i \geq 2$*, one has*

$$dim(V_{\lambda_i}) = n_i$$

- **Recall:** $\dim(V_{\lambda_i})$ is the geometric multiplicity of λ_i.
- **Remark**: AS is the most important property in engineering applications
- **Terminology**: we say that « A is Schur » if all its eigenvalues have modulus < 1.

Exponential Stability

Recall that the equilibrium $\bar{x}$ of $x^+ = Ax$ is **exponentially stable** if there are $\alpha > 0, \rho \in [0,1)$ such that

$$\|\tilde{x}(k) - \bar{x}\| \leq \alpha\rho^k \|\tilde{x}_0 - \bar{x}\|, \quad \forall \tilde{x}_0 \in \mathbb{R}^n$$

Lemma

An LTI systems is AS **if and only if** it is ES.

Sketch of the Proof

AS $\iff$ all modes of the system go to zero as $k \to \infty$. But if a mode goes to zero, it does so exponentially fast. This implies that $\exists\ \alpha, \rho$ verifying the definition of ES

Examples

Analyse the stability of $x^+ = Ax + Bu$.

For $x \in \mathbb{R}^2$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \implies \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 0 \end{cases} \implies \text{unstable}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases} \implies \text{stable but not AS}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \implies \lambda_1 = \lambda_2 = 1 \implies \text{alg. multiplictity } n_1 = 2$$

$$V_1 = \left\{ v : A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} = \left\{ v : \begin{array}{r} v_1 + v_2 = v_1 \\ v_2 = v_2 \end{array} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$

$$\implies \dim(V_1) = 1 < n_1, \text{ therefore the system is unstable}$$

Section 2

Reachability and controllability

Key properties of dynamical systems

Reachability

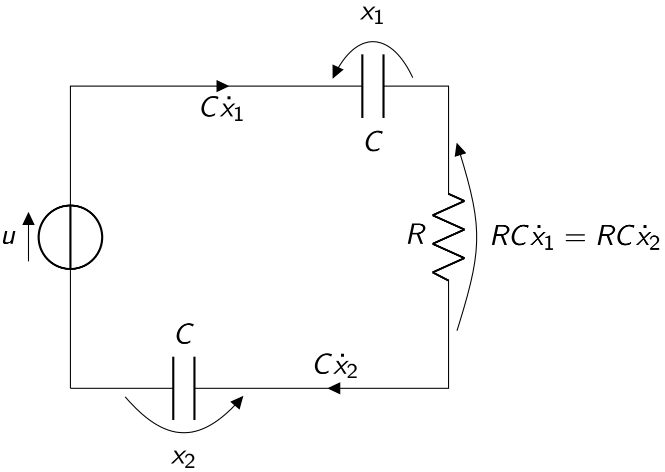
Is it possible to steer the state $x_0 = 0$ **to a desired value** by acting on the inputs?

Controllability

Is it possible to steer the state $x_0 \in \mathbb{R}^n$ **to the origin** by acting on the inputs?

Reachability

Example



$$C = 1$$
$$R = 1$$

Reachability

Example

Model

$$\begin{cases} u = x_1 + x_2 + \dot{x}_1 \\ u = x_1 + x_2 + \dot{x}_2 \\ y = x_1 \end{cases}$$

Discretization

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x(k+1) - x(k)}{T}, \quad T = 0.1$$

$$\Longrightarrow \begin{cases} x_1^+ = -9x_1 - 10x_2 + 10u \\ x_2^+ = -10x_1 - 9x_2 + 10u \\ y = x_1 \end{cases}$$

Can we modify x_1 independently of x_2? It seems not: in the CT model the currents in the upper and lower branches are identical.

Change of coordinates to highlight this phenomenon

$$\hat{x} = Tx, \quad T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \implies \begin{cases} \hat{x}_1 = x_1 - x_2 \\ \hat{x}_2 = x_1 + x_2 \end{cases}$$

$$\hat{x}^+ = TAT^{-1}\hat{x} + TBu, \quad T^{-1} = \begin{bmatrix} +0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

By direct calculation:

$$\begin{aligned} \hat{x}_1^+ &= \hat{x}_1 \\ \hat{x}_2^+ &= -19\hat{x}_2 + 20u \end{aligned}$$

The difference of the voltages is constant and it cannot be affected by the input. $\implies$ states $\tilde{x} \in \mathbb{R}^2$ with $\tilde{x}_1 - \tilde{x}_2 \neq x_1(0) - x_2(0)$ cannot be reached.

Reachability: definitions

$$x^{+} = Ax + Bu \tag{3}$$

$$y = Cx + Du \tag{4}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$

Definition

A state $\tilde{x}$ is **reachable** if $\exists\, \tilde{k} > 0$ and $\tilde{u}(k),\ k = 0, 1, \ldots, \tilde{k}$ such that

$$x(\tilde{k}) = \phi(\tilde{k}, 0, 0, \tilde{u}) = \tilde{x} \tag{5}$$

If all states are reachable, then the system is termed « reachable ».

Reachability: definitions

Remarks

- Reachability = reachability from the origin as $x(0) = 0$ in (5)
- Reachability = property of the pair (A, B) only
- Problem: Difficult to check if a system is reachable using the definition (infinitely many $\tilde{x}$ should be tested)

Reachability test

Definition

The reachability matrix is defined as

$$M_r = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times mn} \tag{6}$$

Remark: powers of A from 0 to $n-1$ only.

Theorem

1. *$\tilde{x} \in \mathbb{R}^n$ is reachable only if it belongs to*

$$X_r = span(M_r) \tag{7}$$

   where $span(M_r)$ is the subspace spanned by the columns of M_r.

2. *If $\tilde{x}$ is reachable, it can be reached in $\tilde{k} \leq n$ steps.*
3. *The system is reachable if and only if $rank(M_r) = n$*

Reachability test

Remarks

- For LTI systems, reachability is a finitely determined property.
- The orthogonal subspace $X_r^{\perp}$ is termed the « unreachable subspace ».
- The point 3 in the above theorem is a maximal rank condition

Reachability test

Proof

Setting $x(0) = 0$ one has

$$
\begin{aligned}
x(1) &= Bu(0) \implies \text{lin. comb. of columns of } B \\
x(2) &= Bu(1) + ABu(0) = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \\
&\implies \text{lin. comb. of columns of } \begin{bmatrix} B & AB \end{bmatrix} \\
&\vdots \\
x(k) &= \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}
\end{aligned}
$$

Reachability test

Proof cont.

From the theorem of Cayley-Hamilton, if $\psi(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n$ is the characteristic polynomial of A, then $\psi(A) = 0$, *i.e.*

$$A^n = -(\alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \ldots + \alpha_{n-1} A + \alpha_n I)$$

Therefore, the columns of $A^n B$ are a linear combination of the columns of matrices $A^i B,\ i = 0, 1, \ldots, n-1$.
This shows that a state is reachable only if it can be reached in at most n steps and that the set of reachable states is given by (7).

Unreachable systems

If the system is not reachable, the unreachable part can be isolated.

Theorem

Let $n_r = rank(M_r) \geq 1$. *There is a suitable and non-unique change of state coordinates*

$$\hat{x}(k) = T_r x(k), \quad \det(T_r) \neq 0$$

such that $x^+ = Ax + Bu$ *is equivalent to*

$$\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u$$

where

$$\hat{A} = \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix} \qquad \hat{A}_a \in \mathbb{R}^{n_r \times n_r}$$

$$\hat{B} = \begin{bmatrix} \hat{B}_a \\ 0 \end{bmatrix} \qquad \hat{B}_a \in \mathbb{R}^{n_r \times m}$$

$$rank\left(\begin{bmatrix} \hat{B}_a & \hat{A}_a\hat{B}_a & \cdots & \hat{A}_a^{n_r-1}\hat{B}_a \end{bmatrix}\right) = n_r \tag{8}$$

Terminology: $(\hat{A}, \hat{B})$ is called the « reachability form » of (A, B)

Unreachable systems

The zero blocks in $(\hat{A}, \hat{B})$ reveal the **unreachable** part. Setting $\hat{x} = \begin{bmatrix} \hat{x}_a^T & \hat{x}_b^T \end{bmatrix}^T$, $\hat{x}_a \in \mathbb{R}^{n_r}$, we have

$$\hat{x}_a^+ = \hat{A}_a \hat{x}_a + \hat{A}_{ab} \hat{x}_b + \hat{B}_a u \tag{9}$$

$$\hat{x}_b^+ = \hat{A}_b \hat{x}_b \tag{10}$$

- (9) is the reachable part. Since $(\hat{A}_a, \hat{B}_a)$ is reachable, one can steer $\hat{x}_a$ to an arbitrary position; (*c.f.* (8)).
- (10) is the unreachable part: $\hat{x}_b$ is not affected by u, neither directly, nor through $\hat{x}_a$.
- Terminology: the eigenvalues of $\hat{A}_a$ are termed « reachable ». Those of $\hat{A}_b$, « unreachable ». The same for the corresponding modes.

How to build T_r?

- Build $M_r = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, $\operatorname{rank}[M_r] = n_r$. Let $v_1, v_2, \ldots, v_{n_r}$ be linearly independent columns of M_r.
- Build $T_r^{-1} = \left[\begin{array}{ccc|ccc} v_1 & \cdots & v_{n_r} & z_1 & \cdots & z_{n-n_r} \end{array}\right]$, where z_i are arbitrary vectors guaranteeing that $\det\left(T_r^{-1}\right) \neq 0$.

Example (ctd)

$$x^+ = \begin{bmatrix} -9 & -10 \\ -10 & -9 \end{bmatrix} x + \begin{bmatrix} 10 \\ 10 \end{bmatrix} u$$

$$M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 10 & -190 \\ 10 & -190 \end{bmatrix}, \quad n_r = 1$$

$$T_r^{-1} = \left[\begin{array}{c|c} 10 & -10 \\ 10 & 10 \end{array}\right] \quad \Longrightarrow \quad T_r = \frac{1}{20}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = T_r A T_r^{-1} = \begin{bmatrix} -19 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{B} = T_r B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Section 3

Controllability

Controllability

$$x^+ = Ax + Bu \tag{11}$$

$$y = Cx + Du \tag{12}$$

Definition

A state $\hat{x}$ is **controllable** if $\exists\, \hat{k} > 0$ and $\hat{u}(k),\ k = 0, 1, \dots, \hat{k}$ such that

$$0 = \phi(\hat{k}, 0, \hat{x}, \hat{u})$$

If all states are controllable, then the system is termed «controllable».

Remarks

- Controllability = controllability to the origin
- Property of (A, B) only

Controllability

Controllability and reachability: do they coincide?

Example

$$x^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad \begin{array}{l} x(k) \in \mathbb{R}^2 \\ u(k) \in \mathbb{R} \end{array}$$

- Every state $\hat{x}$ is controllable using $u(\cdot) = \hat{u}(\cdot) = 0$

$$x(0) = \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \implies x(1) = \begin{bmatrix} \hat{x}_2 \\ 0 \end{bmatrix} \implies x(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The state $\tilde{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ cannot be reached by $x(0) = 0$.

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x(1) = \begin{bmatrix} u(0) \\ 0 \end{bmatrix} \implies x(2) = \begin{bmatrix} u(1) \\ 0 \end{bmatrix} \implies \ldots$$

- Intuition: all eigenvalues of A are zero $\implies$ free states go naturally to zero.

Controllability and reachability: do they coincide?

Lemma

One has:

1. (A, B) reachable $\implies$ (A, B) controllable
2. if $\det(A) \neq 0$:

$$(A, B) \text{ controllable} \implies (A, B) \text{ reachable}$$

Remark

LTI systems with $\det(A) \neq 0$ are termed **reversible**.

Controllability and reachability: do they coincide?

Controllability and reachability: do they coincide?

Proof of (i)

For $x(0) = \hat{x}$, setting $\boldsymbol{u}^k = \begin{bmatrix} u^T(k-1) & \cdots & u^T(0) \end{bmatrix}^T$, one has

$$x(k) = A^k \hat{x} + \underbrace{\begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix}}_{M_r^k} \boldsymbol{u}^k$$

By setting $x(k) = 0$ one has

$$\hat{x} \text{ is controllable} \Leftrightarrow \exists \boldsymbol{u}^k \text{ such that} - A^k \hat{x} = \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \boldsymbol{u}^k \tag{13}$$

Formula (13) is also the same as requiring that $-A^k \hat{x}$ is reachable from the origin in k steps. Equivalently,

$$- A^k \hat{x} \in \underbrace{\text{span}(M_r^k)}_{X_r^k} \tag{14}$$

If (A, B) is reachable, then $X_r^n = \mathbb{R}^n$ and (14) is verified for all $\hat{x} \in \mathbb{R}^n$ if $k = n$. This proves point (i).

Controllability and reachability: do they coincide?

Proof of (ii)

As for point (ii), by assumption, for any $\hat{x} \in \mathbb{R}^n$, $\exists\, k, \boldsymbol{u}^k$ s.t.

$$-A^k \hat{x} = M_r^k \boldsymbol{u}^k$$

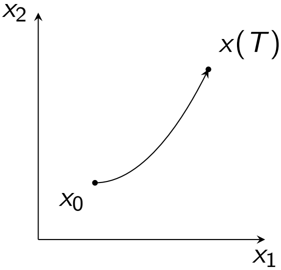
Therefore, any $\bar{x}$ that can be written as

$$\bar{x} = -A^k \hat{x} \quad \text{for some } \hat{x}$$

is reachable. But since $\det(A) \neq 0$, the previous equation always has a solution $\hat{x}$ for any given $\bar{x} \in \mathbb{R}^n$.

Final remarks

A continuous-time LTI system is always reversible, meaning that $\det\left(e^{At}\right) \neq 0$ for any $A \in \mathbb{R}^{n \times n}$.



If $\exists\, u(t) \in [0, T]$ transferring x_0 into $x(T)$, then there is $u'(t) \in [0, T]$ transferring $x(T)$ into x_0.

- Controllability and reachability coincide for continuous-time LTI systems
- Discrete-time LTI systems are substantially different