Lecture 1

Introduction to Multivariable Control

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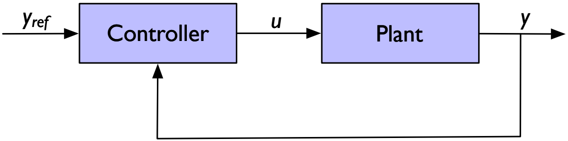
Outline

- Motivations and course topics
- Course organization, supporting material, exams, ...
- Review of system theory

Classic feedback control



y_{ref} : setpoint
u : input
y : output

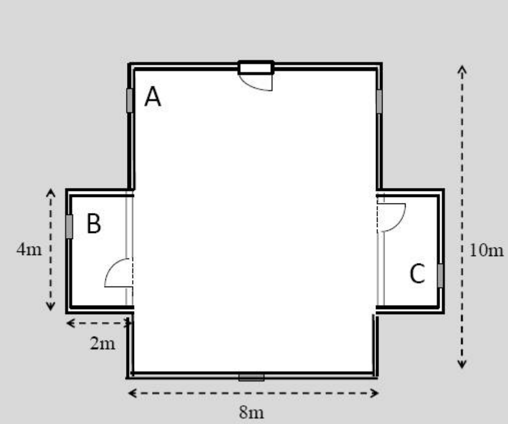
The block diagram

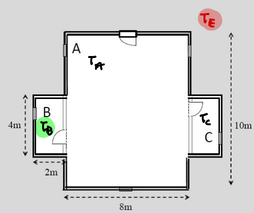
- Summarizes relations between variables
- First class in control theory: y_{ref}, u, y are scalar
  - Single-Input Single-Output (SISO) plant

Always appropriate ?

Example: Green building

Problem: model the evolution of temperatures in the rooms





Model

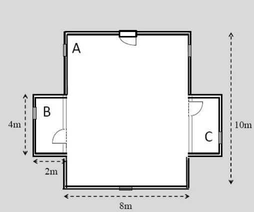
$$
\begin{aligned}
c\rho V_A \dot{T}_A &= s_r\omega(T_B - T_A) + s_r\omega(T_C - T_A) + s_A\Omega(T_E - T_A) + u_A \\
c\rho V_{B,C} \dot{T}_B &= s_r\omega(T_A - T_B) + s_{B,C}\Omega(T_E - T_B) + u_B \\
c\rho V_{B,C} \dot{T}_C &= s_r\omega(T_A - T_C) + s_{B,C}\Omega(T_E - T_C) + u_C
\end{aligned}
$$

Variables:

$T_i,\ i = A, B, C$	temperature in room i (states)
$u_i,\ i = A, B, C$	heating/cooling powers (inputs)

Parameters:

T_E	external temperature (assumed to be zero)
$c, \rho > 0$	specific heat capacity and density of the air
$V_i > 0,\ i = A, B, C$	volume of room i
$s_r > 0$	wall surface between A and B (and C)
$s_A > 0$	wall surface between A and the environment
$s_{B,C} > 0$	wall surface between B (and C) and the environment
$\omega > 0, \Omega > 0$	transmittances (internal and perimeter wall)



Model

$$\begin{bmatrix} \dot{T}_B \\ \dot{T}_A \\ \dot{T}_C \end{bmatrix} = \begin{bmatrix} -(\Gamma+\gamma_r) & \Gamma & 0 \\ \gamma & -(2\gamma+\gamma_A) & \gamma \\ 0 & \Gamma & -(\Gamma+\gamma_r) \end{bmatrix} \begin{bmatrix} T_B \\ T_A \\ T_C \end{bmatrix} + \begin{bmatrix} \frac{1}{c\rho V_{B,C}} & 0 & 0 \\ 0 & \frac{1}{c\rho V_A} & 0 \\ 0 & 0 & \frac{1}{c\rho V_{B,C}} \end{bmatrix} \begin{bmatrix} u_B \\ u_A \\ u_C \end{bmatrix}$$

$\gamma = \frac{s_r \omega}{c\rho V_A}$, $\Gamma = \frac{s_r \omega}{c\rho V_{B,C}}$, $\gamma_A = \frac{s_A \Omega}{c\rho V_A}$, and $\gamma_r = \frac{s_{B,C} \omega}{c\rho V_{B,C}}$.

- 3 inputs and 3 outputs (temperatures in the rooms)
- coupling between inputs and outputs through thermal diffusion
  - changing a control input impacts on the temperature in each room
  - need of considering all scalar inputs and outputs simultaneously
    - ... or a single input and output but *multivariable*

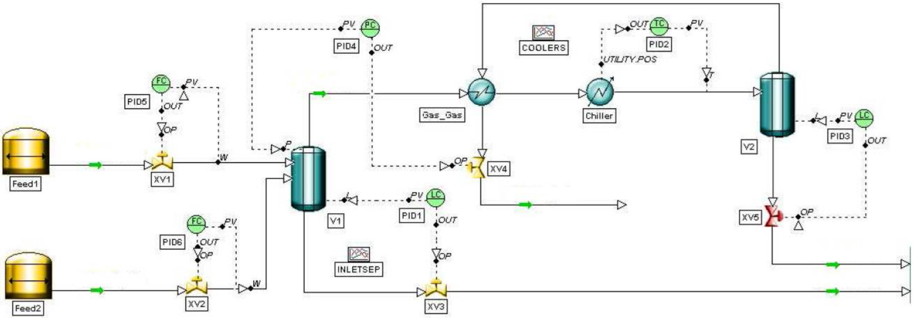
Example: natural gas refrigeration plant

Introduction

- **Natural gas:** key source for the production of energy and chemical commodities
- Gas processing:
  - remove any impurity
  - separate and purify some of its components
- Processing plant typically structured in three sections:
  1. refrigeration
  2. distillation
  3. storage
- We focus on the **refrigeration section**, aiming at separating the *undesirable incondensable* gas components (nitrogen and carbon dioxide, but also methane and a good portion of ethane) from the *desired ones* (propane and butanes).

Example: Natural gas refrigeration plant

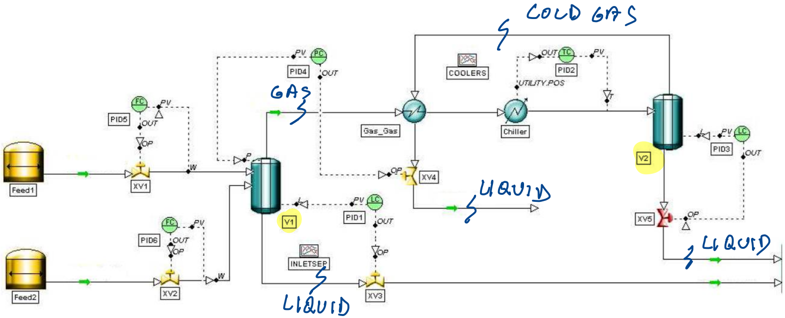
The plant



[M. Farina, G.P. Ferrari, F. Manenti, B. Pizzi, 2016]

Example: Natural gas refrigeration plant

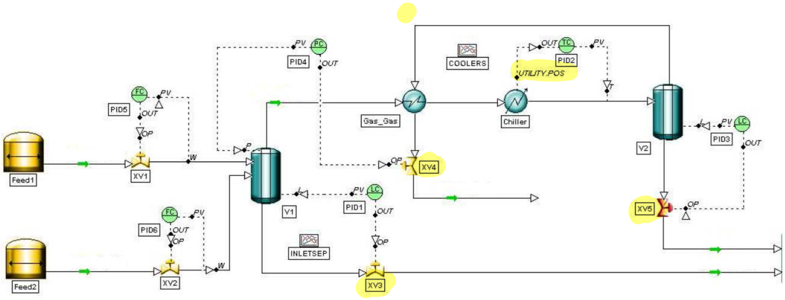
The plant



- The plant is fed by **two streams of natural gas**, mixed up into a **inlet separator** (V1). The heaviest components condensate and separate from incondensable gases. The gas phase exiting V1 still contains amount of propane and butanes
- It passes through a gas/gas **heat exchanger** where it encounters gases already refrigerated, coming from the **LTS** (Low Temperature Separator) V2
- The stream achieves, by means of a **chiller**, -15° C, which allows the almost complete recovery of propane and butanes, which condensate in V2

Example: Natural gas refrigeration plant

The variables



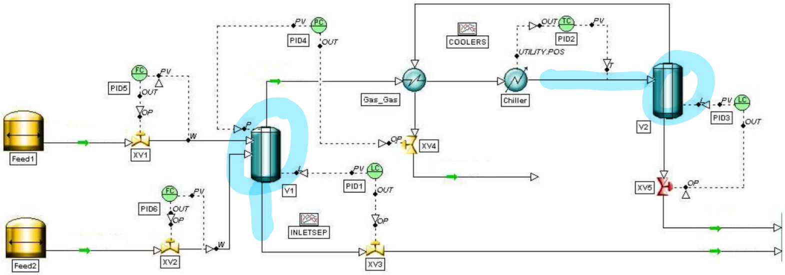
Scalar input variables (valve opening levels):

- *XV3*: opening of the valve regulating the liquid flowrate pouring out from the inlet separator V1
- *XV4*: opening of the valve regulating the pressure in V1[1]
- *UTILITY.POS*: opening of the valve regulating the coolant flowrate in the chiller
- *XV5*: opening of the valve regulating the liquid flowrate pouring out from the LTS separator

[1] Because the gas mass is conserved in the heat exchanger

Example: Natural gas refrigeration plant

The variables

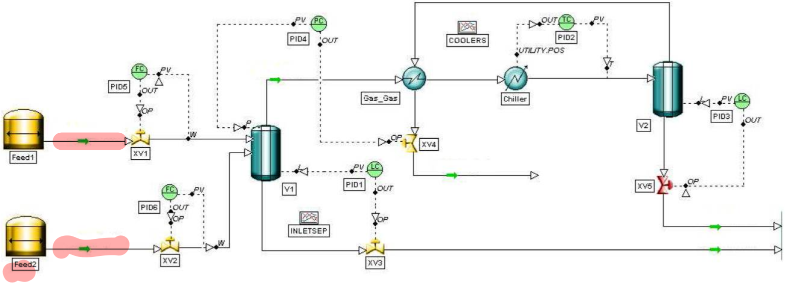


Output variables

- *InletSep.L*: liquid holdup in the inlet separator
- *InletSep.P*: pressure in the inlet separator
- *s8.T*: temperature of the liquid downstream the chiller
- *LTS.L*: liquid holdup in the LTS separator

Example: Natural gas refrigeration plant

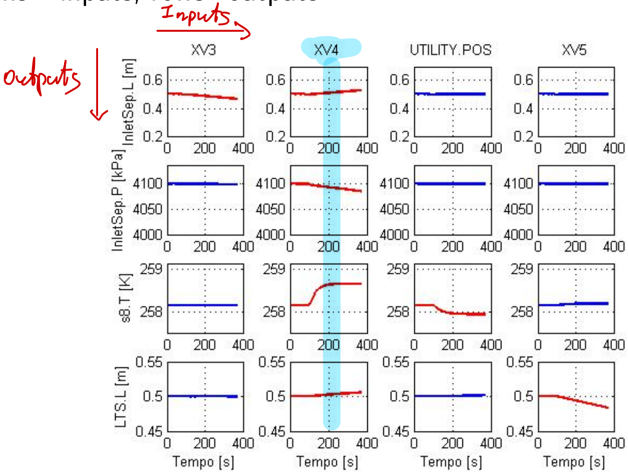
The variables



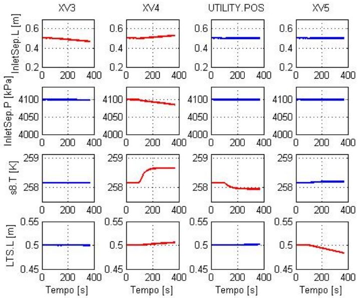
Disturbance: inlet mass flowrate of natural gas from each of the two gas deposits

Experimental step responses

Columns= inputs, rows=outputs



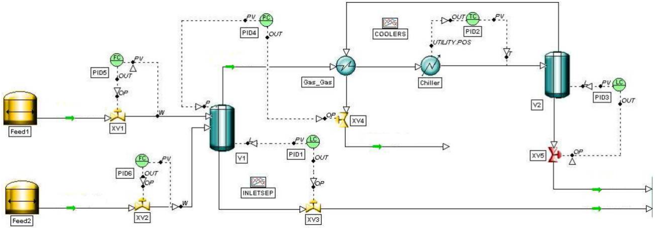
Step responses - coupling



Relevant couplings in red

- XV3 influences InletSep.L only
- UTILITY.POS influences s8.T only
- XV5 influences LTS.L only
- **... but XV4 influences every output**

Need of considering all scalar inputs and outputs simultaneously



State-space model of the refrigeration plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t) \\ y(t) &= Cx(t) \end{aligned} \quad \text{where} \quad \begin{cases} x \in \mathbb{R}^{13} \\ u \in \mathbb{R}^{4} \\ y \in \mathbb{R}^{4} \\ w \in \mathbb{R}^{2} \text{ (disturbance)} \end{cases}$$

Multiple-Input Multiple-Output (MIMO) system

Cyberphysical systems in modern technology

- Enabled by Internet of Things, Industry 4.0 *etc.*
- Naturally MIMO
- Naturally interacting!
  → need of multivariable control

Course focus

Analysis and control of multivariable discrete-time state-space models

Why discrete time?

- Standard control technologies: microcontrollers, control stations *etc.*

→ Receive/send discrete-time electric signals

- **Common practice:** discretize the plant and design a discrete-time controller
- However, most of the concepts also apply to continuous-time systems

Why state-space?

Transfer functions

- exist only for linear systems
- are I/O models → missing description of internal variables
- nontrivial to analyze and manipulate in the multivariable case

Course focus

Linear systems

- For simplicity
- Linearisation about an equilibrium
- Linear models are essential for understanding **nonlinear systems**

Course topics

Part 1: Analysis of multivariable systems

- Introduction to linear discrete-time systems in the state-space
- Stability and modal analysis
- Reachability and observability
- Discretization of continuous-time systems

Part 2: Control of multivariable systems

- Eigenvalue assignment
- Luenberger observers
- Offset-free tracking
- Optimal control: the Linear Quadratic Regulator (LQR)
- Optimal state estimation: the Kalman filter (KF)
- Linear Quadratic Gaussian Control (LQG)
- Distributed LQR

Course organization, supporting material, exams

Course information

- Professor: Giancarlo Ferrari Trecate, Room ME C2 398, `giancarlo.ferraritrecate@epfl.ch`
- Lectures: Tue 9:00-10:00 and 10:00-11:00 CE 1 100
  - **Course slides on Moodle, videos of 2021 available**
  - Probably, a couple ~~two~~ of lectures will be exceptionally pre-recorded. This will be properly notified on Moodle in advance
- Exercises: *Session A*: Tue 11:00-12:00. *Session B*: Tue 12:00-13:00. Both sessions in CE 1 100. Matlab/Simulink required!
  - Session A
    - Focus more on theory
    - This week, no session A but lecture until 12:00
  - Session B
    - Focus more on applications
    - This week, session B will cover Simulink and MatLab Live Script, *both essential for the graded group assignments*

Course information

- Assistants:
  Simone Baratto, Laura Meroi, Jakob Nylöf, Julien Pallage




- Forums
  - Students can post questions anytime on the 'Discussions' forum. Students can also (and are encouraged to!) answer their colleagues. The TAs will check once a week.

... and the teaching team can be always contacted via email!

Exams and grading

Final grade given by graded group assignments (10%) and a final written exam (90%)

Graded group assignments

- 3 sessions during the semester, each replacing the corresponding exercise session B from 12h00 to 13h00.
  - **Dates: 7/10/25, 11/11/25, and 9/12/25.**
- Each assignment will focus on the exercise sessions since the last assignment
- You will work in groups of 3 people, formed at the beginning of the semester (details will follow on Moodle)
- The test is open books, open notes. You will be required to submit the solutions as Matlab Live Scripts, Simulink models, and .m files

Exams and grading

Final exam (90% of the final grade)

Written exam: 2 hours

- 5 sections, 1 multiple choice - example copy available on Moodle
- Closed book, closed notes, **no computers**. Bring with you a pen, an eraser, an ID and a non-programmable calculator
- You are also permitted to bring **one crib sheet**, formatted on A4 paper. The sheet must be **handwritten only** (no tablet-generated content or copies of the slides), and you may use both sides
- Each problem will give a maximal number of points, clearly indicated. The total is 90 points. Example (NOT the real numbers):

Problem	1	2	3	4	5	Total
Value	20	20	15	15	20	90
Grade						

- **Final grade (graded group assignments + final exam):**

Points	96-100	91-95	···	56-60	51-55	···	6-10	1-5	0
Grade	6.00	5.75	···	4.00	3.75	···	1.50	1.25	1.00

Literature

- Supporting textbooks (none is required)
  - G.F. Franklin, J.D. Powell & M. Workman, *Digital Control of Dynamic Systems*, 3rd edition, Addison-Wesley, 1997
  - K. Ogata, *Discrete-Time Control Systems*, 1st edition, Prentice-Hall, 1987
  - H. Kwakernaak & R. Sivan, *Linear Optimal Control Systems*, Available online
  - J. P. Hespanha, *Linear Systems Theory*, 2nd edition, Princeton University Press, 2018

Review of system theory

Dynamical systems in discrete time

Nonlinear (NL) state-space models

$$x(k+1) = f(x(k), u(k), k) \quad (1)$$
$$y(k) = g(x(k), u(k), k) \quad (2)$$
$$x(k_0) = x_0 \quad (3)$$

$x(k) \in \mathbb{R}^n$ state

$u(k) \in \mathbb{R}^m$ input

$y(k) \in \mathbb{R}^p$ output

$\begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix}$

- (1): state equation
- (2): output equation
- n: system order
- $k \in \mathbb{N}$: Discrete Time (DT)
- (1)-(3) is invariant if f and g do not depend on time

Definition

A state trajectory is a function $x(k), k \geq k_0$ verifying (1) and (3). For highlighting the dependence on the input, initial time and initial states, we write $x(k) = \phi(k, k_0, x_0, u)$ and ϕ is called *transition map*

$u \to \{u(k_0) \ldots u(k-1)\}$

Review - linear systems

A system is *linear* if f and g are linear functions of x and u

Linear time-varying (LTV) system

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k) \\
y(k) &= C(k)x(k) + D(k)u(k)
\end{aligned}
\qquad
\begin{aligned}
& x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p \\
& A(k),\ B(k),\ C(k),\ D(k) \text{ matrices}
\end{aligned}
$$

Linear Time-Invariant (LTI) system

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
\qquad A,\ B,\ C,\ D \text{ matrices}
$$

- Alternative notation: $x_{k+1} = x(k+1)$
- Drop the k and define $x^+ = x_{k+1} \rightarrow$ LTI state equation: $x^+ = Ax + Bu$
- Initial time k_0:
  - for LTV models, k_0 is important
  - for LTI models, one can set $k_0 = 0$ without loss of generality

$x(k) \in \mathbb{R}$ $\quad x(k+1) = k\, x(k)$

↳ Exp 1: $k_0 = 0$, $x_0 = 1 \rightarrow x(1) = 0$, $x(2) = 0$, …, $x(k) = 0$

↳ Exp 2: $k_0 = 100$, $x_0 = 1 \rightarrow x(101) = 100$, $x(102) = 101 \cdot 100$

Review - linear systems

A system is linear if f and g are linear functions of x and u

Linear time-varying (LTV) system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

$x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$

$A(k),\ B(k),\ C(k),\ D(k)$ matrices

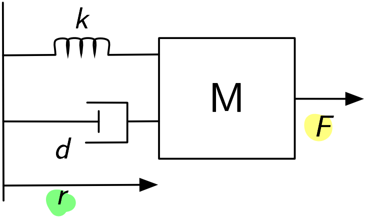
Linear Time-Invariant (LTI) system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

$A,\ B,\ C,\ D$ matrices

- Single-Input Single-Output (SISO) system if $m = p = 1$
- Multivariable or Multi-input Multi-Output (MIMO) system otherwise
- An LTI system is often denoted by the tuple (A, B, C, D)

Example - mass/spring/damper



- $k > 0$: elastic coefficient
- $d > 0$: damping coefficient
- F: external force (input)
- r: position (output)

Set $x_1 = r, x_2 = \dot{r}, u = F, y = x_1 \rightarrow M\ddot{x}_1 = -kx_1 - d\dot{x}_1 + u$

For M=1, d=1, k=1

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + u \\ y &= x_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

Example ctd. - time discretization

- uniform sampling at $t_k = kT$, ($T > 0$: sampling time)
- define $x(k)$, $y(k)$, $u(k)$ as $x(t_k)$, $y(t_k)$, $u(t_k)$
- discretization rule $\left.\frac{dx}{dt}\right|_{t=t_k} \simeq \frac{x(k+1)-x(k)}{T}$ (Euler)

$\dot{x} = Ax + Bu$

$\frac{x(k+1)-x(k)}{T} = Ax(k) + Bu(k)$

DT model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{(TA+I)}_{A_D} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{TB}_{B_D} u(k)$$

$$y(k) = Cx(k) + Du(k)$$

Approximation of the continuous-time dynamics

$$A_D = \begin{bmatrix} 1 & T \\ -T & -T+1 \end{bmatrix} \quad B_D = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

C, D unchanged

- DT LTI SISO system

2h 41

Linear systems: superposition principle

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

For $\alpha, \beta \in \mathbb{R}$, let

- $x_a(k) = \phi(k, k_0, x_{0,a}, u_a)$ and $y_a(k)$ the corresponding output
- $x_b(k) = \phi(k, k_0, x_{0,b}, u_b)$ and $y_b(k)$ the corresponding output
- $x(k) = \phi(k, k_0, \alpha x_{0,a} + \beta x_{0,b}, \alpha u_a + \beta u_b)$ and $y(k)$ the corresponding output

Then, $\forall k \geq k_0$

- $x(k) = \alpha x_a(k) + \beta x_b(k)$
- $y(k) = \alpha y_a(k) + \beta y_b(k)$

LTI systems: Lagrange formula

How the transition map looks like for an LTI system?

$$x(k+1) = Ax(k) + Bu(k) \qquad\qquad x(0) = x_0$$
$$y(k) = Cx(k) + Du(k)$$

By recursive substitution one has

$$x(1) = Ax_0 + Bu(0)$$
$$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$
$$x(3) = Ax(2) + Bu(2) = \cdots = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)$$

- $$x(k) = \phi(k, 0, x_0, u) = \underbrace{A^k x_0}_{\phi(k,0,x_0,0)=free\ \ state} + \underbrace{\sum_{i=0}^{k-1} A^{(k-i-1)}Bu(i)}_{\phi(k,0,0,u)=forced\ \ response}$$

  $\rightarrow k=3 : A^2Bu(0) + ABu(1) + Bu(3)$

- $$y(k) = \underbrace{CA^k x_0}_{free\ \ output} + \underbrace{C\sum_{i=0}^{k-1} A^{(k-i-1)}Bu(i) + Du(k)}_{forced\ \ output}$$

- Easy to generalize to an initial time $k_0 \neq 0$ (just more complex formulas)

Equilibria of LTI systems

Given $u(k) = \bar{u}$, $k \geq 0$, the vector $\bar{x} \in \mathbb{R}^n$ is an *equilibrium state* for $x^+ = Ax + Bu$ if

$$A\bar{x} + B\bar{u} = \bar{x} \implies (A - I)\bar{x} + B\bar{u} = 0$$

and the pair $(\bar{x}, \bar{u})$ is called *an equilibrium*

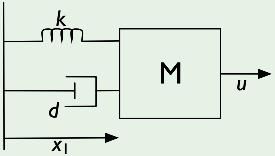
For an LTI system

- $\bar{u} = 0$, $\bar{x} = 0$ is always an equilibrium
- if $\bar{u} \in \mathbb{R}^m$, there might be one/none/infinitely many equilibria

Equilibria of LTI systems

Example

Example - mass spring damper with $k = d = 0$, $M = 1$



$$\begin{cases} x_1^+ = x_1 + T x_2 \\ x_2^+ = 0x_1 + x_2 + Tu \end{cases}$$

$\rightarrow A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ T \end{bmatrix}$

$$(A - I) = T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(A - I)\bar{x} = -B\bar{u} \implies \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ T \end{bmatrix} \bar{u} \rightarrow \begin{cases} T\bar{x}_2 = 0 \\ 0 = T\bar{u} \end{cases}$$

- $\bar{u} = 0 \implies$ all $\bar{x} = [\alpha \ 0]^T$, $\alpha \in \mathbb{R}$ are equilibrium states
- $\bar{u} \neq 0 \implies$ no $\bar{x} \in \mathbb{R}^2$ is a equilibrium state