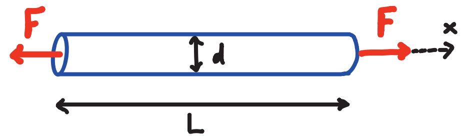
Lecture 1(c): Traction, Bending, Torsion and Buckling of Beams

Slender structures:



$L \gg d$

(1D) Axial Load/Traction of Bar:

$$\boxed{F = EA\frac{du}{dx}}$$

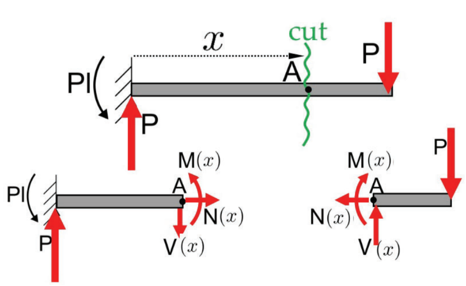
E - Young's Modulus
A - Cross-sectional Area
u - Displacement

Using:
- $\epsilon = \dfrac{du}{dx}$ Strain displacement relation
- $\sigma = E\epsilon$ 1D constitutive relation

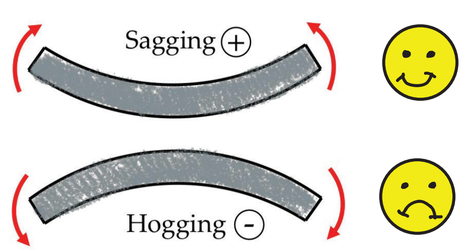
Beams: Elongated structural elements that are submitted to one (or more) internal load(s), offering resistance to bending (and/or torsion, and/or shear) due to applied forces (and/or moments).

Internal Loads:

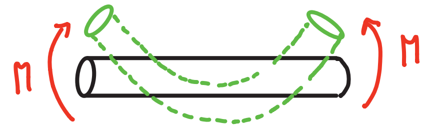
Axial load: $N(x)$
Shear Force: $V(x)$
Moment: $M(x)$



Directions Convention:



How is M related to the deformation?



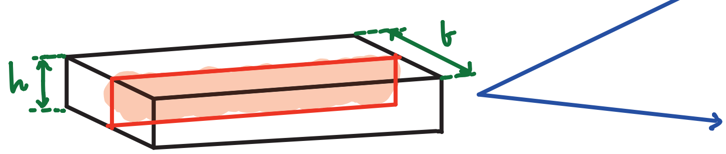
Step 1: Make kinematic approximation that allows us to assume simple motion.

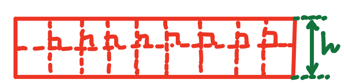
Step 2: Calculate stress field given the assumed simple motion

Step 3: Integrate stress to get $\begin{cases}\text{forces}\\ \text{moments}\end{cases}$

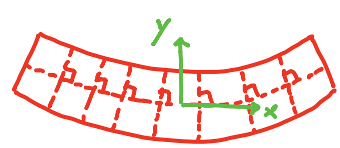
Step 1: Kinematics

Take cross-section of rectangular beam





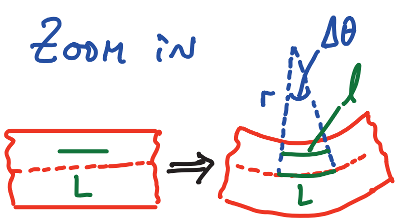
Undeformed shape



Bent shape:

"Pure bending" assumption:
- Perpendiculars stay perpendicular.
- Centerline does not stretch.

Zoom in



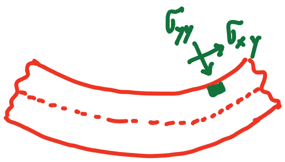
r is radius of curvature

Geometry: $\dfrac{l}{L} = \dfrac{\cancel{\Delta\theta}(r-y)}{\cancel{\Delta\theta}\, r} = 1 - \dfrac{y}{r}$

$\therefore$ Axial strain is x-direction:

$$\boxed{\epsilon_{xx} = -\frac{y}{r}} \qquad \left[\epsilon_{xx} = \frac{l-L}{L} = \frac{\cancel{L}\cdot\left(1-\frac{y}{r}\right)-\cancel{L}^{\,1}}{\cancel{L}} = -\frac{y}{r}\right]$$

STEP 2: STRESS FIELD : (GIVEN MOTION ASSUMED IN STEP 1)



$$\left.\begin{array}{l}\sigma_{yy}\left(y=\pm\frac{h}{2}\right)=0\\ \sigma_{xy}\left(y=\pm\frac{h}{2}\right)=0\end{array}\right\}$$ NOTHING TOUCHING ON BOTH TOP AND BOTTOM SURFACES

↓

BEAM IS THIN

↓

σ_{xy} AND σ_{yy} REMAIN SMALL THROUGH THICKNESS

$\Rightarrow \boxed{\sigma_{xy},\ \sigma_{yy} \ll \sigma_{xx}}$

EACH FIBER IN THE BEAM IS IN STATE OF PURE TENSION (NO SHEAR)

$$\boxed{\sigma_{xx} = -\frac{y}{r}E} \leftarrow \sigma_{xx} = E\epsilon_{xx} \leftarrow$$

STEP 3: FORCES, MOMENTS

$$F_x = \int_{\substack{\text{CROSS}\\ \text{SECTION}}} \sigma_{xx}\, dA = \int_0^b \int_{-h/2}^{h/2} -E\frac{y}{r}\, dy\, dz = 0\ !$$

$$M = \int_{\substack{\text{CROSS}\\ \text{SECTION}}} r \times F = \int_A \overbrace{\langle 0,y\rangle \times \langle 1,0\rangle}^{y\hat{z}} \sigma_{xx}\, dA$$

$$= \hat{z}\int_A y \cdot E\frac{y}{r}\, dA = \hat{z}\int_A Ey^2\frac{1}{r}\, dx\, dy = \hat{z}\frac{E}{r}\boxed{\int_A y^2 dA}$$

$= I =$ "MOMENT OF INERTIA" (OR 2nd MOMENT OF AREA)

FOR RECTANGULAR CROSS SECTION : $I_{\square_z} = \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\, dy\, dz = \frac{bh^3}{12}$

FOR CIRCULAR " " : $I_{\bigcirc} = \frac{\pi R^4}{4}$

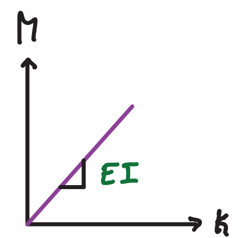
LIKE

MOMENT-CURVATURE RELATION (BIG RESULT)

$$\boxed{M = \frac{EI}{r} = EI\kappa}$$

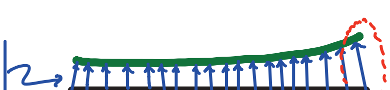
$\kappa = \frac{1}{r}$ is CURVATURE

EI = BENDING STIFFNESS

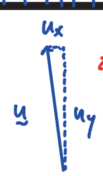


SMALL-DEFLECTION APPROXIMATION

DISPLACEMENT FIELD $\underline{u}(x)$



ZOOM:



$u_x \ll u_y$

$\therefore \underline{u} \simeq (0, u_y(x))$

ASSUME:

- POINTS ON DEFORMED CENTERLINE REMAIN CLOSE TO THEIR ORIGINAL POSIT. (DEFLECTION OF BEAM IS SMALL)
- BEFORE / AFTER, CENTERLINE HAS NEARLY THE SAME LENGTH. (INEXTENSIBILITY ASSUMPTION)

OFTEN, USE "$w(x)$" INSTEAD OF "$u_y(x)$" FOR DEFLECTION.

<u>CURVATURE:</u> $K = \dfrac{w''}{(1+w'^2)^{3/2}} \xrightarrow{\text{WHEN } w' \ll 1} \simeq w''(x)$

$$\begin{cases} w'(x) = \dfrac{dw}{dx} \\ w''(x) = \dfrac{d^2w}{dx^2} \end{cases}$$

$\therefore$ $\boxed{M = EI \dfrac{d^2w}{dx^2}}$ BEAM EQUATION FOR SMALL DEFLECTIONS

FROM ABOVE: $\left.\begin{cases} \epsilon_{xx}(y) = -\dfrac{y}{r} = -Ky \\ \sigma_{xx}(y) = E\epsilon_{xx}(y) = -KEy \end{cases}\right\} \Rightarrow \sigma_{xx} = -\Big(\overbrace{\dfrac{M}{EI}}^{K}\Big)Ey$

$\boxed{\sigma_{xx} = -\dfrac{My}{I}}$ AXIAL STRESS ON BEAM GIVEN A MOMENT

<u>ELASTIC ENERGY STORED DUE TO BENDING OF AN ORIGINALLY STRAIGHT BEAM?</u>

$$\mathcal{E} = \frac{1}{2}\int dx \int dA\, \sigma_{ij}\epsilon_{ij} \Rightarrow \mathcal{E} = \frac{1}{2}\int dx \int dA\, \underline{\sigma_{xx}}\, \underline{\epsilon_{xx}}$$

GIVEN THE ASSUMPTIONS THAT WE HAVE TAKEN THUS FAR.

$\sigma_{xx} = E\epsilon_{xx}$ $\qquad \epsilon_{xx} = -\dfrac{y}{r}$

$$\therefore \mathcal{E} = \frac{E}{2}\int dx \int dA\,(\epsilon_{xx})^2$$

$$= \frac{E}{2r^2}\int dx \underbrace{\int dA\, y^2}$$

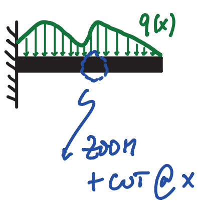
$$= \frac{1}{2}\int \frac{E\,dx}{r^2} \underbrace{\int dA\, y^2}_{\equiv I_z}$$

AREAL MOMENT OF INERTIA

$\Rightarrow \boxed{\mathcal{E} = \displaystyle\int_L \frac{EI}{2r^2}\,dx}$ BENDING ENERGY OF BEAM (W/ LENGTH L)

<u>NOTE:</u> IN THE FUTURE WILL ALSO USE U_b (TO CONTRAST WITH STRETCHING ENERGY U_s)

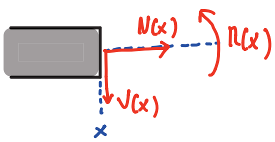
If have distributed load $q(x)$:



$$\Sigma F_y = 0 \Rightarrow \frac{dV(x)}{dx} = -q(x)$$

$$\Sigma M_0 = 0 \Rightarrow \frac{dM(x)}{dx} = V(x)$$

$$\frac{d^2M}{dx^2} = -q(x) \quad \text{BUT} \quad \boxed{M = EI\,\omega''(x)}$$



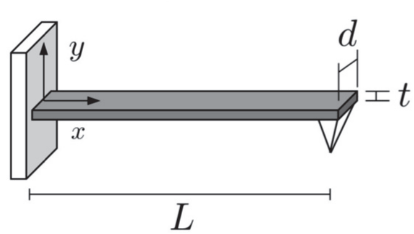
$$\Rightarrow \boxed{q(x) = -\frac{d^2}{d^2}\left(EI\frac{d^2\omega}{dx^2}\right)}$$

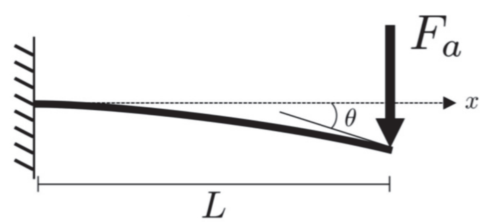
Beam Equation

ODEs that can be solved given B.C.s

Support	w	w'	M	V
Pin	0	$\neq 0$	0	$\neq 0$
Parallel motion	$\neq 0$	0	$\neq 0$	0
Fixed end	0	0	$\neq 0$	$\neq 0$
Free end	$\neq 0$	$\neq 0$	0	0

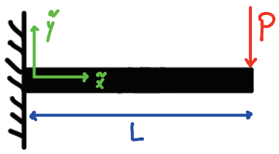
Example: An Atomic Force Microscope (AFM) can be used to detect / locate individual atoms. The key mechanical element of an AFM is a silicon cantilever beam, at whose end an atomically sharp tip is attached. When the tip is close enough to the surface, there is an attractive force, F_a, between the atom at the very end of the tip and the atom below it. The attractive force induces a state of bending in the beam, whose deflection can be measured through optical means.



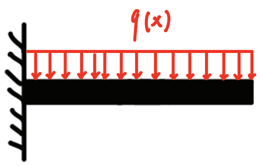


Effective spring constant? $\Rightarrow$ Check notes for solutions.

Some fundamental beam solutions:



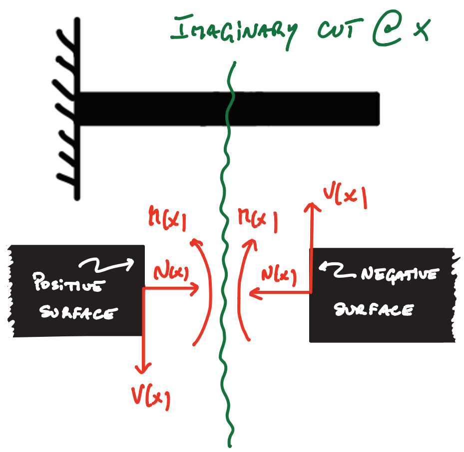
$$\Longrightarrow \quad w(x) = -\frac{Px^2}{6EI}(3L-x)$$

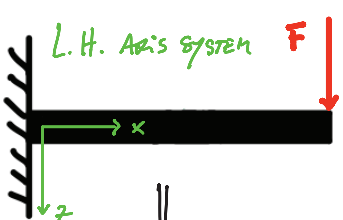


$$\Longrightarrow \quad w(x) = -\frac{qx^2}{24EI}(x^2-4Lx-6L^2)$$

$$\Longrightarrow \quad w(x) = \frac{M_0 x^2}{2EI}$$

Comment on choice of sign convention:



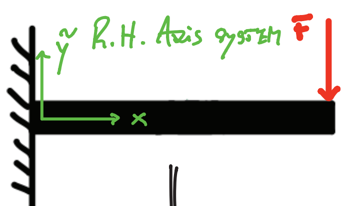


$$\Downarrow$$

$$M = -EI\kappa = -EIw''$$

$w > 0$ for given F

$$w_{max} = \frac{FL^3}{3EI}$$



$$\Downarrow$$

$$M = EI\kappa = EIw''$$

$w < 0$ for given F

$$w_{max} = -\frac{FL^3}{3EI}$$

Be careful w/ your sign convention and be consistent

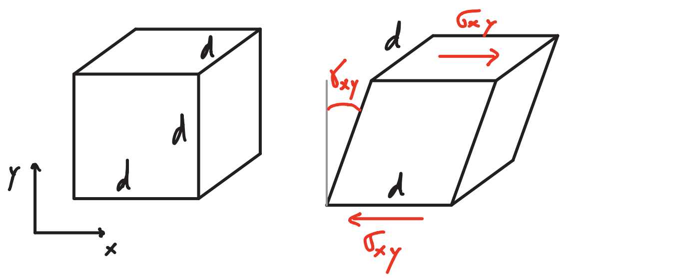
Torsion of Rods / Shafts (will now have to deal w/ shear)

Recall: $\underline{\epsilon} = \frac{1}{E}\left[\underline{\sigma}(1-\nu) - \nu(\mathrm{tr}\,\underline{\sigma})\underline{I}\right] \Rightarrow \epsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$

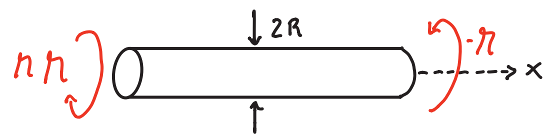
Shear modulus: $G = \frac{E}{2(1+\nu)} \Rightarrow \sigma_{xy} = 2G\epsilon_{xy} = G\gamma_{xy}$

γ_{xy}: Engineering shear strain

Recall simple shear of elastic cube:



Apply opposing torques to ends of rod

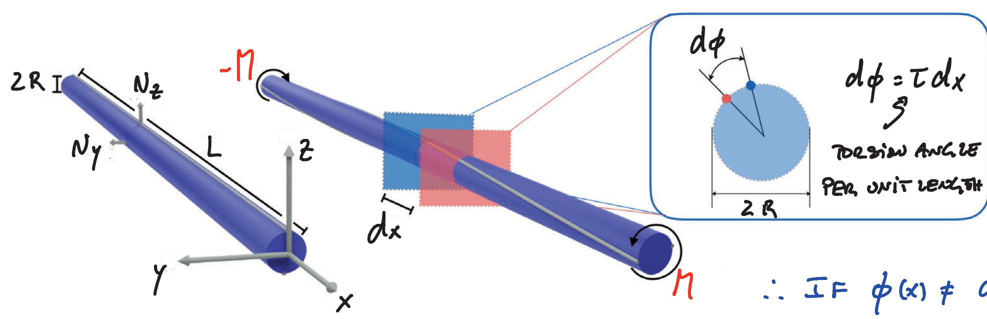


$\Rightarrow$ Torsion (i.e. twisting)

[Coulomb, 1784]

Simplifying Assumptions:

- During torsion, no displacements in axial direction $\Rightarrow \epsilon_{xx} = 0$
- Each cross-section moves only by rotation in its plane



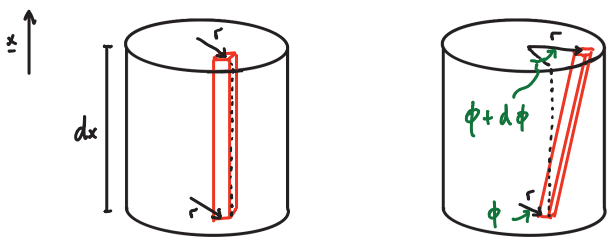
Assumption: $\tau R \ll 1$

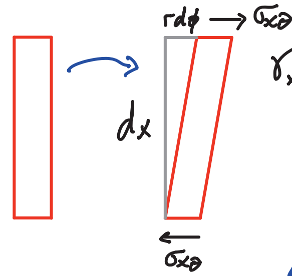
$\therefore$ If $\phi(x) \neq$ constant, shaft is twisted

- In cylindrical basis: $\epsilon_{rr} = 0$, $\epsilon_{\theta\theta} = 0$, $\epsilon_{r\theta} = 0$

i.e. in-plane angles do not change and in-plane lines do not stretch.

Before — After Torsion : For small deformations, column is put under state of simple shear.





$$\gamma_{x\theta} = \tan^{-1}\left(\frac{r\,d\phi}{dx}\right) \quad \text{small}$$

$$\approx r \cdot \underbrace{\frac{d\phi}{dx}}_{\tau \,\equiv\, \text{twist per length}}$$

$\therefore$ $\gamma_{x\theta}$ increases with r.

$$\sigma_{x\theta} = G\gamma_{x\theta} = Gr\frac{d\phi}{dx}$$

Now want to compute internal torque on each cross-section:

$$M(x) = \int_{\underset{\text{cross-section through } x}{A}} \underbrace{r}_{\text{distance from axis}} \cdot \underbrace{\sigma_{x\theta}\, dA}_{\text{force}} = \int_A r\, G\, r \frac{d\phi(x)}{dx}\, dA$$

$$= G\frac{d\phi(x)}{dx} \cdot \underbrace{\int_A r^2 dA}_{\equiv\, J \,=\, \text{"polar moment of inertia"}}$$

- For a solid circular cross section:

$$J = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r \cdot r\,dr\,d\theta = 2\pi\frac{R^4}{4} = \frac{\pi R^4}{2}$$

- For a thin tube (radius R, thickness t)

$$J = \int_A r^2 dA \cong R^2 \cdot 2\pi R t = 2\pi R^3 t$$

Moment-twist relation:

$$\boxed{M(x) = \underbrace{GJ}_{GJ:\ \text{torsional stiffness}}\frac{d\phi(x)}{dx} = GJ\tau}$$

c.f. with moment-curvature relation for pure bending

$$M(x) = EI\frac{d^2 w(x)}{dx^2} = EI\kappa$$

Elastic energy stored in pure torsion of rod (Small deformations assumption)

Total strain energy in rod of
- Length L
- Cross-section area A
- Volume V

From above: $\sigma_{\theta x} = G\gamma_{\theta x}$, $\quad \gamma_{\theta x} = r\dfrac{d\phi}{dx} = r\tau$

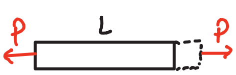
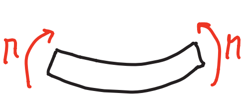
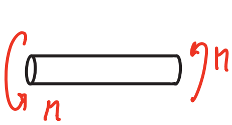
$$\mathcal{E}_t = \frac{1}{2}\int_V \sigma_{x\theta}\,\epsilon_{\theta x}\,dV = \frac{1}{2}\int G\tau^2 r^2\,dV = \frac{1}{2}\int G\tau^2\,dx \underbrace{\int_A r^2\,dA}_{\equiv J = \text{"Polar moment of inertia"}}$$

$$\Rightarrow \mathcal{E}_t = \frac{1}{2}\int_L GJ\tau^2\,dx$$

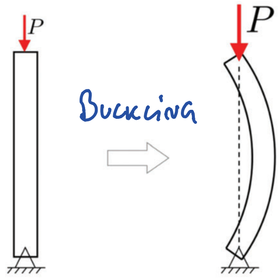
But showed that $M = (GJ)\tau$

$$\Rightarrow \boxed{\mathcal{E}_t = \frac{1}{2}\int_L \frac{M^2}{GJ}\,dx}$$

Summary: Stretching, bending and torsion of rods

	EA Axial stiffness	$F = EA\dfrac{du}{dx}$	$\mathcal{E}_s = \frac{1}{2}\int_L EA\,\epsilon^2\,dx = \frac{1}{2}\int_L \frac{P^2}{EA}\,dx$
	EI Bending stiffness	$M = EI\dfrac{d^2w}{dx^2}$	$\mathcal{E}_b = \frac{1}{2}\int_L \frac{EI}{R^2}\,dx = \frac{1}{2}\int_L \frac{M^2}{EI}\,dx$
	GJ Torsional stiffness	$M = GJ\dfrac{d\phi}{dx}$	$\mathcal{E}_t = \frac{1}{2}\int_L GJ\tau^2\,dx = \frac{1}{2}\int_L \frac{M^2}{GJ}\,dx$

Buckling of Beams (Elastic Failure)

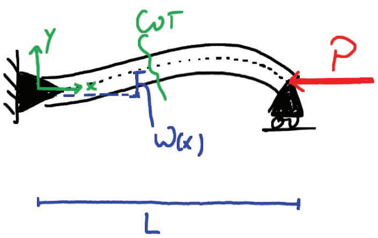


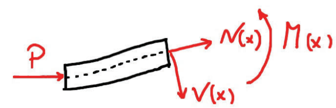
Buckling: Instability of elastic structures under compression that causes bending

Will expand beam theory to permit a compressive load but assume no compressive deformation:

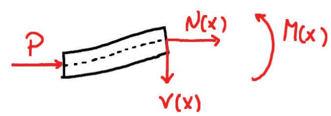
$\Rightarrow$ All motion still just due to bending

Consider:
Specific case with Pin-Pin B.C.s





Since deformations are small, assume:



$\underline{V} = -V(x)\,\hat{\underline{y}}$

$\underline{N} = N(x)\,\hat{\underline{x}}$

Force/Moment Balance:

$\sum F_x = 0 \quad : \quad P + N = 0 \qquad \Rightarrow N(x) = -P$

$\sum F_y = 0 \quad : \qquad V(x) = 0$

$\sum M_o = 0 \quad : \quad M - \omega(x) N - x V(x) \quad \Rightarrow M(x) = -P\omega(x)$

Moment-Curvature Relation:

$$EI \frac{d^2\omega}{dx^2} = M = -P\omega \Rightarrow \boxed{EI \frac{d^2\omega}{dx^2} + P\omega = 0}$$

ODE for ω

General solution:

$$\omega(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

B.C.s (Pin - Pin):

$$\omega(0)=0 \implies C_2=0 \implies C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right)=0$$

$$\omega(L)=0 \implies C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right)=0$$

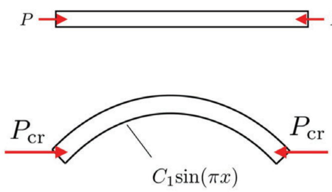
- $C_1=0 \implies \omega(x)=0$ ---- STRAIGHT SOLUTION

or

- $\underbrace{\sin\left(\sqrt{\frac{P}{EI}}L\right)}=0$ ------ BUCKLED SOLUTION

$$\left(\sqrt{\frac{P}{EI}}L\right)=n\pi \implies P=\frac{n^2\pi^2EI}{L^2} \quad n \text{ is integer}$$

LOWEST LOAD WHEN $n=1$

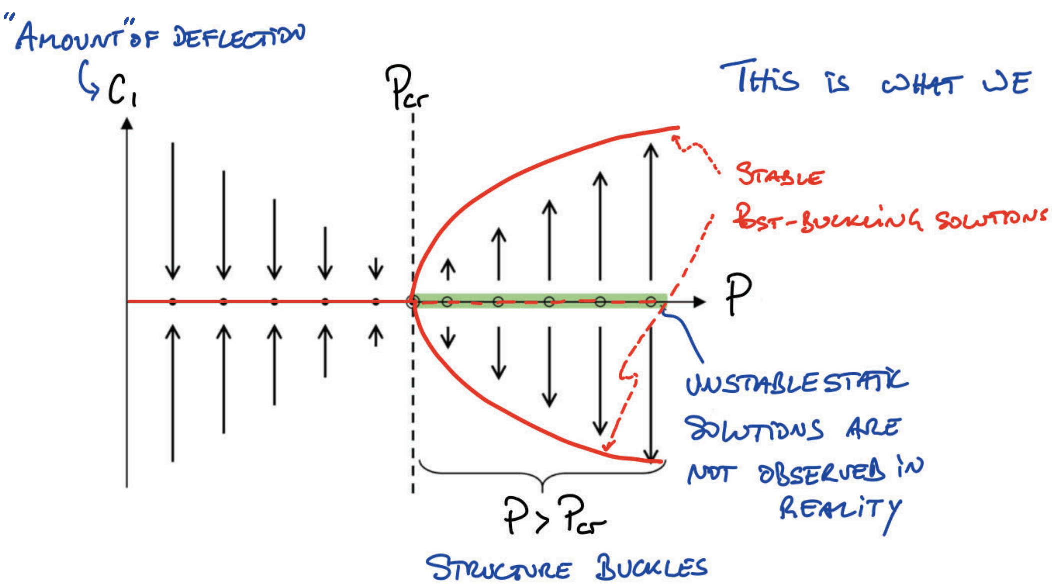


$$\boxed{P_{cr}=\frac{\pi^2EI}{L^2}}$$

P_{cr} is EULER BUCKLING LOAD

$P<P_{cr} \implies$ STRAIGHT SOLUTION

$P>P_{cr} \implies$ BUCKLING



THIS IS WHAT WE CALL A BIFURCATION DIAGRAM

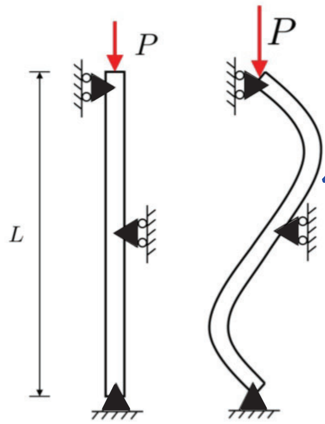
Higher order buckling loads:

Before got general solution:

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad n \text{ is integer}$$

How can we get the $n=2$ solution?

$\Rightarrow$ Modify system to prevent $n=1$ solution



Buckling

$$P_{(n=2)} = (2\pi)^2 \frac{EI}{L^2} > \text{Euler load}$$

$$w_2 = C_1 \sin\left(\sqrt{\frac{P_{(n=2)}}{EI}}\, x\right) = C_1 \sin\left(\frac{2\pi x}{L}\right)$$

To get $n=3$ solution, prevent $n=1$ and $n=2$ solution

$n=N$ " " all $1 \leq n \leq N-1$ solutions

$\boxed{Q:}$ What if ends are constrained by supports other than pins?

$\Rightarrow$ Following similar above procedure but $\neq$ B.C.s would give:

$$P_{cr} = \frac{\pi^2 EI}{(K L)^2}$$

K is effective length factor, which depends on the type of end constraints.