

Problem 1. (Naive Bayes)

We aim to predict the probability of success for a data point $x = (x_1, \dots, x_d)$ with features $x_i \in \mathbb{R}$ for $i = 1, \dots, d$. Success corresponds to class 1 and failure corresponds to class 0. Let $P(1)$ denote the probability of success determined based on a training set.

1. Suppose you are given a test point $x^t = (x_1^t, \dots, x_d^t)$ and $f_{x_i|1}$ is the conditional probability density of feature x_i given success. Give the expression of the probability that the test point is a success, by filling in the blanks below and using feature independence with the Naive Bayes assumption.

$$P(1 | x^t) = \frac{\dots\dots\dots P(1)}{\dots\dots\dots}$$

2. Suppose the number of features is $d = 1$ and the test data point is $x^t = 4$. Assume the following is known to you: $f_{x|1}(4) = 0.17$, $f_{x|0}(4) = 0.24$ and $P(1) = 0.6$. Using the Naive Bayes approach to determine whether the test point $x^t = 4$ is classified as a success or not.

Exercise 2. (Naive Bayes)

In this problem we aim to predict the likelihood of failure during manufacturing of an item based on the pressure and temperature applied to the object. Consider the following hypothetical dataset consisting of 10 measurements:

Failure (F)	1	0	0	0	0	1	0	0	0	0
Temperature (T) °C	170	100	80	140	60	230	100	60	140	120
Pressure (P) MPa	430	220	200	160	240	370	200	180	240	160

1. Based on this dataset, what is the empirical probability for an item to be defective ($F = 1$)?
2. Given each of the two classes, 1 (failure) and 0 (no failure), for each feature vector, determine the empirical means $\mu_{1,T}, \mu_{1,P}, \mu_{0,T}, \mu_{0,P}$ and variances $\sigma_{1,T}^2, \sigma_{1,P}^2, \sigma_{0,T}^2, \sigma_{0,P}^2$. Based on this, determine the corresponding two Gaussian distributions for each of the two features.
3. Suppose you have the choice between two classification models that use the means and variances derived previously to predict a new datapoint $x_{test} = (T', P')$:
 - Model A: Naive Bayes Classifier.
 - Model B: Classify (T', P') according to its Euclidean (L_2) distance to the class means. For example, classify (T', P') as failure if

$$(T' - \mu_{1,T})^2 + (P' - \mu_{1,P})^2 < (T' - \mu_{0,T})^2 + (P' - \mu_{0,P})^2.$$

Now suppose you are given the datapoint $x_{test} = (T', P') = (151, 302)$.

- (a) Predict the output of Model A.
- (b) Predict the output of Model B.
- (c) Are the above two outputs different? If yes, why?

Problem 3. (Dynamical systems, discretization, RNN)

Consider the following dynamical system: $\ddot{x}(t) = -x(t)$.

1. Verify that $x(t) = a \sin(t) + b \cos(t)$ is a solution to this system for any $a, b \in \mathbb{R}$.
2. Given $x(0) = c_1$, $\dot{x}(0) = c_2$, determine $a, b \in \mathbb{R}$ in terms of c_1, c_2 .
3. By defining $s_1 = x(t)$ and $s_2 = \dot{x}(t)$ verify that the system can be put in state-space form as follows.

$$\begin{bmatrix} \dot{s}_1(t) \\ \dot{s}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \quad (0.1)$$

What are the eigenvalues of the system matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

4. Consider the phase plot of the system, that is, plotting $s_1(t)$ on the x-axis and $s_2(t)$ on the y-axis. For a given initial condition $s_1(0) = c_1$, $s_2(0) = c_2$, verify that phase-plot will be a circle centered at $(0, 0)$ with radius $\sqrt{c_1^2 + c_2^2}$.
5. Derive the Euler discretizations of the continuous-time system in differential equation (0.1) with discretizations parameter $\delta > 0$. Denote the discretized state by $(s_{1,k}, s_{2,k})$. What are the eigenvalues of the resulting system matrix?

$$\begin{bmatrix} s_{1,k+1} \\ s_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix}$$

6. Provide the phase plot of the discretized system, namely, plot $s_2(k)$ against $s_1(k)$. You may use $\delta = 0.05$, $c_1 = 1, c_2 = 2$. Observe that in this phase plot, we see spirals growing out in contrast to the purely rotational dynamics in the continuous-time system. Thus, the trajectories of the system grow unbounded.
7. Let us consider the so-called *backward* Euler discretization, where $\dot{s}(t) \approx \frac{s(t) - s(t-\delta)}{\delta}$. Verify that with the above discretization, the system dynamics can be written as

$$\begin{bmatrix} s_{1,k+1} \\ s_{2,k+1} \end{bmatrix} = \frac{1}{1 + \delta^2} \begin{bmatrix} 1 & \delta \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix}$$

Hint: you will have to solve for $s_{1,k}, s_{2,k}$ in terms of $s_{1,k-1}, s_{2,k-1}$. After solving for s_k in terms of s_{k-1} , we shift the index $k \mapsto k + 1$ so that the dynamics are expressed in the usual form $s_{k+1} = A_b s_k$.

8. What are the eigenvalues of the discretized system dynamics $A_b = \frac{1}{1 + \delta^2} \begin{bmatrix} 1 & \delta \\ -\delta & 1 \end{bmatrix}$, corresponding to backward Euler discretization approach?

Provide the phase plot of the discretized system above, for $\delta = 0.05$, $c_1 = 1, c_2 = 2$. Observe that in this phase plot, we see spirals growing in, in contrast to the purely rotational dynamics in the continuous time system. Thus, the trajectories of the system are approaching the equilibrium at $(0, 0)$.

9. As you can observe from the above two subparts, neither forward, nor backward Euler discretization is effective in capturing the perfect oscillations of the continuous-time signal $y(t) = a \sin(t) + b \cos(t)$. The issue is that the eigenvalues of the continuous-time system matrix A are purely imaginary and the forward Euler discretization results in eigenvalues with magnitude slightly larger than one (and thus, growing oscillations and unbounded growth of the solution), whereas the backward Euler discretization results in eigenvalues with magnitude slightly smaller than 1 (thus, damping oscillations and convergence to zero of the solutions). Another discretization approach is using central difference discretization $\dot{s}(t) \approx \frac{s(t+\delta) - s(t-\delta)}{2\delta}$. Verify that with the central difference discretization, the discrete-time system can be written as

$$\begin{bmatrix} s_{1,k+1} \\ s_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 - \delta^2 & \delta \\ -\delta & 1 \end{bmatrix} \begin{bmatrix} s_{1,k} \\ s_{2,k} \end{bmatrix}$$

What are the eigenvalues of $A_c = \begin{bmatrix} 1 - \delta^2 & \delta \\ -\delta & 1 \end{bmatrix}$? Provide the phase plot of the above discretized system. As you can see the phase plot now matches that of the continuous-time system much better.

10. Now, letting $y_k = s_{1,k}$ verify that any of the discretized dynamics above can be written as a recurrent neural network without any input, with two hidden states and the identity function as the activation.

Remark: The system in differential equation 0.1 is a Hamiltonian, and the energy is preserved (see Part 4 of the question). For such systems, Euler discretization is not effective as it does not preserve the energy. The so-called Symplectic discretization ensures energy preservation and will result in a discretized dynamics that are identical (up to a linear transformation) to that of the central difference approach.