Problem 1. Dynamical systems

Consider an inverted pendulum mounted to a motorized cart, referred to as cart-pole as shown in Figure 1. The objective of the control system is to balance the inverted pendulum by applying a force to the cart that the pendulum is attached to. You can think of this as trying to balance a long stick on your palm by moving your palm in different directions. This is also a simple model of attitude control of a booster rocket at takeoff.

The parameters of the system are as follows: M denotes mass of the cart, m mass of the pendulum, b coefficient of friction for the cart, l length to pendulum center of mass, I moment of inertia of the pendulum. The *states* of the system are the angular position of the pendulum from vertical θ, the angular velocity $\dot{\theta}$, the horizontal position of the cart x and the horizontal velocity of the cart $\dot{x}$. The input to the system is denoted by a and is the force applied to the cart (on the figure it is shown as F).

The equations of motion of the dynamical system are derived as below (see the derivation here).

$$(I + ml^2)\ddot{\theta} - mgl\theta = ml\ddot{x},$$
$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\theta} = a$$

1. Let $s_1 = \theta$, $s_2 = \dot{\theta}$, $s_3 = x$, $s_4 = \dot{x}$, and let $\boldsymbol{s}$ denote $(s_1, s_2, s_3, s_4)^T \in \mathbb{R}^4$. Write the dynamics in state-space form.

   The state-space equation can be written

$$\frac{d\boldsymbol{s}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 & 0 & -\frac{mlb}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 1 \\ \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 & 0 & -\frac{(I+ml^2)b}{I(M+m)+Mml^2} \end{pmatrix} s + \begin{pmatrix} 0 \\ \frac{ml}{I(M+m)+Mml^2} \\ 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \end{pmatrix} a = A\boldsymbol{s} + Ba$$

2. Use Euler discretization to discretize the dynamics, with the stepsize denoted by δ.

$$\boldsymbol{s}_{t+\delta} = \boldsymbol{s}_t + \delta\frac{d\boldsymbol{s}}{dt} = (I + \delta A)\boldsymbol{s}_t + \delta B a_t$$

   Note that we neglected the second order terms

3. Now, consider process noise w_i affecting state s_i. Such noise can capture either modelling uncertainties (e.g. imprecise values for the parameters) or arise from disturbances acting on the system (e.g. wind). It can also capture actuation noise (e.g. the force applied may not be exactly as the one planned due to actuation mechanism imperfection). Assuming $w_i \sim N(0, \sigma_i)$, with $N(0, \sigma_i)$ denoting the Normal probability distribution with mean 0 and variance σ_i^2, what is the probability distribution on state $\boldsymbol{s}_{t+1}$ given state s_t and input a_t?

$$\boldsymbol{s}_{t+1} = (I + \delta A)\boldsymbol{s}_t + \delta B a_t + \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = A_d \boldsymbol{s}_t + B_d a_t + \boldsymbol{\omega}$$

4. Suppose our goal is to bring the pendulum to the upright position at a final timestep T, namely, $\theta_T = \pi$. Write an objective function that helps capture this goal.

$$C_t(\boldsymbol{s}, a) = \begin{cases} 0 & \text{if } t < T \\ (s_{1,T} - \pi)^2 & \text{else} \end{cases}$$

Note that any function that is minimized by $\theta_T = \pi$ would work. It is also intersting to note that as we linearized the dynamics around $s_1 = \pi$, we need to stay close to this region at all time, so using

$$C_t(\boldsymbol{s}, a) = (s_{1,t} - \pi)^2 \ \forall \ t \leq T$$

might be a better cost.

5. Let the control policy be time-varying and denoted by $\pi_t : \mathbb{R}^4 \to \mathbb{R}$. Write the closed-loop transition dynamics.

$$\boldsymbol{s}_{t+1} = A_d\boldsymbol{s}_t + B_d a_t + \boldsymbol{\omega}_t = A_d\boldsymbol{s}_t + B_d\pi_t(\boldsymbol{s}_t) + \boldsymbol{\omega}$$

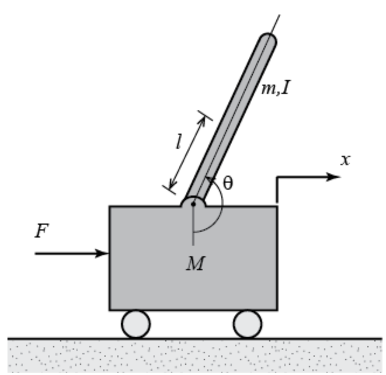

Figure 1: The inverted pendulum on a cart. The parameters, the state and input are shown. The figure is from here.