Student seminar exercise sheet Week 13

1. In this exercise, you will prove the 'Artin Lemma' that was stated during the lecture. Its proof requires in its entirety multiple technical lemmas that build on each other. To simplify your work, we state here the last of those lemmas which will be the only one you use.

   Lemma 0.1. *Let F be a number field, $\mathcal{S}$ a finite set of primes in $\mathbb{Z}$, $\mathfrak{p}$ a prime of $\mathcal{O}_F$. Then for any integer $n > 1$, there exists $m \in \mathbb{Z}$, prime to $\mathcal{S}$ and to $\mathfrak{p}$, such that if ζ_m is a primitive m^{th}-root of unity, then*

   i. $Gal(F(\zeta_m)/F) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$.

   ii. $\left(\frac{p}{F(\zeta_m)/F}\right)$ *has order divisible by n in $Gal(F(\zeta_m)/F)$.*

   iii. There is some $\tau \in Gal(F(\zeta_m)/F)$ of order divisible by n, such that τ is independent to $\left(\frac{p}{F(\zeta_m)/F}\right)$. Note that this means here that $\langle\tau\rangle \cap \langle\left(\frac{p}{F(\zeta_m)/F}\right)\rangle = 1$.

   We recall here the statement of Artin's Lemma as you will prove it:

   Lemma 0.2. *Let K/F be a cyclic extension of number fields of degree n, $\mathcal{S}$ a finite set of primes of $\mathbb{Z}$, $\mathfrak{p}$ a prime of $\mathcal{O}_F$. Then, there is some $m \in \mathbb{Z}_+$ prime to the elements of $\mathcal{S}$ and to $\mathfrak{p}$, and an extension E/F such that:*

   i. $K \cap E = F$

   ii. $K(\zeta_m) = E(\zeta_m)$, *i.e.* $KE \subseteq K(\zeta_m) = E(\zeta_m)$

   iii. $K \cap F(\zeta_m) = F$

   iv. $\mathfrak{p}$ *splits completely in E/F.*

   With all the setup being clear, we start the exercises:

1) Use the given lemma to construct an adequate m and to prove *iii.* above, i.e. $K \cap F(\zeta_m) = F$. You may use freely the fact that there does not exist an everywhere unramified extension of $\mathbb{Q}$.

2) Denote $G = \mathrm{Gal}(K/F) = \langle \sigma \rangle$. Describe the Galois group $\mathrm{Gal}(K(\zeta_m)/F)$ in terms of G and $(\mathbb{Z}/m\mathbb{Z})^\times \simeq \mathrm{Gal}(F(\zeta_m)/F)$.

3) Let τ be the morphism as obtained from 1). Define the subgroup H in $\mathrm{Gal}(K(\zeta_m)/F)$ to be the one generated by $\sigma \times \tau$ and $\left(\frac{\mathfrak{p}}{K(\zeta_m)/F}\right)$, and E to be its fixed field.

   3.a) Prove that $\mathfrak{p}$ splits completely in E. (**Hint**: The decomposition group might be useful.)

   3.b) Prove that $K \cap E = F$, by showing that $\sigma \times 1$ fixes $K \cap E$.

4) By 2., a generic element of H writes as

$$(\sigma \times \tau)^i \left(\left(\frac{\mathfrak{p}}{K/F} \right) \times \left(\frac{\mathfrak{p}}{F(\zeta_m)/F} \right) \right)^j$$

   and a generic element in $G \times 1$ writes as $\sigma^a \times 1$.

   4.a) Compute the fixed field of $H \cap (G \times 1)$.

   4.b) Let $b \in H \cap (G \times 1)$. By comparing coordinate-wise, show that $b = 1$.

   4.c) Conclude.

2. Let $m > 0$. Let E/F be an extension of number fields such that $E \subset F(\zeta_m)$. Prove that Artin reciprocity holds for E assuming that Artin reciprocity holds for quasi-cyclotomic extensions.

3. Let K/F be an Abelian extension of number fields with $\mathrm{Gal}(K/F) = G$. Let $\sigma \in G$. Define

$$\mathcal{S}_\sigma = \left\{ \text{primes } \mathfrak{p} \text{ of } \mathcal{O}_F : \ \mathfrak{p} \text{ is unramified in } K/F \text{ and } \left(\frac{\mathfrak{p}}{K/F} \right) = \sigma \right\}.$$

   Show that $\delta_F(\mathcal{S}_\sigma) = \frac{1}{[K:F]}$.

   (This exercise is one of the ingredients in the proof of the Chebotarev Density Theorem, see Exercise 5.10 for a proof of the Density Theorem and Exercise 5.11 for a nice application.)

4. Let K/F be an Abelian extension of number fields. We showed during lecture that the idelic Artin map is surjective with kernel $F^\times \tilde{N}_{K/F} J_K$.

   Reformulate the idelic Artin map such that the domain becomes $C_F = J_F/F^\times$. In particular, show that the new Artin map $C_F \to \mathrm{Gal}(K/F)$ is still surjective, but the kernel becomes $\tilde{N}_{K/F} C_K$.