Student seminar exercise sheet Week 8

For this exercise sheet, recall that the Hilbert class field over F can be characterized as the maximal unramified abelian extension of F (unramified extension means each *prime* ideal in F is unramified in the extension).

1. Find the Hilbert class field over $\mathbb{Q}(\sqrt{15})$.
2. As mentioned in class, the Principal Ideal Theorem states that any ideal of a number field F becomes principal in its Hilbert class field F_1. This exercise is a partial proof of it, assuming one group theory key fact.

   (I) Let F_2 be the Hilbert class field over F_1. Show that F_2 is Galois over F.

   (II) Let $A := \mathrm{Gal}(F_2/F_1)$ and $G := \mathrm{Gal}(F_2/F)$. Show that $A = G' = [G, G]$ and that

   $$\gamma : \mathcal{I}_F/\mathcal{P}_F \longrightarrow \mathcal{I}_{F_1}/\mathcal{P}_{F_1} : \mathfrak{a}\mathcal{P}_F \mapsto (a\mathcal{O}_{F_1})\mathcal{P}_{F_1}$$

   is a well defined group homomorphism.

   (III) Let $V : G/G' \to G'$ (called the transfer map, ***V****erlagerung* in german) be the map making the following diagram commute.

   $$\begin{array}{ccc} \mathcal{C}_F & \xrightarrow{\cong} & G/G' \\ \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle V} \\ \mathcal{C}_{F_1} & \xrightarrow[\cong]{} & G \end{array}$$

   Prove that the upper and lower morphisms are isomorphisms and show that proving the Principal Ideal Theorem amounts to proving the map V to be trivial.

   (IV) Artin was the one able to reduce our initial statement to the group theoretical problem of showing that V is trivial. Fortunately, there is a result in group theory (also called the Principal Ideal Theorem!) which precisely answers our needs.
   If G is a finite group and G' is its commutator subgroup, then the transfer $V : G/G' \to G'/(G')'$ is the trivial map.
   Using the previous points and this result, conclude.

3. We now recall the definition of *absolute value* on an algebraic number field F. It is a map

$$\| \bullet \| : F \to [0, \infty)$$

satisfying $\|0\| = 0$ and such that its restriction to $F^\times$ is an multiplicative group homeomorphism, satisfying

$$\|1 + x\| \leq c, \quad \text{whenever } \|x\| \leq 1$$

and some suitable constant c.
For any such absolute value $\| \bullet \|$, show that there exists $\lambda \in \mathbb{R}_+^\times$ such that $\| \bullet \|^\lambda$ satisfies the triangular inequality, or equivalently $\| \bullet \|^\lambda$ is a standard absolute value, as defined for instance in in Chapter 1. of Childress' book.

4. Let $F = \mathbb{Q}(i)$ and $x = 2 - i$. Compute $\|x\|_v$, for all $v \in V_F$ and verify that the product formula $\prod_{v \in V_F} \|x\| = 1$ holds.

5. Let H be a subgroup of the topological group G. Show the following:
   (a) If H is open in G, then H is also closed.
   (b) If H is closed with finite index in G, then H is also open.
   (c) If G is compact and H is open in G, then $[G : H]$ is finite.