Student seminar exercise sheet Week 3

1. Let $F = \mathbb{Q}(\sqrt{3}, \sqrt{7})$. Compute r and s, the numbers of real embeddings and pairs of conplex conjugate embeddings, respectively, and then give an explicit description of the unit group $\mathcal{O}_F^\times$. Do the same for $F' = \mathbb{Q}(\sqrt{3}, \sqrt{7}, i)$.

2. (a) Find the p-adic expansion of $\frac{1}{6} \in \mathbb{Q}_7$.

   (b) For which $p = 2, 3, 5, 7, 11, 13$ does -1 have a square root in $\mathbb{Q}_p$?

3. Let F be a number field. Let $\mathfrak{p} \neq \mathfrak{q}$ be non-zero prime ideals of $\mathcal{O}_F$. Prove that the norms $|\cdot|_\mathfrak{p}$ and $|\cdot|_\mathfrak{q}$ are not equivalent.

4. Let ${}^{K}\!/_{F}$ be a number field extension. And let $\mathfrak{P}, \mathfrak{p}$ be primes of K and F resp. such that $\mathfrak{P}$ is above $\mathfrak{p}$. Then consider the extension of local fields ${}^{K_\mathfrak{P}}\!/_{F_\mathfrak{p}}$. Show that for $\pi_\mathfrak{p}$ an uniformizer of $F_\mathfrak{p}$, we have:

$$|\pi_\mathfrak{p}|_\mathfrak{P} = |\pi_\mathfrak{P}|_\mathfrak{P}^e$$

   Where $\pi_\mathfrak{P}$ is an uniformizer of $K_\mathfrak{P}$ and e is the ramification index $e = e({}^{\mathfrak{P}}\!/_{\mathfrak{p}})$. This means that the ramification index of the field extension and their completion coincides.