Student seminar exercise sheet Week 11

1. (Convergence of the p-adic Exponential)

   Consider the p-adic exponential power series defined over the field of p-adic numbers $\mathbb{Q}_p$:
   $$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$
   Let v_p denote the standard p-adic valuation normalized such that $v_p(p) = 1$. By analyzing the p-adic valuation of the coefficients $a_n = \frac{1}{n!}$, prove that the radius of convergence of this power series is $p^{-\frac{1}{p-1}}$. In other words, show that the series converges on the open disc:
   $$D = \left\{ x \in \mathbb{C}_p \;\middle|\; |x|_p < p^{-\frac{1}{p-1}} \right\}$$

2. (Continuity of the Norm Map)

   Let k_2/k_1 be an extension of local fields above $\mathbb{Q}_p$. Show that, with respect to the p-adic topology, the norm map
   $$N_{k_2/k_1} : k_2 \to k_1$$
   is continuous.

3. (The map α)

   The map $\eta_{K,\mathfrak{m}} : \mathcal{J}_{K,\mathfrak{m}}^{+} \to \mathcal{I}_K(\mathfrak{m})$ is defined by
   $$\eta_{K,\mathfrak{m}} : a \mapsto \prod_{v \nmid \infty} \mathfrak{p}_v^{\mathrm{ord}_v(a_v)}.$$
   We "extend" $\eta_{K,\mathfrak{m}}$ to all ideals by considering the map
   $$\alpha : \; \mathcal{J}_K \twoheadrightarrow \mathcal{J}_K / F^{\times} \xrightarrow{\cong} \mathcal{J}_{K,\mathfrak{m}}^{+} / F_{\mathfrak{m}}^{+} \xrightarrow{\bar{\eta}_{K,\mathfrak{m}}} \mathcal{I}_K(\mathfrak{m}) / \mathcal{P}_{K,\mathfrak{m}}^{+} = \mathcal{R}_{K,\mathfrak{m}}^{+}.$$
   Explicitly, it is given by :
   $$\alpha : \; a \longmapsto aF^{\times} \overset{\cong}{\longmapsto} xaF_{\mathfrak{m}}^{+} \overset{\bar{\eta}_{K,\mathfrak{m}}}{\longmapsto} \left(\prod_{v \nmid \infty} \mathfrak{p}^{\mathrm{ord}_v(x_v a_v)}\right) \mathcal{P}_{K,\mathfrak{m}}^{+}.$$
   where $x \in F^{\times}$ is any unit such that $xa \in \mathcal{J}_{F,\mathfrak{m}}^{+}$.

(i) Show that $\eta_{K,\mathfrak{m}}$ is surjective (This is the missing step in the proof of the idèle-ideal correspondence).

(ii) The definition of α depends on the choice of $x \in F^\times$ by weak approximation. Show that α is well-defined.

(iii) Compute the kernel of α.

4. (A more general case)

   Let E/F be a generic number extension (not necessarily Galois), and let $\mathcal{H} = F^\times \tilde{\mathrm{N}}_{E/F}\mathcal{J}_E$. Show that $\mathcal{H}$ is open in $\mathcal{J}_F$.