Student seminar exercise sheet Week 7

October 28, 2025

1. The explicit formula for the strict ray class number modulo $\mathfrak{m}$ is

$$|\mathcal{R}^+_{F,\mathfrak{m}}| = \frac{h_F 2^{r_1} \varphi(\mathfrak{m})}{[\mathcal{U}_F : \mathcal{U}^+_{F,\mathfrak{m}}]} \tag{1}$$

where :

- h_F is the class number
- r_1 is the number of real embeddings
- $\varphi(\mathfrak{m})$ is the cardinal of $\left(\mathcal{O}_F/\mathfrak{m}\right)^\times$
- $\mathcal{U}_F = \mathcal{O}_F^\times$, the units of $\mathcal{O}_F$
- $\mathcal{U}^+_{F,\mathfrak{m}} = \{\varepsilon \in \mathcal{U}_F \mid \varepsilon \gg 0, \varepsilon \equiv 1 \pmod{\mathfrak{m}}\}$

(a) Let $F = \mathbb{Q}(\sqrt{3})$ and $\mathfrak{m} = \mathcal{O}_F$. Compute the ray strict ray class number and deduce what the ray class group $\mathcal{R}^+_{\mathbb{Q}(\sqrt{3}),\mathbb{Z}[\sqrt{3}]}$ is up to isomorphism.

(b) Same question for $F = \mathbb{Q}(i)$, $\mathfrak{m} = \mathcal{O}_F$.

(c) Compare $\mathcal{R}^+_{\mathbb{Q}(i),\mathbb{Z}[i]}$ and $\mathcal{R}_{\mathbb{Q}(i),\mathbb{Z}[i]}$. How are they related ?

2. Let F be a number field and let $\mathfrak{n}, \mathfrak{m}$ be (not necessarily distinct) ideals of $\mathcal{O}_F$. Is it possible to find $\mathcal{H}_1 \neq \mathcal{H}_2$ with $\mathcal{P}^+_{F,\mathfrak{n}} \leq \mathcal{H}_1 < \mathcal{I}_F(\mathfrak{n})$ and $\mathcal{P}^+_{F,\mathfrak{m}} \leq \mathcal{H}_2 < \mathcal{I}_F(\mathfrak{m})$ such that they have the same class field over F ?

3. Let K be the class field of $\mathcal{H}$ over F where $\mathcal{P}^+_{F,\mathfrak{m}} \leq \mathcal{H} < \mathcal{I}_F(\mathfrak{m})$.
Prove that $[\mathcal{I}_F(\mathfrak{m}) : \mathcal{H}] = [K : F]$.

4. Let F be a number field. If we take $\mathfrak{m} = \mathcal{O}_F$, then the class field of $\mathcal{P}_{F,\mathfrak{m}}$ over F is called the *Hilbert class field*.

(a) Find the Hilbert class field of $\mathbb{Q}$.

(b) Find the Hilbert class field of $\mathbb{Q}(i)$.

5. *(Algebraic bonus !)* Prove that there is an exact sequence

$$0 \longrightarrow \mathcal{O}_F^\times \hookrightarrow F^{\mathfrak{m}} \longrightarrow \mathcal{I}(\mathfrak{m}) \twoheadrightarrow \mathcal{C}_F \longrightarrow 0$$

where $F^{\mathfrak{m}} := \{a \in F^\times \mid \langle a \rangle \in \mathcal{I}_F(\mathfrak{m})\}$.

This is lemma 1.1 on page 144 of the book Class Field Theory*, by J. S. Milne. You can refer to it for a proof !*