Student seminar exercise sheet Week 4

1. Review the concepts introduced in the lecture by consulting the Jupyter Notebook uploaded on Moodle (you can open it, for example, with CoCalc). The notebook summarizes the key notions from the lecture and demonstrates how they can be computed using SageMath. Run all code cells to observe how the computations are carried out. Experiment by modifying the examples (e.g., changing input values) to strengthen your understanding of the concepts and see how the results change.

2. (a) Let χ and ψ be primitive characters. Show that if $(f_\chi, f_\psi) = 1$, then $f_{\chi\psi} = f_\chi f_\psi$. Find a counterexample in the case where $(f_\chi, f_\psi) \neq 1$.

   (b) Show that the set of all even Dirichlet characters is a subgroup of the group of all Dirichlet characters.

3. Let $p > 2$ be a prime and let $n > 0$ be an integer. Calculate the absolute discriminant of $\mathbb{Q}(\zeta_{p^n})$.

   Hint: Use the Conductor Discriminant Formula

4. Let X_1 and X_2 be the group of Dirichlet characters corresponding to the fields L_1 and L_2, respectively. Show that:

   (a) The group generated by X_1 and X_2 corresponds to the compositum $L_1 L_2$.

   (b) The group $X_1 \cap X_2$ corresponds to $L_1 \cap L_2$.

5. In this exercise we study quadratic Dirichlet characters and their associated fields.

   (a) Let m be an odd positive integer. How many quadratic Dirichlet characters modulo m are there? How many of them are primitive?

   Hint: Consider the decomposition of a quadratic character. What is the group structure of $(\mathbb{Z}/p^a\mathbb{Z})^\times$?

   (b) What does your answer to part (a) tell you about the quadratic subfield(s) of $\mathbb{Q}(\zeta_p)$, where p is an odd prime? Does a quadratic subfield always exist? Is it unique? When is it real? Calculate the quadratic subfield(s) and the discriminant(s).

   Hint: Is the quadratic character even/odd? Use the conductor discriminant formula and [Sh2,Ex4].

(c) Let p be an odd prime. How many quadratic Dirichlet characters modulo $4p$ are there? How many of them are primitive? What does this tell you about the quadratic subfield(s) of $\mathbb{Q}(\zeta_{4p})$, where p is an odd prime? Calculate all quadratic subfield(s).

(d) Answer similar questions about the quadratic subfield(s) of $\mathbb{Q}(\zeta_8)$.

(e) Conclude that for any odd prime p, $\mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(\zeta_m)$ for $m = p$ or $4p$. Use this to show (without Kronecker–Weber) given any integer d, there is some m such that $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_m)$.