MATH562 – Fall 2025

Problem Set: Week 2

1. Investigate exponential tilting of a baseline density function $f_0(y)$ that is uniform on the interval $\mathcal{Y} = (0, 1)$, when the tilting functions are (a) $s(y) = y$ and (b) $s(y)^\mathsf{T} = (\log y, \log(1 - y))$.

   Solution: (a) In this case

$$k(\varphi) = \log \int_0^1 e^{\varphi y}\, \mathrm{d}y = \log\left(\frac{e^\varphi - 1}{\varphi}\right), \quad \varphi \in \mathbb{N} = \mathbb{R},$$

   with $k(0) = 0$ defined by continuity as $\varphi \to 0$. Hence the resulting exponential family is

$$f(y; \varphi) = e^{\varphi y - k(\varphi)} = \frac{\varphi e^{\varphi y}}{e^\varphi - 1}, \quad y \in \mathcal{Y}, \varphi \in \mathbb{R}.$$

   (b) In this case

$$k(\varphi) = \log \int_0^1 e^{\varphi_1 \log y + \varphi_2 \log(1-y)}\, \mathrm{d}y = \log \int_0^1 y^{\varphi_1}(1 - y)^{\varphi_2}\, \mathrm{d}y$$

   and we recognise this as a beta integral, defined for $\varphi_1, \varphi_2 > -1$, and then equal to $\Gamma(1 + \varphi_1)\Gamma(1 + \varphi_2)/\Gamma(2 + \varphi_1 + \varphi_2)$. The resulting density is therefore

$$f(y; \varphi) = \frac{\Gamma(2 + \varphi_1 + \varphi_2)}{\Gamma(1 + \varphi_1)\Gamma(1 + \varphi_2)} y^{\varphi_1}(1 - y)^{\varphi_2}, \quad y \in \mathcal{Y}, \varphi = (\varphi_1, \varphi_2) \in (-1, \infty)^2,$$

   which is just a rewriting of the beta density, for which $\alpha = \varphi_1 + 1$ and $\beta = \varphi_2 + 1$, and $\alpha, \beta > 0$, in the usual notation.

*2. For $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathbb{N}(\mu, \sigma^2)$, derive the limiting distribution of $Y = 1/\bar{X}$ as $n \to \infty$. Why can the event $\bar{X} = 0$ be neglected and what does the result tell us in practice?
   Hint: In the derivation, the cases $\mu \neq 0$ and $\mu = 0$ have to be treated differently.

   Solution:

   Using the results on linear combinations of normal variables, $\bar{X} \overset{d}{=} \mu + \sigma n^{-1/2} Z$, where $Z \sim \mathbb{N}(0, 1)$. Hence $Y \overset{d}{=} 1/(\mu + \sigma n^{-1/2} Z)$. If we apply the delta method with $g(u) = 1/u$, we have $g'(u) = -1/u^2$, provided $u \neq 0$. If $\mu \neq 0$ and as $n \to \infty$, therefore,

$$Y = g(\bar{X}) \overset{d}{=} g(\mu + \sigma n^{-1/2} Z) \doteq g(\mu) + \sigma n^{-1/2} Z g'(\mu) \sim \mathbb{N}\left\{g(\mu), g'(\mu)^2 \times \sigma^2/n\right\} = \mathbb{N}\{1/\mu, \sigma^2/(n\mu^4)\}.$$

   Note that if X has units of length (say), then its mean and its variance have units of length and length^2, so $1/X$ has units of 1/length and its variance has units of 1/length^2, agreeing with the distribution here.

   If $\mu = 0$, then for any n,

$$\Pr(\bar{X} \leq 0) = \Pr(\bar{X} > 0) = \Pr(\mu + \sigma n^{-1/2} Z > 0) = \Pr(Z > 0) = 1/2,$$

   and if $y > 0$ then

$$\begin{aligned}
\Pr(Y > y \mid \bar{X} > 0) &= \Pr\{1/(\mu + \sigma n^{-1/2} Z) > y \mid Z > 0\} \\
&= \Pr\{Z < n^{1/2}/(\sigma y) \mid Z > 0\} \\
&= 2\left[\Phi\left\{n^{1/2}/(y\sigma)\right\} - \Phi(0)\right] \\
&\to 1, \quad n \to \infty,
\end{aligned}$$

and

$$\Pr(Y > y \mid \bar{X} < 0) = \Pr\{1/(\mu + \sigma n^{-1/2} Z) > y \mid Z < 0\} = \Pr\{n^{1/2}/(\sigma Z) > y \mid Z < 0\} = 0.$$

The event $\bar{X} = 0$ has zero probability for any n, so for any $y > 0$ we have

$$\begin{aligned} \Pr(Y > y) &= \Pr(Y > y \mid \bar{X} < 0)\Pr(\bar{X} < 0) + \Pr(Y > y \mid \bar{X} > 0)\Pr(\bar{X} > 0) \\ &= 0 \times \frac{1}{2} + 2\left[\Phi\left\{n^{1/2}/(y\sigma)\right\} - 1/2\right] \times \frac{1}{2} \to \frac{1}{2}, \quad n \to \infty, \end{aligned}$$

and for any $y < 0$ we have $\Pr(Y < y) \to \frac{1}{2}$ by symmetry. Thus the limiting distribution of Y concentrates at $\pm\infty$ with equal probabilities.

3. In a simple model for the spread of an epidemic in a large closed population of n identical individuals, it can be shown that the the expression $1 - \tau = e^{-R_0\tau}$ relates the basic reproduction number R_0 (the number of susceptible persons infected by a single infective person at the start of the epidemic) to the ultimate fraction τ of the population who are infected.

   (a) Show that if $R_0 \le 1$ then there is only one possible value for τ, but that if $R_0 > 1$ then there are two possible values, and explain this heuristically.

   Solution: Clearly $\tau = 0$ is one possible solution for any $R_0 > 0$. The slopes of $1 - \tau$ and $e^{-R_0\tau}$ at $\tau = 0$ are respectively -1 and $-R_0$, and it is clear from a plot of these two functions against τ that there is a second, positive, solution to the equation within the interval $\tau \in (0, 1)$ if $R_0 > 1$. The intuition is that if $R_0 \le 1$, then the epidemic is certain to be of negligible size, but that if $R_0 > 1$ then it may be of negligible size (if by chance it dies out immediately) but otherwise will affect a positive fraction of the population.

   (b) If it is positive, the final proportion infected can be estimated by $\hat{\tau}$, which has an approximate normal distribution with mean τ and variance σ^2/n, where

   $$\sigma^2 = \frac{\tau(1-\tau)}{\{1 - (1-\tau)R_0\}^2}\{1 + c^2(1-\tau)R_0^2\},$$

   where $c = \mathrm{Var}(T)^{1/2}/\,\mathrm{E}(T)$ is the coefficient of variation of the infectious period for an individual, T. Hence obtain an estimator of R_0, and show that this is approximately normal with mean R_0 and variance σ_R^2/n, where $\sigma_R^2 = \{1 + c^2(1-\tau)R_0^2\}/\{\tau(1-\tau)\}$.

   Solution: The equation $1 - \tau = e^{-R_0\tau}$ gives $R_0 = -\tau^{-1}\log(1-\tau) = g(\tau)$, say, which is a smooth function of τ for $\tau \in (0, 1)$. Hence the delta method applies, and $\hat{R}_0 = g(\hat{\tau})$ is asymptotically normal with mean $g(\tau) = R_0$ and with variance given using

   $$g'(\tau) = \log(1-\tau)/\tau^2 + 1/\{\tau(1-\tau)\} = -R_0/\tau + 1/\{\tau(1-\tau)\} = \{1 - R_0(1-\tau)\}/\{\tau(1-\tau)\};$$

   the delta method variance formula $\sigma_R^2 = g'(\tau)^2\sigma^2$ gives the required result.

   (c) Find c^2 when T is constant, uniform on some interval, and exponential. Hence suggest why an upper bound for the variance might be obtained by setting $c = 1$.

   Solution: The variance of a constant is zero, so if the constant itself is non-zero, $c^2 = 0/T = 0$. If $T \sim U(0, d)$ then $\mathrm{E}(T) = d/2$ and $\mathrm{Var}(T) = d^2/12$, leading to $c^2 = 1/3$. An exponential variable with mean μ has variance μ^2, giving $c^2 = 1$. The exponential distribution has high coefficient of variation (for example, the gamma distribution has $c = \alpha^{-1/2}$, where α is the shape parameter, and taking $\alpha > 1$ gives a unimodal distribution that seems potentially suitable for T), so taking $c = 1$ should give an upper bound for σ^2 and hence for σ_R^2.

4. (a) The random variable Y follows a Lomax distribution, i.e.,

$$\Pr(Y \le y) = \begin{cases} 1 - \dfrac{\theta^\alpha}{(\theta+y)^\alpha}, & y > 0, \\ 0, & y \le 0, \end{cases}$$

where $\alpha, \theta > 0$ are unknown. Is this an exponential family distribution?

Solution: The density is

$$f(y) = \frac{\alpha\theta^\alpha}{(\theta+y)^{\alpha+1}} = \exp\left\{-(\alpha+1)\log(\theta+y) + \log\alpha + \alpha\log\theta\right\}, \quad y > 0, \quad \alpha, \theta > 0.$$

This is not of exponential family form, because no term in the log density can be written as $s(y)^\mathsf{T}\varphi(\theta)$ for some function $s(y)$ that depends only on y.

From here on $Y_1, \ldots, Y_n$ represent independent identically distributed variables from the Lomax distribution with unknown θ and known $\alpha > 2$.

(b) Given that

$$\mathrm{E}(Y) = \frac{\theta}{\alpha-1}, \quad \mathrm{Var}(Y) = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)},$$

use $Y_1, \ldots, Y_n$ to obtain a method-of-moments estimator $\tilde\theta$ of θ, and compute $\mathrm{Var}(\tilde\theta)$. Is $\tilde\theta$ biased? **Solution**:

The method-of-moments estimator satisfies $\bar Y = \tilde\theta/(\alpha-1)$, i.e., $\tilde\theta = (\alpha-1)\bar Y$, and this is easily checked to be unbiased using the given formula for $\mathrm{E}(Y)$ and the fact that $\mathrm{E}(\bar Y) = \mathrm{E}(Y)$.

Moreover $\mathrm{Var}(\bar Y) = \mathrm{Var}(Y)/n$, so

$$\mathrm{Var}(\tilde\theta) = (\alpha-1)^2\mathrm{Var}(\bar Y) = \frac{(\alpha-1)^2}{n}\mathrm{Var}(Y) = \frac{\theta^2\alpha}{n(\alpha-2)}.$$

(c) Another estimator of θ is $\tilde\theta_c = c\bar Y$, where $\bar Y = n^{-1}(Y_1 + \cdots + Y_n)$ and $c > 0$. Compute the bias and variance of $\tilde\theta_c$. What value of c minimises the mean square error of $\tilde\theta_c$?

Solution:

The bias of of $\tilde\theta_c$ is

$$b(\tilde\theta_c;\theta) = \mathrm{E}(\tilde\theta_c) - \theta = c\frac{\theta}{\alpha-1} - \theta = \frac{(\alpha-1-c)\theta}{\alpha-1},$$

and $\mathrm{Var}(\tilde\theta_c) = c^2\mathrm{Var}(\bar Y)$, so (after a little algebra) the mean squared error of $\tilde\theta_c$ is

$$b(\tilde\theta_c;\theta)^2 + c^2\mathrm{Var}(\bar Y) = \frac{\theta^2}{n(\alpha-1)^2(\alpha-2)}\left\{n(\alpha-2)(\alpha-1-c)^2 + c^2\alpha\right\}.$$

This is minimised by differentiating the expression $\{\cdot\}$ here with respect to c, giving

$$\frac{\mathrm{d}\{\cdot\}}{\mathrm{d}c} = -2n(\alpha-2)(\alpha-1-c) + 2c\alpha = 0,$$

which results in

$$c = \frac{n(\alpha-1)(\alpha-2)}{n(\alpha-2)+\alpha} \to \alpha - 1, \quad n \to \infty.$$

Note that for any finite n the estimator is biased, but it is asymptotically unbiased, and that the second derivative $\mathrm{d}^2\{\cdot\}/\,\mathrm{d}c^2 = 2n(\alpha-1)(\alpha-2) + 2\alpha$ is positive — which is obvious because the MSE is a sum of positive quadratic expressions in c.