MATH562 – Fall 2025

Problem Set: Week 2

1. Investigate exponential tilting of a baseline density function $f_0(y)$ that is uniform on the interval $\mathcal{Y} = (0, 1)$, when the tilting functions are (a) $s(y) = y$ and (b) $s(y)^\mathsf{T} = (\log y, \log(1 - y))$.

*2. For $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathbb{N}(\mu, \sigma^2)$, derive the limiting distribution of $Y = 1/\bar{X}$ as $n \to \infty$. Why can the event $\bar{X} = 0$ be neglected and what does the result tell us in practice?
Hint: In the derivation, the cases $\mu \neq 0$ and $\mu = 0$ have to be treated differently.

3. In a simple model for the spread of an epidemic in a large closed population of n identical individuals, it can be shown that the the expression $1 - \tau = e^{-R_0 \tau}$ relates the basic reproduction number R_0 (the number of susceptible persons infected by a single infective person at the start of the epidemic) to the ultimate fraction τ of the population who are infected.

   (a) Show that if $R_0 \leq 1$ then there is only one possible value for τ, but that if $R_0 > 1$ then there are two possible values, and explain this heuristically.

   (b) If it is positive, the final proportion infected can be estimated by $\hat{\tau}$, which has an approximate normal distribution with mean τ and variance σ^2/n, where

   $$\sigma^2 = \frac{\tau(1 - \tau)}{\{1 - (1 - \tau)R_0\}^2}\{1 + c^2(1 - \tau)R_0^2\},$$

   where $c = \mathrm{Var}(T)^{1/2}/\,\mathrm{E}(T)$ is the coefficient of variation of the infectious period for an individual, T. Hence obtain an estimator of R_0, and show that this is approximately normal with mean R_0 and variance σ_R^2/n, where $\sigma_R^2 = \{1 + c^2(1 - \tau)R_0^2\}/\{\tau(1 - \tau)\}$.

   (c) Find c^2 when T is constant, uniform on some interval, and exponential. Hence suggest why an upper bound for the variance might be obtained by setting $c = 1$.

4. (a) The random variable Y follows a Lomax distribution, i.e.,

   $$\Pr(Y \leq y) = \begin{cases} 1 - \dfrac{\theta^\alpha}{(\theta + y)^\alpha}, & y > 0, \\ 0, & y \leq 0, \end{cases}$$

   where $\alpha, \theta > 0$ are unknown. Is this an exponential family distribution?

   From here on $Y_1, \ldots, Y_n$ represent independent identically distributed variables from the Lomax distribution with unknown θ and known $\alpha > 2$.

   (b) Given that

   $$\mathrm{E}(Y) = \frac{\theta}{\alpha - 1}, \quad \mathrm{Var}(Y) = \frac{\alpha\theta^2}{(\alpha - 1)^2(\alpha - 2)},$$

   use $Y_1, \ldots, Y_n$ to obtain a method-of-moments estimator $\tilde{\theta}$ of θ, and compute $\mathrm{Var}(\tilde{\theta})$. Is $\tilde{\theta}$ biased?

   (c) Another estimator of θ is $\tilde{\theta}_c = c\bar{Y}$, where $\bar{Y} = n^{-1}(Y_1 + \cdots + Y_n)$ and $c > 0$. Compute the bias and variance of $\tilde{\theta}_c$. What value of c minimises the mean square error of $\tilde{\theta}_c$?